Horizons of Combinatorics

Budapest 2006

The Erdős-Rényi Phase Transition

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Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.

– Fan Chung

Paul Erdős and Alfred Rényi

On the Evolution of Random Graphs

Magyar Tud. Akad. Mat. Kutató Int. Közl

volume 8, 17-61, 1960

$\Gamma_{n,N(n)}$: n vertices, random $N(n)$ edges

[...] the largest component of $\Gamma_{n,N(n)}$ is of order $\log n$ for $\frac{N(n)}{n} \sim c < \frac{1}{2}$, of order $n^{2/3}$ for $\frac{N(n)}{n} \sim \frac{1}{2}$ and of order n for $\frac{N(n)}{n} \sim c > \frac{1}{2}$. This double "jump" when c passes the value $\frac{1}{2}$ is one of the most striking facts concerning random graphs.

The "Double Jump"

$G(n,p)$, $p = \frac{c}{n}$ (or $\sim \frac{c}{2}n$ edges)

- $c < 1$

  Biggest Component $O(\ln n)$

  $|C_1| \sim |C_2| \sim \ldots$

  All Components simple (= tree/unicyclic)

- $c = 1$

  Biggest Component $\Theta(n^{2/3})$

  $|C_1|n^{-2/3}$ nontrivial distribution

  $|C_2|/|C_1|$ nontrivial distribution

  Complexity of C_1 nontrivial distribution

- $c > 1$

  Giant Component $|C_1| \sim yn$, $y = y(c) > 0$

  All other $|C_i| = O(\ln n)$ and simple

The Critical Window $p = \frac{1}{n} + \lambda n^{-4/3}$

- $\lambda(n) \to -\infty$ Subcritical

  Biggest Component $o(n^{2/3})$

  $|C_1| \sim |C_2| \sim \ldots$

  All Components simple

- λ constant. The Critical Window

  Biggest Component $\Theta(n^{2/3})$

  $|C_1| n^{-2/3}$ nontrivial distribution

  $|C_2|/|C_1|$ nontrivial distribution

  Complexity of C_1 nontrivial distribution

- $\lambda(n) \to +\infty$ Supercritical

  Dominant Component $|C_1| \gg n^{2/3}$

  High Complexity

  All other $|C_i| = o(n^{2/3})$ and simple

What is the Critical Window?

Difficult in General

When Dominant Component is Emerging

Subcritical: Biggest about same size

Supercritical: Biggest $\gg$ second

Susceptibility $\chi(G) = E[|C(0)|] = \frac{1}{n}\sum |C_i|^2$

Largest Component starts to dominate

Subcritical: $\frac{1}{n}|C_1|^2 \ll \chi(G)$

Critical: $\frac{1}{n}|C_1|^2 = O(\chi(G))$

Supercritical: $\frac{1}{n}|C_1|^2 \sim \chi(G)$

Computer Experiment (Try It!)

$n = 50000$ vertices. Start: Empty

Add random edges

Parametrize $e/\binom{n}{2} = (1 + \lambda n^{-1/3})/n$

Merge-Find for Component Size/Complexity

$-4 \leq \lambda \leq +4$, $|C_i| = c_i n^{2/3}$

See biggest merge into dominant

A Strange Physics

Components $c_i n^{2/3}$, $c_j n^{2/3}$

$\lambda \leftarrow \lambda + d\lambda$, $p \leftarrow p + n^{-4/3} d\lambda$

Merge with probability $c_i c_j d\lambda$

Increment Complexity $\frac{1}{2} c_i^2 d\lambda$

Galton-Watson Birth Process

Root node “Eve”

Each node has $Po(c)$ children

(Poisson: $\Pr[Po(c) = k] = e^{-c}c^k/k!$)

$T = T_c$ is total size

- $c < 1$

  T finite

- $c = 1$

  T finite

  $E[T]$ infinite (heavy tail)

- $c > 1$

  $\Pr[T = \infty] = y = y(c) > 0$

Galton-Watson Exact

$$\Pr[T_c = u] = \frac{e^{-uc}(uc)^{u-1}}{u!}$$

$$\Pr[T_1 = u] = \frac{e^{-u}u^{u-1}}{u!} = \Theta(u^{-3/2})$$

For $c > 1$, $\Pr[T = \infty] = y = y(c) > 0$ where

$$1 - y = e^{-cy}$$

Duality: $d < 1 < c$ with $de^{-d} = c^{-c}$

Conditioning on

$Po(c)$ process being finite

gives the $Po(d)$ process

Galton-Watson Near Criticality

$\Pr[T_1 \geq u] \sim cu^{-1/2}$

$\Pr[T_{1+\epsilon} = \infty] \sim 2\epsilon$

Conditioning on finite, $T_{1+\epsilon}$ becomes $T_{1-\epsilon+o(\epsilon)}$

$\Pr[T_{1-\epsilon} \geq u] \sim \Pr[\infty > T_{1+\epsilon} \geq u]$

If $u = o(\epsilon^{-2})$ (can't see ϵ):

