FAMILIES OF VARIETIES OF GENERAL TYPE: THE SHAFAREVICH CONJECTURE AND RELATED PROBLEMS

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Let C be a smooth projective curve of genus g, $\Delta \subset C$ a finite subset, $\delta = \#\Delta$, and $q > 1$ a positive integer. Shafarevich conjectured at the 1962 ICM in Stockholm that the set, $\mathfrak{S}$, of non-isotrivial families of smooth projective curves of genus q over $C \setminus \Delta$ is finite and that if $2q-2+\delta \leq 0$, then no such families exist.

This was confirmed by Parshin for the case $\Delta = \emptyset$ and by Arakelov in general. Their method was to divide the problem into the following parts:

(1) "Boundedness": There are only finitely many deformation types of families in $\mathfrak{S}$.
(2) "Rigidity": There are no non-trivial deformations within $\mathfrak{S}$.
(3) "Hyperbolicity": If $2g-2+\delta \leq 0$, then $\mathfrak{S} = \emptyset$.

A basic question is whether (an appropriate variant of) Shafarevich's conjecture holds in higher dimensions. In fact it seems natural to try to work with these parts individually as they may generalize in different ways.

It is actually more convenient to work with a compactification of the family over C, understanding that later we are free to alter it over Δ. Let $\overline{\mathfrak{S}}$ be the set of non-isotrivial families of projective curves of genus q over C that are smooth over $C \setminus \Delta$ and have a smooth total space. Two such families are equivalent if they agree over $C \setminus \Delta$.

The first important observation is that "boundedness" follows easily from the existence of moduli spaces of curves and

(1') "Weak boundedness": There exist a function $d(g, \delta, q, m)$, and a natural number m_0, such that for any family $f : X \to C$ in $\overline{\mathfrak{S}}$, $\deg(f_* \omega^m_{X/C}) \leq d(g, \delta, q, m)$ for all $m \geq m_0$.

Considering families over a compact base curve leads to a slightly different view on the hyperbolicity problem. One could ask what can be said about the singular fibers of the family. On the simplest level, how many are there? In fact Szpiro did ask this: Is there a lower bound on the number of singular fibers if $C \simeq \mathbb{P}^1$?

Beauville gave the following answer: there are always at least 3 singular fibers and there are families with exactly 3. In fact Beauville's proof also shows that there is at least 1 singular fiber if the base curve is elliptic. In short $2g-2+\delta > 0$, i.e., "hyperbolicity" holds. Note that Kodaira surfaces show that there are families over high genus curves without any singular fibers.

More recently Catanese and Schneider asked if the same is true with higher dimensional fibers, and a conjecture of Shokurov translates to the same: Is it true that for a familily of varieties of general type $\delta \geq 3$ if $g = 0$ and $\delta \geq 1$ if $g = 1$, or equivalently: Is $2g-2+\delta > 0$? This is again simply asking if "hyperbolicity" also holds in higher dimensions.

It is interesting to note the wide range of applications this question relates to: Catanese and Schneider wanted to use this to obtain good estimates for the size of the automorphism group of a variety of general type, while Shokurov needed it for proving quasi-projectivity of certain moduli spaces.

The lectures will concentrate on recent results relating to these questions.

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