Definitions and Contextualism for Topologies on the Space of Spacetimes

Samuel C. Fletcher

Munich Center for Mathematical Philosophy
Ludwig-Maximilians-Universität

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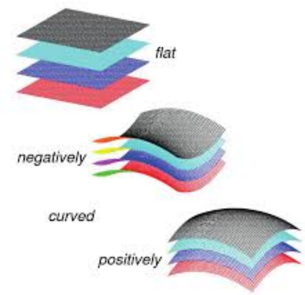
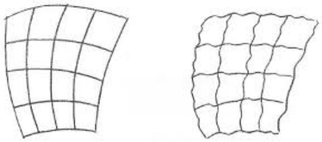
Spacetime Similarity





(L) Daniel Baumann. *Cosmology (Part III Mathematical Tripos)*
`http://www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf`.
(R) Hannu Kurki-Suonio. *Cosmological Perurbation Theory*.
`http://www.helsinki.fi/~hkurkisu/CosPer.pdf`

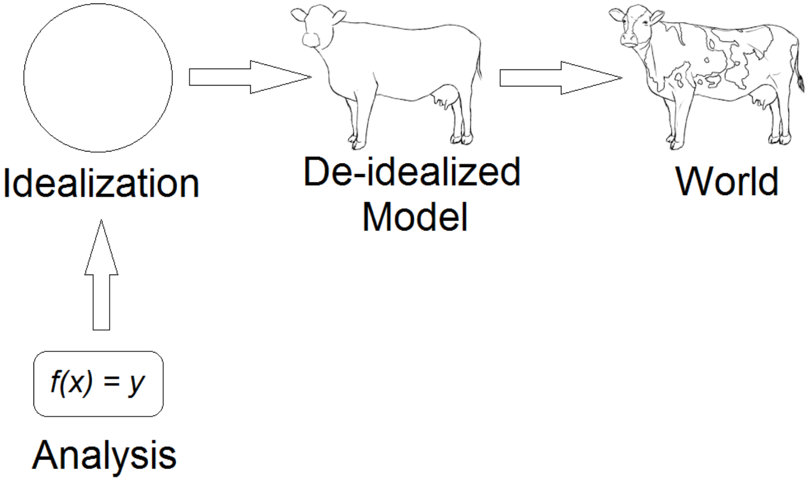
Lerner on Perturbations

The question naturally arises as to how prevalent these singular solutions are. Is it true that "nearly all" physically reasonable, spatially closed cosmologies are singular? And if a given space-time is singular, does it remain so under small perturbations of the metric tensor (are the singularities stable, in other words)? What do we mean by "perturbing the metric"?

It is evident that the precise mathematical formulation of these questions proceeds in the following manner: we must topologize the set of [relativistic spacetimes]. (1–2)

Lerner, David E. *The Space of Lorentz Metrics on a Non-Compact Manifold*. PhD dissertation: University of Pittsburgh, 1972.

Spacetime Similarity: Justified Inference



Geroch: Stability as Nec. Condition

It is a general feature of the description of physical systems by mathematics that only conclusions which are stable, in an appropriate sense, are of physical interest.
. . . To obtain a precise notion of stability in general relativity we must say what “sufficiently small perturbation” means, i.e., we must find a suitable topology on the space of solutions of Einstein’s equations. (70)

Geroch, R. “Singularities,” *Relativity*. Ed. Moshe Carmeli, Stuart I. Fickler, and Louis Witten. New York: Plenum, 1970.

Hawking: Stability as Nec. Condition

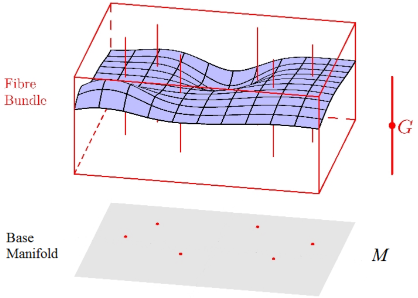
. . . the only properties of space-time that are physically significant are those that are stable in some appropriate topology. (395)

Hawking, S.W. “Stable and Generic Properties in General Relativity,” *General Relativity and Gravitation* 1.4: 393–400 (1971).

Metrics as Sections

Models

- (M, g), M fixed
- $T_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$



Adapted from Henrique de A. Gomez, “The Dynamics of Shape," arXiv:1108.4837.

Topology from Observables

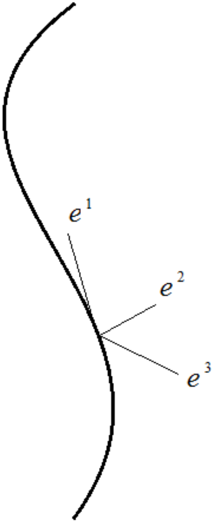
Idea

"Closeness" of spacetimes = "closeness" of certain classes of observables.

Observers

1. Congruences of observers with frame fields covering M
2. Construct (inverse) Riemannian metric

$$h = \sum_{i=1}^{n} e^i \otimes e^i$$



Observables

Real-valued fields constructed from g and possibly its derivatives:

Regions	Points, compact regions, the whole of M
Derivatives	Zero, one, two, . . . , compatible with h.

