Rational points on fibrations

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Let k be a number field, X be a smooth, projective, geometrically connected variety over k and p be a surjective morphism from X to the projective line $\mathbf{P}^1_k$, with geometrically integral generic fibre. Assume that X has points in all the completions of k. Does there exist a k-rational point m of $\mathbf{P}^1_k$ with smooth fibre $X_m = p^{-1}(m)$ having points in all completions of k ? This may fail, but known counterexamples can be interpreted by means of the subgroup of the Brauer group of X consisting of elements whose restriction to the generic fibre comes from the Brauer group of the function field of $\mathbf{P}^1_k$. This fibred version of the Brauer-Manin obstruction is at the heart of recent investigations on the Hasse principle and weak approximation.

Let us further assume that the generic fibre of p is a homogeneous space of a connected algebraic group G (defined over the function field of $\mathbf{P}^1_k$). If the fibration is not too degenerate, and X satisfies the so-called Brauer-Manin condition, then one conjectures that X contains a k-rational point.

When G is a linear algebraic group, evidence for this conjecture has accumulated over the years, even though we are very far from a satisfactory answer. In that case the variety X is geometrically unirational. But if G is an abelian variety, the geometry of the space X can be quite complicated. For example, X might be a $K3$-surface. Conditional results on the existence of a k-rational point in that case have nevertheless been recently established. These results depend on some well-known, difficult conjectures. I will try to describe the method. If time permits, I will explain how Swinnerton-Dyer very recently used it to establish the Hasse principle for diagonal cubic threefolds over the rationals, under the sole assumption that Tate-Shafarevich groups of elliptic curves are finite.

On the number theory side, some basic results from class field theory will be recalled.

On the algebraic geometry side, basics on Grothendieck's Brauer group will be covered. Examples of explicit computation of that group will be given.

A number of concrete equations (diophantine problems) will come up in the lectures.

Bibliography:

J.-L. Colliot-Thélène, *The Hasse principle in a pencil of algebraic varieties*, in Contemporary Mathematics **210** (1998)

A. N. Skorobogatov, *Torsors and rational points*, Cambridge University Press, to appear.