$\Pr[\infty > T_{1+\epsilon} \geq u] \sim \Pr[T_1 \geq u] \sim cu^{-1/2}$

If $u = \Theta(\epsilon^{-2})$ (somewhat see ϵ):

$\Pr[\infty > T_{1+\epsilon} \geq u] = \Theta(\Pr[T_1 \geq u]) = \Theta(u^{-1/2})$

If $u \gg \epsilon^{-2}$ (strong ϵ effect):

$\Pr[\infty > T_{1+\epsilon} \geq u] \sim \Pr[T_1 \geq u] e^{-u\epsilon^2/2}$

$$\begin{aligned}
\frac{\Pr[T_{1\pm\epsilon} = u]}{\Pr[T_1 = u]} &= [e^{\mp\epsilon}]^u (1 \pm \epsilon)^{u-1} \\
&\sim [(1 \pm \epsilon) e^{\mp\epsilon}]^u \\
&= e^{(1+o(1))u\epsilon^2/2}
\end{aligned}$$

Galton-Watson as Walk

$Z_i \sim Po(c)$, $i = 1, 2, \ldots$

$Y_0 = 1$ (Eve)

$Y_i = Y_{i-1} + Z_i - 1$ (Z_i children and dies)

Fictitious Continuation

$T =$ min t with $Y_t = 0$

(If no such t, $T = \infty$)

$c < 1$ negative drift, T finite

$c > 1$ positive drift, maybe T infinite

$c = 1$ zero drift, delicate

$C(v)$ in $G(n,p)$ as BFS Walk

$Y_0 = 1$ (Root v)

$Y_i = Y_{i-1} + Z_i - 1$ (pop queue/add Z_i new)

where $Z_i \sim \text{BIN}[n-(i-1)-Y_{i-1}, p]$

The Link:

When $p \sim \frac{c}{n}$

and $i - 1 + Y_{i-1} = o(n)$

Z_i is roughly $Po(c)$

$|C(v)|$, T_c similar while small

Ecological Limitation: Success in BFS in $G(n,p)$

is selflimiting. “Eating your seed corn”

Rough (but Accurate) Link

$p = \frac{c}{n}$, $c > 1$

$C(v)$ like T_c if finite

With probability y, T_c infinite

Corresponding $C(v)$ become large

All merge to form giant $\sim yn$ component

$p = \frac{1+\epsilon}{n}$, $o(1) = \epsilon \gg n^{-1/3}$

With probability $\sim 2\epsilon$, T_c infinite

Corresponding $C(v)$ become large

All merge to form dominant $\sim 2\epsilon n$ component

Finite T_c have small $|C(v)| < \epsilon^{-2+}$

$\epsilon \gg n^{-1/3}$ small/dominant dichotomy

Why $\Theta(n^{2/3})$ at $p = \frac{1}{n}$

Ignore Ecological Limitation (so rough!)

$\Pr[|C(v)| \geq u] \sim \Pr[T_1 \geq u] = \Theta(u^{-1/2})$

$X_u :=$ number v with $|C(v)| \geq u$

$E[X_u] = \Theta(nu^{-1/2})$

$X_u \neq 0 \Rightarrow X_u \geq u$

$\Pr[X_u \neq 0] = O(nu^{-3/2}) = O(1)$ when $u = \Theta(n^{2/3})$

$Y_t \sim 1 - t + \text{BIN}[n-1, 1-(1-p)^t]$

$|C(v)| = t \Rightarrow Y_t = 0$ (Converse False!)

