Decidability of Temporal Logic of Two Deimensional Minkowski Spacetime

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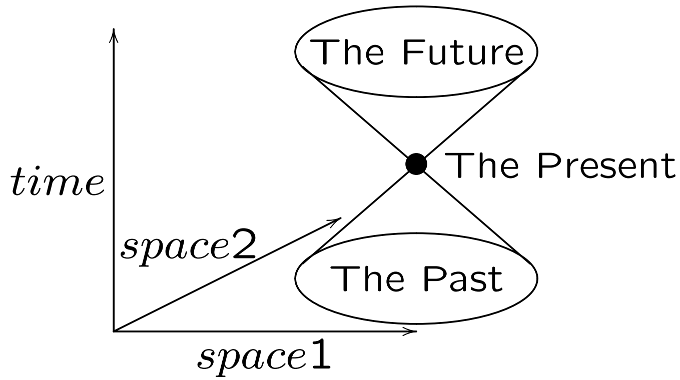
Minkowski Spacetime

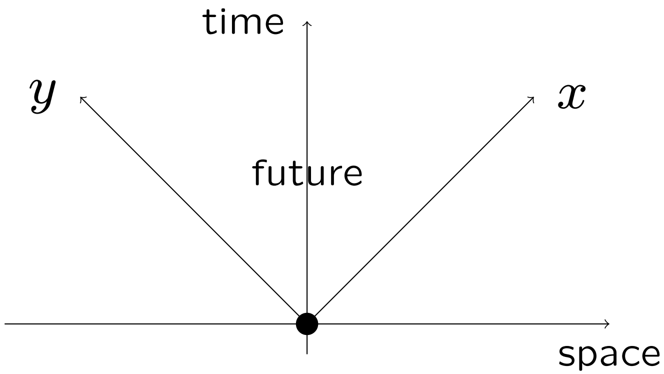


$$(x_0, x_1, \ldots, x_n) \leq (y_0, y_1, \ldots, y_n) \iff \sqrt{\sum_{i<n} (y_i - x_i)^2} \leq y_n - x_n$$

$$\mathcal{F}, \mathsf{x} \models \mathbf{F}\psi \iff (\exists \mathsf{y} \geq \mathsf{x})\ \mathcal{F}, \mathsf{y} \models \psi$$

$$\mathcal{F}, \mathsf{x} \models \mathbf{P}\psi \iff (\exists \mathsf{y} \leq \mathsf{x})\ \mathcal{F}, \mathsf{y} \models \psi$$

Two dimensional coordinates

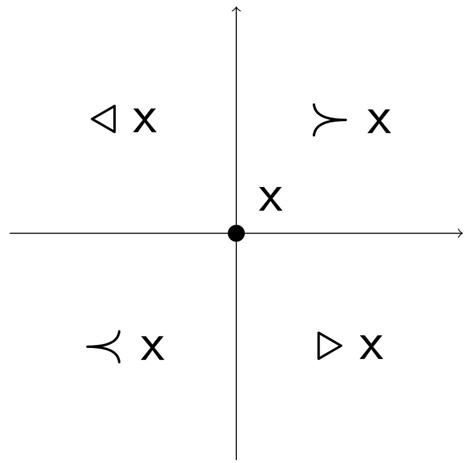


$$(x,y) \leq (x',y') \iff x \leq x' \wedge y \leq y'$$

$$(x,y) < (x',y') \iff (x,y) \leq (x',y') \wedge (x,y) \neq (x',y')$$

$$(x,y) \prec (x',y') \iff x < x' \wedge y < y'$$

$$(x,y) \lhd (x',y') \iff x \leq x' \wedge y \geq y' \wedge (x,y) \neq (x',y')$$


◁ x
≻ x
x
≺ x
▷ x

Modal Axioms S4.2

$$\mathbf{G}(\psi \to \theta) \to (\mathbf{G}\psi \to \mathbf{G}\theta)$$
$$\mathbf{G}\psi \to \mathbf{GG}\psi$$
$$\mathbf{FG}\psi \to \mathbf{GF}\psi$$
$$\mathbf{G}\psi \to \psi$$

+ Duals + **Temporal Axioms**

$$\psi \to \mathbf{GP}\psi$$
$$\psi \to \mathbf{HF}\psi$$
$$\mathbf{FP}\psi \leftrightarrow \mathbf{PF}\psi$$

S4.2 is Sound and Complete for modal logic of . . .

