Graduate Texts in Mathematics

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Graph Theory

An Introductory Course

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CHAPTER VI

Ramsey Theory

Show that in a party of six people there is always a group of three who either all know each other or are all strangers to each other. This well known puzzle is a special case of a theorem proved by Ramsey in 1928. The theorem has many deep extensions which are important not only in graph theory and combinatorics but in set theory (logic) and analysis as well. In this chapter we prove the original theorems of Ramsey, indicate some variations and present some applications of the results.

§1 The Fundamental Ramsey Theorems

We shall consider partitions of the *edges* of graphs and hypergraphs. For the sake of convenience a partition will be called a *colouring*, but one should bear in mind that a colouring in this sense has nothing to do with the edge colourings considered in Chapter V. Adjacent edges may have the same colour and, indeed, our aim is to show that there are large subgraphs all of whose edges have the same colour. In a 2-colouring we shall always choose red and blue as colours; a *subgraph* is *red* (*blue*) if all its edges are red (blue).

Given a natural number s, there is an $n(s)$ such that if $n \geq n(s)$ then every colouring of the edges of K^n with red and blue contains either a red K^s or a blue K^s. In order to show this and to give a bound on $n(s)$, we introduce the following notation: $R(s, t)$, called Ramsey number, is the minimum of n for which every red-blue colouring of K^n yields a red K^s or a blue K^t. (We assume that $s, t \geq 2$, for we adopt the reasonable convention that every K^1 is both red and blue since it has no edges.) A priori it is not clear that $R(s, t)$

is finite for every s and t. However, it is obvious that

$$R(s,t) = R(t,s) \quad \text{for every } s, t \geq 2$$

and

$$R(s,2) = R(2,s) = s,$$

since in a red-blue colouring of K^s either there is a blue edge or else every edge is red. The following result shows that $R(s,t)$ is finite for every s and t, and at the same time it gives a bound on $R(s,t)$.

Theorem 1. *If* $s > 2$ *and* $t > 2$ *then*

$$R(s,t) \leq R(s-1,t) + R(s,t-1) \tag{1}$$

and

$$R(s,t) \leq \binom{s+t-2}{s-1}. \tag{2}$$

PROOF. (i) When proving (1) we may assume that $R(s-1,t)$ and $R(s,t-1)$ are finite. Let $n = R(s-1,t) + R(s,t-1)$ and consider a colouring of the edges of K^n with red and blue. We have to show that this colouring contains either a red K^s or a blue K^t. Let x be a vertex of K^n. Since $d(x) = n - 1 = R(s-1,t) + R(s,t-1) - 1$, either there are at least $n_1 = R(s-1,t)$ red edges incident with x or there are at least $n_2 = R(s,t-1)$ blue edges incident with x. By symmetry we may assume that the first case holds. Consider a subgraph K^{n_1} of K^n spanned by n_1 vertices joined to x by a red edge. If K^{n_1} has a blue K^t, we are home. Otherwise K^{n_1} contains a red K^{s-1} which forms a red K^s with x.

(ii) Inequality (2) holds if $s = 2$ or $t = 2$ (in fact, we have equality since $R(s,2) = R(2,s) = s$). Assume now that $s > 2$, $t > 2$ and (2) holds for every pair (s', t') with $2 \leq s', t'$ and $s' + t' < s + t$. Then by (1) we have

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$
$$\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}. \qquad \square$$

The result easily extends to colourings with arbitrarily (but finitely) many colours: given k and $s_1, s_2, \ldots, s_k$, if n is sufficiently large then every colour of K^n with k colours is such that for some i, $1 \leq i \leq k$, it contains a K^{s_i} coloured with the i-th colour. (The minimal value of n for which this holds is usually denoted by $R_k(s_1, \ldots, s_k)$.) Indeed, if we know this for $k - 1$ colours, then in a k-colouring of K^n we replace the first two colours by a new colour. If n is sufficiently large (depending on $s_1, s_2, \ldots, s_k$) then either there is a K^{s_i} coloured with the i-th colour for some i, $3 \leq i \leq k$, or else there is a

K^m, $m = R(s_1, s_2)$, coloured
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Theorem 2. *Let* $1 < r < \min\{$

$R^{(r)}(s,t) \leq R^{(r-1}$

PROOF. Both assertions follow
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also finite.

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Theorem 3. *Let* $c\colon A^{(r)} \to \{1, \ldots$
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K^m, $m = R(s_1, s_2)$, coloured with the new colour, that is coloured with the first two (original) colours. In the first case we are home and in the second, for $i = 1$ or 2 we can find a K^{s_i} in K^m coloured with the i-th colour.

In fact, Theorem 1 also extends to hypergraphs, that is to colourings of the set $X^{(r)}$ *of all r-tuples* of a finite set X with k colours. This is one of the theorems proved by Ramsey.

Denote by $R^{(r)}(s, t)$ the minimum of n for which every red-blue colouring of $X^{(r)}$ yields a red s-set or a blue t-set, provided $|X| = n$. Of course, a set $Y \subset X$ is called red (blue) if every element of $Y^{(r)}$ is red (blue). Note that $R(s, t) = R^{(2)}(s, t)$. As in the case of Theorem 1, the next result not only guarantees that $R^{(r)}(s, t)$ is finite for all values of the parameters (which is not at all obvious a priori), but also gives an upper bound on $R^{(r)}(s, t)$. The proof is an almost exact replica of the proof of Theorem 1. Note that if $r > \min\{s, t\}$ then $R^{(r)}(s, t) = \min\{s, t\}$ and if $r = s \le t$ then $R^{(r)}(s, t) = t$.

