A CLT for the L^2 norm of increments of local times of random walks as time goes to infinity, with applications to polymers

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(Joint work with J. Rosen.) The expression

$$H_n = \sum_{i,j=1,\, i\neq j}^{n} 1_{\{S_i=S_j\}} - \frac{1}{2}\sum_{i,j=1,\, i\neq j}^{n} 1_{\{|S_i-S_j|=1\}},$$

is the Hamiltonian in a model for the critical attractive random polymer in one dimension. Here $S = \{S_n\,;\, n = 0, 1, 2, \dots\}$ is a simple random walk on Z^1. It is easy to see that

$$H_n = \sum_{x\in Z^1} \left(L^x_n - L^{x+1}_n\right)^2,$$

where $L^x_n = \sum_{i=1}^n 1_{\{S_i=x\}}$ is the local time for S. We show that

$$\frac{H_n - 2n}{n^{3/4}} \xrightarrow{\mathcal{L}} (12)^{1/2} \left(\int_{-\infty}^{\infty} (L^x_1)^2\, dx\right)^{1/2} \eta, \qquad \text{as } n\to\infty,$$

where $\{L^x_1, x \in R^1\}$ is the local time of Brownian motion at time 1. (A similar formula should hold for symmetric random walks with variance 1.)

In general let $X := \{X_n\,;\, n = 0, 1, 2, \dots\}$ be a mean zero random walk on Z^1 in the domain of attraction of a non-degenerate stable process, that is there exists a function $b(n)$ of regular variation of index $1/\beta$, with $1 < \beta \le 2$, such that

$$\lim_{n\to\infty} X_n/b(n) = U_{\beta,a}, \tag{1}$$

where $U_{\beta,a}$ is a non-degenerate stable random variable with Lévy exponent of the form

$$\psi_{\beta,a}(\lambda) = (1 - ia(\text{ sign } \lambda)\tan(\pi\beta/2))|\lambda|^\beta.$$

In addition assume that almost surely the range of X is not contained in a proper subgroup of Z^1. Let $\phi(\lambda) = E(e^{i\lambda X_1})$ denote the characteristic function of X_1 and, as above, let $L^x_n := \sum_{i=1}^n 1_{\{X_i=x\}}$ denote the local time of X. Set

$$c_{\phi,1} := \frac{8}{\pi}\int_{-\pi}^{\pi} \sin^4(p/2) \left(\frac{1-|\phi(p)|^2}{|1-\phi(p)|^2}\right)^2 dp.$$

Let $L_{\beta,a} = \{L^x_{\beta,a,t}\,;\,(x,t) \in R^1 \times R^1_+\}$ denote the local time of the stable process $\{U_t, t \in R_+\}$ that is uniquely determined by $U_1 = U_{\beta,a}$.

We obtain the following central limit theorem in which we do not assume that X_1 is symmetric:

Theorem. Let $\{L^x_n\,;\,(x,n) \in Z^1 \times Z^1_+\}$ be the local time of a strongly aperiodic random walk X that satisfies (1). Then

$$\frac{\sum_{x\in Z^1}(L^{x+1}_n - L^x_n)^2 - E(\sum_{x\in Z^1}(L^{x+1}_n - L^x_n)^2)}{n/\sqrt{b(n)}} \stackrel{\mathcal{L}}{\Longrightarrow} (2c_{\phi,1})^{1/2}\left(\int \left(L^x_{\beta,a,1}\right)^2\,dx\right)^{1/2}\eta \tag{2}$$

as $n \to \infty$, where $L_{\beta,a}$ and η are independent. Furthermore, the mean in (2) can be replaced by

$$\frac{4n}{\pi}\int_0^\pi \sin^2(p/2)\frac{1+\phi(p)}{1-\phi(p)}\,dp.$$