- Confluent partial orders

- $(\mathbb{R}^n, \leq)$ (any $n \geq 2$)

- $(\mathbb{Q}^n, \leq)$

S4.2 + Temporal Axioms is complete for confluent partial orders.
Proof, by filtration.

Problems

1. Axiomatise temporal logic of $(\mathbb{R}^2, \leq)$, $(\mathbb{R}^2, <)$, $(\mathbb{Z}^2, \leq)$, $(\mathbb{Q}^3, \prec), \ldots$

2. Prove decidability of $(\mathbb{R}^2, \leq)$, $(\mathbb{R}^2, <)$, $(\mathbb{R}^2, \prec)$,

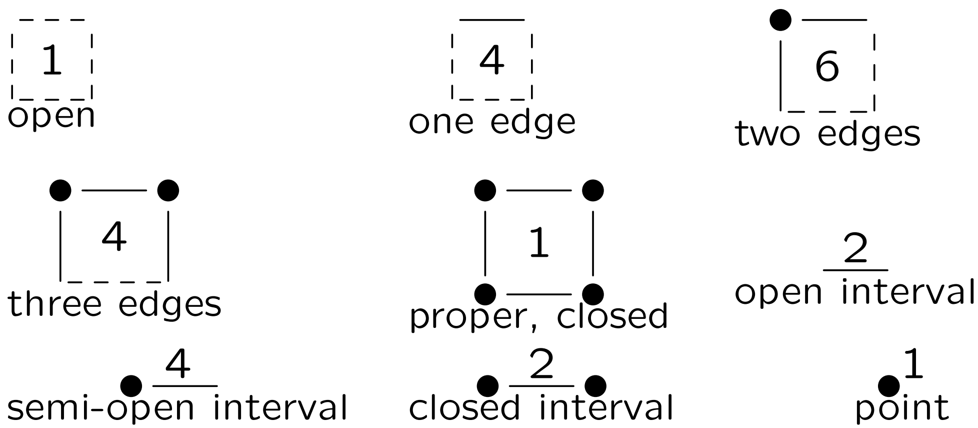
3. Prove undecidability of $(\mathbb{R}^3, \leq)$, $(\mathbb{R}^2, <)$,

4. Find temporal formula distinguishing $(\mathbb{R}^n, \leq)$ and $(\mathbb{R}^m, \leq)$ for $2 \leq n < m$.

Rectangles

$$x, y \in R,\ x \wedge y \leq z \leq x \vee y \rightarrow z \in R$$

Cartesian product of two convex intervals.



Topology

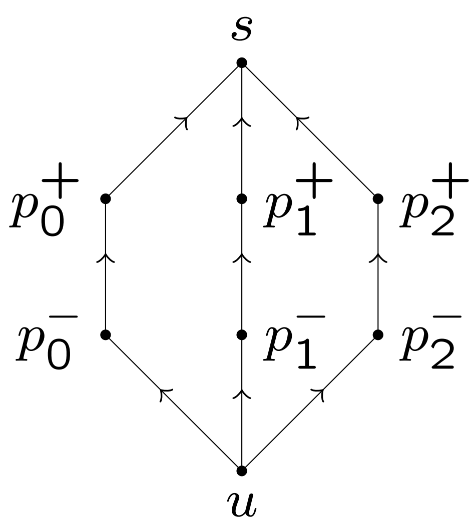
1. Boundary of S is closure minus interior — a closed set.

2. If S, T are closed and bounded subsets of $\mathbb{R}^2$ and $\forall \epsilon > 0 \exists s \in S,\ t \in T\ \ d(s,t) \leq \epsilon$ then $S \cap T \neq \emptyset$.

3. R is a closed rectangle, say $[0,1] \times [0,1]$. If S is closed downward and has non-trivial boundary then boundary is homeomorphic to a closed line segment.

