Problems on zeta functions of varieties

1. Let $X \subset \mathbf{P}^n$ be a smooth projective geometrically connected variety of dimension d over the finite field $\mathbf{F}_q$ with q elements, and set $U = \mathbf{P}^n \setminus X$. The purpose of this exercise is to prove:

(P) The number of points $|U(\mathbf{F}_{q^N})|$ is divisible by q^N for all $N \geq 1$ if and only if all eigenvalues of Frobenius on the cohomology groups $H^i_c(U, \mathbf{Q}_\ell)$ for $0 \leq i \leq 2d$ are divisible by q.

(This property is interesting for various reasons: for instance, $|U(\mathbf{F}_q)|$ divisible by q implies $X(\mathbf{F}_q) \neq \emptyset$.)

We shall use the well-known shape of the ℓ-adic cohomology of $\mathbf{P}^n$ over an algebraic closure: $H^{2i}(\mathbf{P}^n, \mathbf{Q}_\ell) \cong \mathbf{Q}_\ell$ if $0 \leq i \leq n$ and $H^{2i-1}(\mathbf{P}^n, \mathbf{Q}_\ell) = 0$. A generator for $H^{2i}(\mathbf{P}^n, \mathbf{Q}_\ell)$ is given by the i-fold cup-product of the class of a hyperplane in $H^2(\mathbf{P}^n, \mathbf{Q}_\ell)$. Given a smooth closed subvariety $\bar{X} \subset \mathbf{P}^n$, the restriction map $H^{2i}(\mathbf{P}^n, \mathbf{Q}_\ell) \to H^{2i}(\bar{X}, \mathbf{Q}_\ell)$ sends this generator to the i-fold cup-product of the class of a hyperplane section; it is thus nonzero (check this!)

(a) Show that $|U(\mathbf{F}_{q^N})|$ is divisible by q^N for all $N \geq 1$ if and only if the zeta function $Z_U(T)$ lies in $\mathbf{Z}[[qT]]$.

(b) Conclude that $|U(\mathbf{F}_{q^N})|$ is divisible by q^N for all $N \geq 1$ if and only if the reciprocal zeros and poles of $Z_U(T)$ are divisible by q.

(c) Conclude that if the numerator and the denominator of $Z_U(T)$ have no common zeros, then (P) holds.

(d) Show that there is a surjection $H^{i-1}(\bar{X}, \mathbf{Q}_\ell) \twoheadrightarrow H^i_c(\overline{U}, \mathbf{Q}_\ell)$ for all i.

(e) Use (c), (d) and the Weil conjectures to conclude that (P) holds.

2. The goal of this exercise is to show a pretty weak form of the Weil conjecture:

($*$) If X is a separated scheme of finite type over $\mathbf{F}_q$ ($q = p^r$), and $\mathcal{F}$ is an ℓ-adic sheaf on X such that for all closed points $x \in X$ the eigenvalues of Frobenius on the stalk $\mathcal{F}_{\bar{x}}$ have absolute value $\leq q^{r[\kappa(x):\mathbf{F}_q]}$ in every complex embedding for some real number r, then for all i the absolute values of Frobenius on $H^i_c(\bar{X}, \mathcal{F})$ have absolute value $\leq q^{i+r}$ in every complex embedding.

[The general result of Deligne in Weil II gives a much sharper statement, but here we don't need the Weil conjectures at all.]

(a) Reduce to the case when X has dimension 1 and $i = 1$.

(b) In the above case, show that it is enough to verify that the complex function $\log \sigma(Z_X(\mathcal{F}, T))$ is holomorphic in the domain $|T| \leq q^{-1-r}$, where $\sigma : \mathbf{Q}_\ell \to \mathbf{C}$ is an embedding.

(c) Verify this holomorphy by estimating the coefficient of t^N/N in the power series expansion of $\log \sigma(Z_X(\mathcal{F}, T))$. (Use an easy estimate $|X(\mathbf{F}_q^N)| \leq C(q^N)$ with an absolute constant C and the implications of the assumption for the traces of Frobenius on geometric stalks of $\mathcal{F}$.)