Theorem 2. *Let* $1 < r < \min\{s, t\}$. *Then* $R^{(r)}(s, t)$ *is finite and*

$$R^{(r)}(s, t) \le R^{(r-1)}(R^{(r)}(s-1, t), R^{(r)}(s, t-1)) + 1.$$

PROOF. Both assertions follow if we prove the inequality under the assumption that $R^{(r-1)}(u, v)$ is finite for all u, v, and $R^{(r)}(s-1, t)$, $R^{(r)}(s, t-1)$ are also finite.

Let X be a set with $R^{(r-1)}(R^{(r)}(s-1, t), R^{(r)}(s, t-1)) + 1$ elements. Given any red-blue colouring of $X^{(r)}$, pick an $x \in X$ and define a red-blue colouring of the $(r-1)$-sets of $Y = X - \{x\}$ by colouring $\sigma \in Y^{(r-1)}$ the colour of $\{x\} \cup \sigma \in X^{(r)}$. By the definition of the function $R^{(r-1)}(u, v)$ we may assume that Y has a red subset Z with $R^{(r)}(s-1, t)$ elements.

Now let us look at the colouring of $Z^{(r)}$. If it has a blue t-set, we are home, since $Z^{(r)} \subset X^{(r)}$ so a blue t-set of Z is also a blue t-set of X. On the other hand if there is no blue t-set of Z then there is a red $(s-1)$-set, and its union with $\{x\}$ is then a red s-set of X. □

Very few of the non-trivial Ramsey numbers are known, even in the case $r = 2$. It is easily seen that $R(3, 3) = 6$ and with some work one can show that $R(3, 4) = 9$, $R(3, 5) = 14$, $R(3, 6) = 18$, $R(3, 7) = 23$ and $R(4, 4) = 18$. Because of (1) any upper bound on an $R(s, t)$ helps to give an upper bound on every $R(s', t')$, $s' \ge s$, $t' \ge t$. Lower bounds for $R(s, t)$ are not easy to come by either. In Chapter VII we shall apply the method of random graphs to obtain lower bounds on the Ramsey numbers $R(s, t)$.

As a consequence of Theorem 2 we see that every red-blue colouring of the r-tuples of the natural numbers contains arbitrarily large monochromatic subsets; a subset is *monochromatic* if its r-tuples have the same colour. Ramsey proved that, in fact, we can find an *infinite monochromatic* set.

Theorem 3. *Let* $c: A^{(r)} \to \{1, \dots, k\}$ *be a k-colouring of the r-tuples* $(1 \le r < \infty)$ *of an infinite set A. Then A contains a monochromatic infinite set.*

§2 Monochromatic Subgraphs

Let H_1 and H_2 be arbitrary graphs. Given n, is it true that every red-blue colouring of the edges of K^n contains a red H_1 or a blue H_2? Since H_i is a subgraph of K^{s_i}, where $s_i = |H_i|$, the answer is clearly "yes" if $n \geq R(s_1, s_2)$. Denote by $r(H_1, H_2)$ the smallest value of n that will ensure an affirmative answer. Note that this notation is similar to the one introduced earlier: $R(s_1, s_2) = r(K^{s_1}, K^{s_2})$. Clearly $r(H_1, H_2) - 1$ is the maximal value of n for which there is a graph G of order n such that $H_1 \not\subset G$ and $H_2 \not\subset \bar{G}$.

The numbers $r(H_1, H_2)$, sometimes called *generalized Ramsey numbers*, have been investigated fairly extensively in recent years. We shall determine $r(H_1, H_2)$ for some simple pairs (H_1, H_2).

Theorem 4. *Let T be a tree of order t. Then $r(K^s, T) = (s-1)(t-1) + 1$.*

PROOF. The graph $(s-1)K^{t-1}$ does not contain T, its complement, $K_{s-1}(t-1)$, does not contain K^s, so $r(K^s, T) \geq (s-1)(t-1) + 1$.

Let now G be a graph of order $n = (s-1)(t-1) + 1$ whose complement does not contain K^s. Then $\chi(G) \geq \lceil n/(s-1) \rceil = t$ so it contains a critical subgraph H of minimal degree at least $t - 1$ (see Theorem 1 of Chapter V). It is easily seen that H contains (a copy of) T. Indeed, we may assume that $T_1 \subset H$, where $T_1 = T - x$ and x is an endvertex of T, adjacent to a vertex y of T_1 (and of H). Since y has at least $t - 1$ neighbours in H, at least one of its neighbours, say z, does not belong to T_1. Then the subgraph of H spanned by T_1 and z clearly contains (a copy of) T. □

As we know very little about $r(K^s, K^t)$, it is only to be expected that $r(G_1, G_2)$ has been calculated mostly in the cases when both G_1 and G_2 are sparse (have few edges compared to their orders), e.g., when $G_1 = sH_1$ and $G_2 = tH_2$. The following simple lemma shows that for fixed H_1 and H_2 the function $r(sH_1, tH_2)$ is at most $s|H_1| + t|H_2| + c$, where c depends only on H_1 and H_2, and not on s and t,

Lemma 5. $r(G, H_1 \cup H_2) \leq \max\{r(G, H_1) + |H_2|, r(G, H_2)\}$. *In particular,* $r(sH_1, H_2) \leq r(H_1, H_2) + (s-1)|H_1|$.

PROOF. Let n be equal to the right hand side and suppose there is a red-blue colouring of K^n without a red G. Then $n \geq r(G, H_2)$ implies that there is a blue H_2. Remove it. Since $n - |H_2| \geq r(G, H_1)$, the remainder contains a blue H_1. Hence K^n contains a blue $H_1 \cup H_2$. □

Theorem 6. *If $s \geq t \geq 1$ then*

$$r(sK^2, tK^2) = 2s + t - 1.$$