4. If open line segment is partitioned into closed line segments, then at least one is a single point.

Distinguishing Formulas



$$
\begin{aligned}
\Delta(\mathcal{P}) = \mathbf{GH} \quad [\ & \textstyle\bigvee P && \wedge \\
& \textstyle\bigwedge_{p \neq q \in P} \neg(p \wedge q) && \wedge \\
& \textstyle\bigwedge_{p < q \in P} (p \to \mathbf{F}q) \wedge (q \to \mathbf{P}p) && \wedge \\
& \textstyle\bigwedge_{p \not\leq q \in P} (p \to \mathbf{G}\neg q) \wedge (q \to \mathbf{H}\neg p) \\
]\ &
\end{aligned}
$$

$(\mathbb{R}^2, \leq) \models \neg\Delta(\mathcal{P}), \qquad (\mathbb{Q}^2, \leq) \not\models \neg\Delta(\mathcal{P})$

Filtration

ϕ a fixed temporal formula.
$Cl(\phi) = \{$subformulas, single negations of subformulas of $\phi\}$.
MCS is set of maximal consistent subsets of $Cl(\phi)$.

$$m \leq n \iff (\mathbf{G}\psi \in m \to \mathbf{G}\psi \in n \ \wedge \ \mathbf{H}\psi \in n \to \mathbf{H}\psi \in m)$$

$(MCS, \leq)$ is reflexive, transitive, confluent, but not antisymmetric.
Cluster is equivalence class of MCSs. $MCS/\sim$ is a confluent partial order.

Trace

$$(c_0, m_0, c_1, m_1, \ldots, m_{k-1}, c_k)$$

where $c_i \leq m_i \leq c_{i+1}$ and $c_i < c_{i+1}$, for $i < k$.

Rectangle Model

$$h : R \to MCS$$

- $\mathsf{x} \leq \mathsf{y} \in R \to h(\mathsf{x}) \leq h(\mathsf{y})$

- If $\mathbf{F}\psi \in h(\mathsf{x})$ then either
  - $\exists \mathsf{y} \geq \mathsf{x}\ \psi \in h(\mathsf{y})$,
  - R includes boundary point y due East of x and $\mathbf{F}\psi \in h(\mathsf{y})$, or
  - R includes boundary point y due North of x and $\mathbf{F}\psi \in h(\mathsf{y})$.

- Similar for $\mathbf{P}\psi$.

Defects

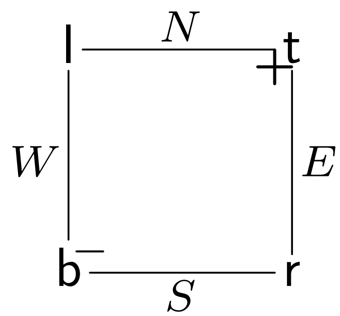
MCS $\mathbf{F}\psi \in m,\ \psi \notin m$,

Cluster $\mathbf{F}\psi \in \bigcup c,\ \psi \notin \bigcup c$,

Trace $\mathbf{F}\psi$ is a defect of c_i but $\mathbf{F}\psi \notin m_i$.

Boundary Maps

$$\partial : \{-,+\} \cup \{\mathsf{b}, \mathsf{t}, \mathsf{l}, \mathsf{r}\} \cup \{N, S, E, W\} \to \{\text{clusters}\} \cup MCS \cup \{\text{traces}\}$$



Rectangle Model to Boundary Map

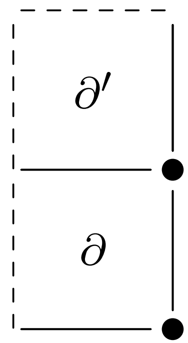
Rectangle model h determines boundary map ∂^h.