Observables: Examples

Observable		Region	Derivatives
Red-shift factor	$g_{ab}(e^0)^a(e^0)^b$	Points	Zero
Curvature	$R^a_{bcd}\xi^b = -2\nabla_{[c}\nabla_{d]}\xi^a$	Points	Two
Proper Time	$\int_\gamma (g_{ab}\xi^a\xi^b)^{1/2} ds$	Lines	Zero
Komar Mass	$\int_M (T_{ab} - 2Tg_{ab})u^a\kappa^b dV$	Manifold	Two

h-Fiber Norms

1. Consider smoothly varying norms on the tensor bundle's fibers, i.e., if K is an (r, s)-tensor field:

$$|K|_h = \begin{cases} |K|, \text{ if } r = s = 0, \\ |K^{a_1 \cdots a_r}_{b_1 \cdots b_s} K^{c_1 \cdots c_r}_{d_1 \cdots d_s} h_{a_1 c_1} \cdots h_{a_r c_r} h^{b_1 d_1} \cdots h^{b_s d_s}|^{1/2}, & \text{o/w.} \end{cases}$$

2. Decide how many derivatives of the metric to take into account:

$$|K|_h, |\nabla K|_h, \ldots, |\nabla^{(n)} K|_h$$

3. Decide if the topology is to determine, e.g., pointwise, compact, or uniform convergence:

$$B^n(g, \epsilon; h, S) = \{g' : (|g - g'|_h)_{|S} < \epsilon_{|S}, \ldots, (|\nabla^{(n)}(g - g')|_h)_{|S} < \epsilon_{|S}\}$$

where $\epsilon : M \to \mathbb{R}$.

Point-Open Topologies

The sets

$$B^n(g,\epsilon;h,p) = \{g' : (|g-g'|_h)_{|p} < \epsilon_{|p}, \ldots, (|\nabla^{(n)}(g-g')|_h)_{|p} < \epsilon_{|p}\},$$

ranging over

- all Lorentz metrics g
- all continuous ϵ
- all $p \in M$
- (all Riemannian h)

form a subbasis for the C^n *point-open topology*.

Theorem

A sequence of spacetimes $\overset{k}{g}$ converges to a spacetime g as $k \to \infty$ in the C^n point-open topology iff the observables for $\overset{k}{g}$ defined on points, depending on at most n derivatives, converge to those for g as $k \to \infty$.

Compact-Open Topologies

The sets

$$B^n(g,\epsilon;h,C) = \{g' : (|g-g'|_h)_{|C} < \epsilon_{|C}, \ldots, (|\nabla^{(n)}(g-g')|_h)_{|C} < \epsilon_{|C}\},$$

ranging over

- all Lorentz metrics g
- all continuous ϵ
- all compact $C \subseteq M$
- (all Riemannian h)

form a subbasis for the C^n *compact-open topology*.

Theorem

A sequence of spacetimes $\overset{k}{g}$ converges to a spacetime g as $k \to \infty$ in the C^n compact-open topology iff the observables of $\overset{k}{g}$ defined on compact sets, depending on at most n derivatives, converge to those of g as $k \to \infty$.

Open Topologies

The sets

$$B^n(g,\epsilon;h,M)=\{g' : |g-g'|_h<\epsilon,\dots,|\nabla^{(k)}(g-g')|_h<\epsilon\},$$

ranging over

- all Lorentz metrics g
- all continuous ϵ
- all Riemannian h (if M is non-compact)

form a subbasis for the C^n *open topology*.

Theorem

A sequence of spacetimes $\overset{k}{g}$ converges to a spacetime g as $k\to\infty$ in the C^n open topology iff there is some compact $C\subseteq M$ such that eventually $\overset{k}{g}_{|M-C}=g_{|M-C}$, and $\overset{k}{g}$ converges to g as $k\to\infty$ in the C^n compact-open topology.

Theorem

The C^n open topology is metrizable iff M is compact.

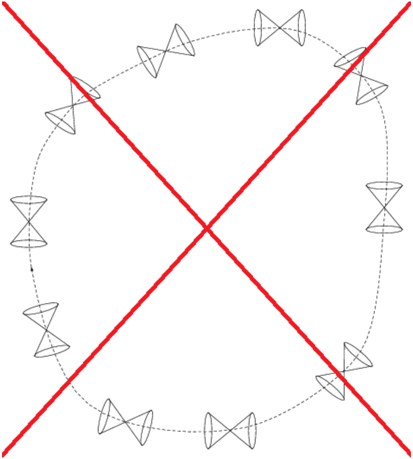
Relevance of Open Topologies

Theorem (Hawking)

The existence of a global time function on a relativistic space-time is equivalent to stable causality, an absence of closed causal curves that is stable in the C^0 open topology.

Theorem

No sequence of relativistic spacetimes converges to a Newtonian spacetime in any open topology.



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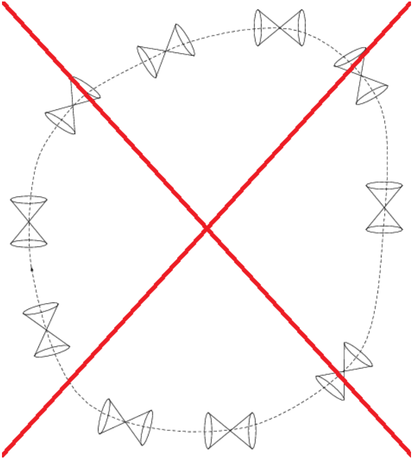

Drawbacks of Compact-Open Topologies

Theorem

If $\dim(M) \leq 3$*, then chronology violating spacetimes are generic in the space of Lorentz metrics on M in any of the* C^n *compact-open topologies.*

Theorem

If $\dim(M) \leq 3$*, no Lorentz metric is stably causal in any of the* C^n *compact-open topologies on the Lorentz metrics of M.*

Theorem

Every spacetime with manifold M containing a closed timelike curve does so stably in any of the C^n *compact-open topologies if* $\dim(M) \leq 3$*.*

A Modest No-Go

Theorem

If there is some Lorentz metric on M that admits of a global time function, then there is no topology on the space of Lorentz metrics on M relative to which

1. *all parameterized family of spacetimes continuous in the C^n compact-open topology are continuous, and*
2. *Hawking's theorem holds.*