Homework 5

due March 19, 2025

1. Prove that line graphs are claw-free, that is, if a graph is a line graph, then it does not contain an induced $K_{1,3}$ subgraph (that is called a claw). ($K_{m,n}$ denotes the complete bipartite graph with m vertices in one and n vertices in the other partite class.)

2. Determine the value of $\chi_e(K_n)$ for every n.

3. Let G be a bipartite graph. Prove that for its complementary graph $\bar{G}$ we have
$$\chi(\bar{G}) = \omega(\bar{G}),$$
that is the chromatic number of $\bar{G}$ is equal to its clique number.

4. Draw some lines in the plane so that no three of them go through the same point. Consider their points of intersection as the vertices of a graph and the segments between neighbouring intersection points on each line as edges. Prove that the chromatic number of the resulting graph is at most 3.

5. Let D be a directed graph (that is, its edges are ordered pairs of distinct vertices) and define its line graph $L(D)$ as follows. The vertices of $L(D)$ are the edges (or *arcs*) of D and two edges $\vec{ab}$ and $\vec{cd}$ are adjacent in $L(D)$ if and only if either $b = c$ or $a = d$. (That is two edges of D form an edge of $L(D)$ iff the head of one of them is the tail of the other.) Let $\chi(D)$ simply denote the chromatic number of the underlying undirected graph of D, that is meant to be the graph we obtain from D if we disregard the orientation of its edges. Prove that we always have
$$\chi(L(D)) \geq \lceil \log_2 \chi(D) \rceil.$$