$p = \frac{c}{n}$, $t = yn$ $\Pr[Y_t = 0]$ tiny unless

$1-t+(n-1)(1-(1-p)^t) \sim 0$ so $y = 1-e^{-cy}$

`whp` either $t = O(1)$ or $t \sim yn$

$t = O(1)$ same as Galton-Watson $\Rightarrow$

$\Pr[|C(v)| = O(\ln n)] \sim \Pr[T_c < \infty] = 1 - y$

Karp Approach: Keep generating components

After $O(1)$ tries get giant

Now $n' = n(1-y)$, $p = d/n'$, $d < 1 < c$

Duality: $G(n, c/n)$ minus giant component is like $G(n', d/n')$ (c, d conjugate)

$p = \frac{1+\epsilon}{n}$, $o(1) = \epsilon$, $\epsilon = \lambda n^{-1/3}$, $\lambda \to +\infty$

$\Pr[Y_t = 0]$ tiny unless

$1 - t + (n-1)(1 - (1-p)^t) \sim 0$ so $t \sim 2\epsilon n$

`whp` either $t = O(\epsilon^{-2+})$ or $t \sim 2\epsilon n = 2\lambda n^{2/3}$

$\lambda \to +\infty \Rightarrow$ small/dominant dichotomy

$t = O(\epsilon^{-2+})$ same as Galton-Watson $\Rightarrow$

$\Pr[|C(v)| = O(\epsilon^{-2+})] \sim \Pr[T_{1+\epsilon} < \infty] = 1 - 2\epsilon$

Karp Approach: Keep generating components

After $O(\epsilon^{-1})$ tries get dominant

"Failures" $\sim T_{1-\epsilon}$ use ϵ^{-1} each

Total $\epsilon^{-2} = \lambda^{-2} n^{2/3}$ used before dominant

$n' = n - \lambda^{-2} n^{2/3}$

$p = \frac{1}{n'} + (\lambda - \lambda^{-2})(n')^{-4/3}$

$\lambda \to +\infty \Rightarrow$ failures not too costly

Duality: Now $G(n', \frac{1-\epsilon}{n'})$

Evolution of n-Cube

Ajtai, Komlos, Szemeredi

Bollobas, Luczak, Kohayakawa

Borgs, Chayes, Slade, JS, van der Hofstad

$p = c/n$

$c < 1$ subcritical

$c > 1$ giant $\Omega(2^n)$ component

Much more!

Achlioptas Processes

Each round random v, w, x, y.

Add $\{v, w\}$ if both isolated

Otherwise add $\{x, y\}$

JS-Wormald:

$tn/2$ rounds susceptibility $\chi(t)$

Diff Eq, $\chi(t) \to \infty$ at $t = t_0$

$t < t_0$ all $|C_i| = O(\ln n)$

$t > t_0$ giant $|C_1| = \Omega(n)$

`Conjecture:` Critical Window like Erdős-Rényi

`Computer Simulation:` Yes!

In Critical Window (e.g.: $p = \frac{1}{n}$)

$\Pr[|C(v)| \geq u] = \Theta(u^{-1/2})$ for $u = O(n^{2/3})$

$\Pr[|C(v)| \geq An^{2/3}] = \Theta(n^{-1/3}e^{-\Theta(A^3)})$

A Conjecture of Yuval Peres

G any graph n vertices, regular degree d

$p = \frac{1}{d-1}$ (e.g.: $d = 3$)

$$\Pr[|C(v)| \geq An^{2/3}] \leq c_1 n^{-1/3} e^{-c_2 A^3} \text{ (??)}$$

Balaton Preview

$G(n,p)$, $p = \frac{c}{n}$, $c > 1$

Vertices $\{v, 1, 2, \ldots, n-1\}$.

$T_j^* = i$: Vertex j joins Breadth First Tree at i-th opportunity. *Fictitious Continuation.*

$\Pr[T_j^* = i] = p(1-p)^{i-1}$

$\Pr[|C(v)| = s] = A_1 A_2$

A_1: Prob. exactly $s-1$ of $T_j^* \leq s$.

Condition $T_j^* \leq s$, $1 \leq j \leq s-1$. (rest $> s$)

$T_j^* \to T_j$ truncated exponential

Tilted Balls into Boxes

$$\Pr[T_j = i] = \frac{p(1-p)^{i-1}}{1-(1-p)^s}, 1 \le i \le s$$

Ball j in Bin T_j. Bin i has Z_i balls

Queue Walk:

$$Y_0 = 1, Y_i = Y_{i-1} + Z_i - 1, Y_s = 0$$

TREE : $Y_t > 0$ for $0 \le t < s$

$A_2 = \Pr[\text{TREE}]$

Asymptotic Analysis: Left and Right edges

It is six in the morning.

The house is asleep.

Nice music is playing.

I prove and conjecture.

– Paul Erdős, in letter to Vera Sós