Simple boundary maps

$\partial \in B_0$ if

- $\partial(-) = \partial(+)$,
- No internal defects

Joining



$$\partial \oplus_N \partial'$$

Limits

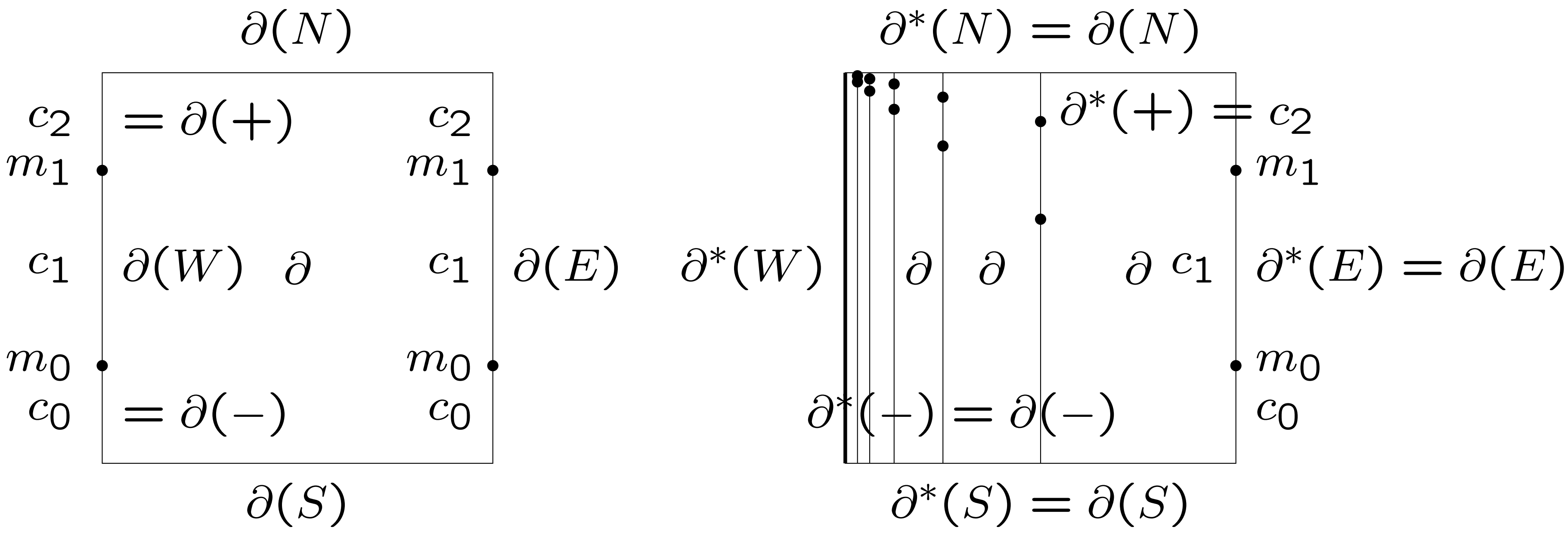
Suppose

- $\partial = \partial \oplus_W \partial$,

- $\exists$ undecomposable rectangle model h where $\partial = \partial^h$

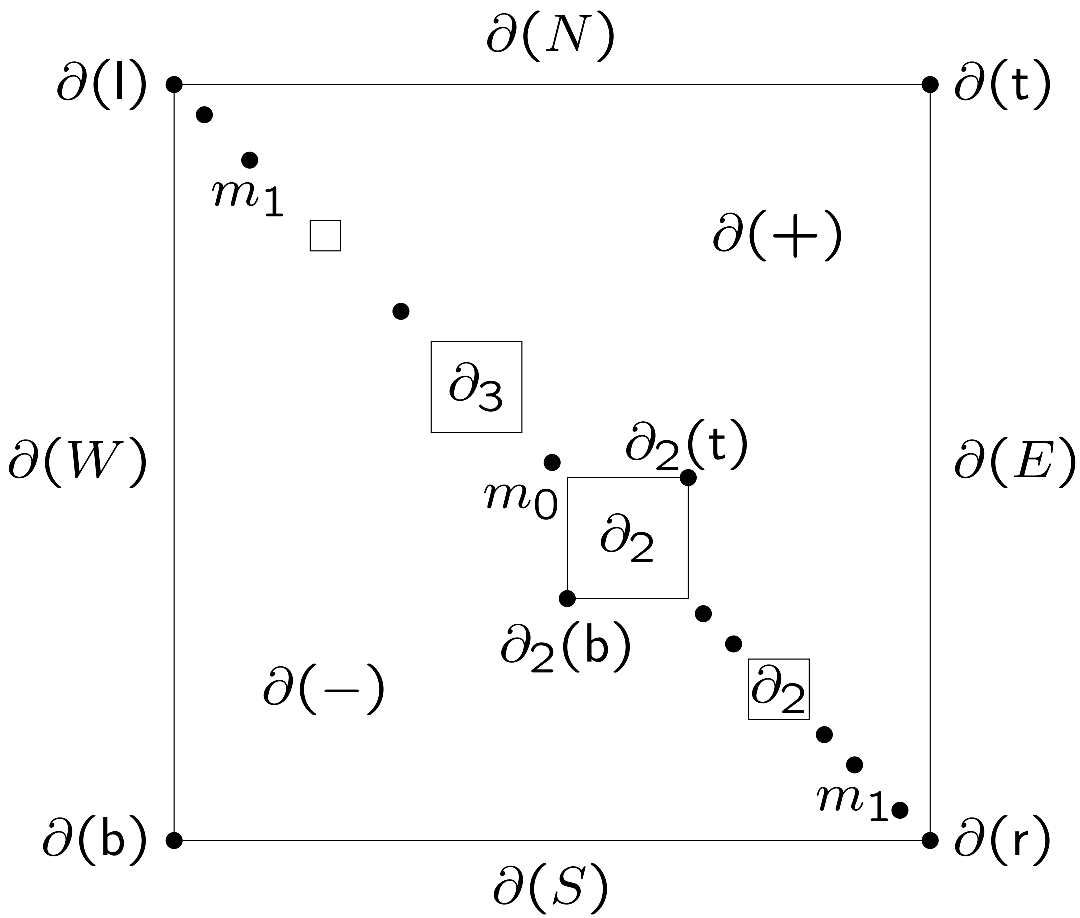
- ∂^* is identical to ∂ except $\partial^*(W)$ is either undefined or it can be a single cluster trace, such that there are no internal defects.

then ∂^* is a Western limit of ∂.



$\partial(N)$
c_2
$= \partial(+)$
c_2
m_1
m_1
c_1
$\partial(W)$
∂
c_1
$\partial(E)$
m_0
m_0
c_0
$= \partial(-)$
c_0
$\partial(S)$
$\partial^*(N) = \partial(N)$
$\partial^*(+) = c_2$
m_1
$\partial^*(W)$
∂
∂
∂
c_1
$\partial^*(E) = \partial(E)$
m_0
$\partial^*(-) = \partial(-)$
c_0
$\partial^*(S) = \partial(S)$

Shuffles


$\partial(N)$
$\partial(\mathsf{l})$
$\partial(\mathsf{t})$
m_1
$\partial(+)$
∂_3
$\partial_2(\mathsf{t})$
$\partial(W)$
$\partial(E)$
m_0
∂_2
$\partial_2(\mathsf{b})$
$\partial(-)$
∂_2
m_1
$\partial(\mathsf{b})$
$\partial(\mathsf{r})$
$\partial(S)$

B

B is closure of B_0 under joins, limits and shuffles.

Algorithm 1 Algorithm to compute B

```
B = B_0
while new elements can be found do
    Add any joins of elements of B to B
    Add any limits of elements of B to B
    Add any suffles from B to B
```

Main Theorem

$$\partial \in B \iff (\exists h)\, \partial = \partial^h$$

$\Rightarrow$ By induction on number of iterations of the while loop in algorithm for B.

$\Leftarrow$ By induction on maximum length of chain of distinct clusters from $\partial^h(-)$ up to $\partial^h(+)$,

