On spacetimes with small exotic smooth structure

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Agenda

Troublemaking.

Relativistic spacetime

$$\langle M, g \rangle,$$

where M is a smooth 4-dimensional manifold, and g is a Lorentz-signature metric.

Indeterminism

Indeterminism: situation when initial data do not uniquely fix their development.

$A = \langle M, g \rangle$ is indeterministic iff there exists $B = \langle M', g' \rangle$ such that A and B are isometric in a large region, but are not isometric.

(where usually one thinks about the large region as everything to the past of a Cauchy surface)

Good news — Choquet-Bruhat and Geroch theorem: uniqueness of maximal globally hyperbolic development.

Examples of indeterministic spacetimes:

1. Some extendible spacetimes
2. Spacetimes with boundary at infinity
3. Spacetimes with holes

(?) Spacetimes constructed using exotic smooth structure

Closer look at M

X - non-empty set

τ - a topology on X

A - maximal smooth atlas

Question

Fix X, τ. Suppose that it admits smooth structure A.

Is A uniquely fixed by the choice of X, τ?

Answer

Sometimes, but not always.

Exotic smoothness

For $\mathbb{R}^n$, $n \neq 4$: one.

For $n = 4$: uncountably many.

Manifold which is homeomorphic but not diffeomorphic to a given (standard) one is called exotic and denoted by subscript Θ.

Two types of $\mathbb{R}^4_\Theta$

Small: can be embedded into $\mathbb{R}^4$

Large: cannot.

Exotic invader

Start with spacetime $\langle \mathbb{R}^4, g \rangle$.

Fix some Cauchy surface S at $t = 0$.

Above $t = 1$, remove upper part of $\mathbb{R}^4$ and insert small $\mathbb{R}^4_\Theta$.

Define in any way Lorentzian metric g^* on the exotic region.

Continue g in any way to the exotic region with g^* to obtain spacetime $\langle \mathbb{R}^4_\Theta, g' \rangle$ (where g' is g up to $t = 1$, g^* above $t = 1$, and whatever is needed to smoothen it inbetween).

Exotic invader is not isometric to the original spacetime

Since $\mathbb{R}^4$ and $\mathbb{R}^4_\Theta$ are not diffeomorphic, there is no isometry between $\langle \mathbb{R}^4, g \rangle$ and $\langle \mathbb{R}^4_\Theta, g' \rangle$. But they are isometric up to S.

Does $\langle \mathbb{R}^4_\Theta, g' \rangle$ have any physical interpretation?

Brans conjecture:

> *exotic smoothness can act as a source for some externally regular field, just as matter or a wormhole can*

Claims in the literature (Asselmeyer-Maluga and Brans 2007): there are both compact and noncompact examples where the conjecture holds.

Summary

It seems, then, that something along the following lines is the case:

For a spacetime $\langle \mathbb{R}^4, g \rangle$ there exists another spacetime $\langle \mathbb{R}^4_\Theta, g' \rangle$, which is:

1. isometric to $\langle M, g \rangle$ up to Cauchy surface S,
2. not isometric to it due to presence of exotic smooth structure.

And assuming that Brans conjecture makes sense and is true,

3. we have spontaneously appearing additional gravitational sources in the spacetime, whose appearance is not determined by fixing data at S.

Which is similar to space invaders scenario (but: does not use infinite acceleration and time reversal).

Good: two nice features of $\langle \mathbb{R}^4_\Theta, g' \rangle$

1. Inextendible.

2. Intuitively, it is hole free, since topological $\mathbb{R}^4$ has no holes.

Bad: two unpleasant features of $\langle \mathbb{R}^4_\Theta, g' \rangle$

1. Not globally hyperbolic:

   Dieckmann 1988: any 4d globally hyperbolic spacetime is diffeomorphic to $\mathbb{R} \times S$, where S is 3d submanifold.

   McMillan 1961: Poincare conjecture $\Rightarrow \mathbb{R}^4_\Theta$ is not diffeomorphic to product of $\mathbb{R}^3 \times \mathbb{R}$.

   Hamilton and Perelman: Poincare conjecture is true.

2. Brans and Randall 1993: any flat metric on $\mathbb{R}^4_\Theta$ is geodesically incomplete.

Unknown

We have our ways around indeterministic examples.

1. Some extendible spacetimes — **are rare**
2. Spacetimes with boundary at infinity — **fix the boundary conditions**
3. Spacetimes with holes — **are unphysical**
4. Spacetimes constructed using exotic smooth structure — **?**