Good sets

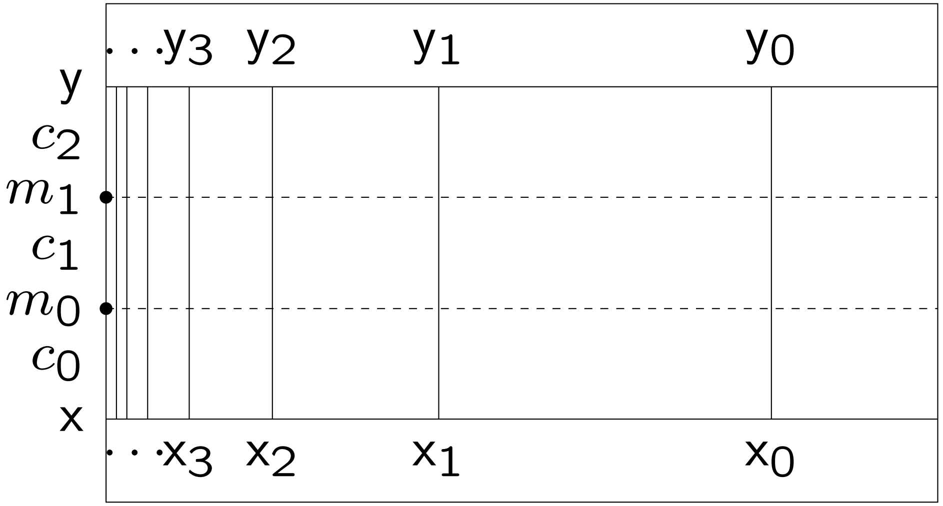
Let h be a rectangle model. Must show $\partial^h \in B$.

Def: $S \subseteq Cl(dom(h))$ is *good* if it is a finite union of rectangles and for every defined subrectangle R of S we have $\partial^{h|_R} \in B$

Main Proof

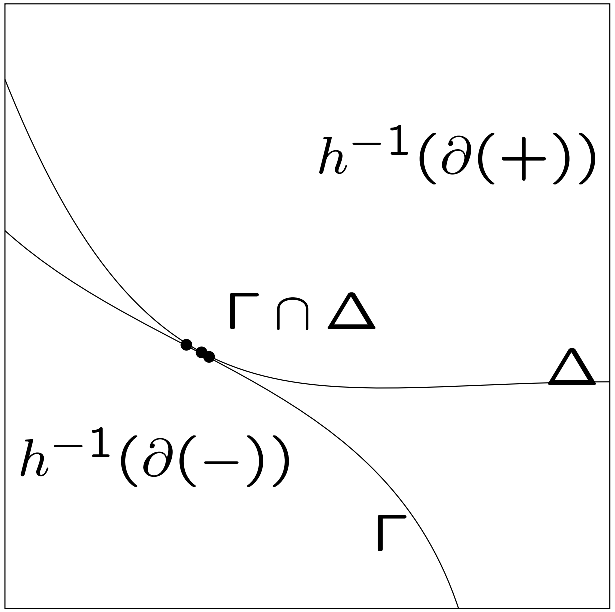
- $G(X), G(Y) \Rightarrow G(X \cup Y)$,

- $(\forall i > 0 G[\mathsf{x}_i, \mathsf{y}_0]) \Rightarrow G[\mathsf{x}, \mathsf{y}_0]$,



Boundaries

Let Γ be boundary of $h^{-1}(\partial(-))$, let Δ be boundary of $h^{-1}(\partial(+))$.



- If $\Gamma \cap \Delta \cap Int(dom(h)) = \emptyset$ then $dom(h)$ is good,

Equivalence Relation $\approx$ over $\Gamma \cap \Delta$

Contains $\triangleleft$ successor relation, and closed under limits.

- $S \subseteq \Gamma \cap \Delta$ a $\approx$ equiv. class, $\triangleleft$-bounded by x, y then $R(S) = [\mathsf{x} \wedge \mathsf{y}, \mathsf{x} \vee \mathsf{y}]$ is good,

- Union of the lower boundaries of the $R(S)$s (for S a $\approx$-class) is homeomorphic to a simple line segment,

- There is a singleton $\approx$ class,

- ∂^h is a shuffle of $\partial^{h|_{R(S)}}$s for $S \in E$.

References

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten V [On infinite, linear point-manifolds (sets)] Mathematische Annalen, 21: 545–591, 1883.

R. Goldblatt. Diodorean modality in Minkowski space-time. *Studia Logica*, 39:219–236, 1980.

A. Prior. *Past, Present and Future*. Oxford University Press, 1967.

A. Prior. The notion of the present. *Studium Generale*, 23:245–248, 1970.

V. Shehtman. Modal logics of the real plane. *Studia Logica*, 42:63–80, 1983.

Ilya Shapirovsky and Valentin B. Shehtman. Chronological future modality in minkowski spacetime. In Philippe Balbiani, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, editors, *Advances in Modal Logic*, 437–460. King's College Publications, 2002.