Symmetry breaking in graphs

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Abstract

This talk is concerned with automorphism and endomorphism breaking of finite and infinite graphs.

Albertson and Collins (1996) introduced the *distinguishing number* $D(G)$ of a graph G as the least cardinal d such that G has a labeling with d labels that is only preserved by the trivial automorphism.

This concept has spawned numerous papers, mostly on finite, but also on infinite graphs. We are interested in the infinite motion conjecture of Tom Tucker. We present partial results pertaining to the conjecture, a generalization to uncountable graphs, and results that support the conjecture.

We also introduce an endomorphism distinguishing number and extend results about automorphisms to endomorphisms.

The starting point is a result of of Russell and Sundaram (1998).

Motion Lemma *Let the motion $m(G)$ of a graph G be the minimum number of vertices that are moved by any nonidentity automorphism of G. Then*

$$2^{\frac{m(G)}{2}} \geq |\operatorname{Aut}(G)|$$

implies $D(G) \leq 2$.

The immediate generalization to infinite graphs yields the Motion Conjecture, with the Infinite Motion Conjecture of Tom Tucker as a special case.

Motion Conjecture *Let G be an infinite, connected graph with infinite motion $m(G)$. Then $2^{m(G)} \geq |Aut(G)|$ implies $D(G) \leq 2$.*

Infinite Motion Conjecture of **Tom Tucker** *Let G be an infinite, connected, locally finite graph with infinite motion $m(G)$. Then $D(G) \leq 2$.*

Both conjectures are open, but partial results and connections to other structures will be presented in the talk. We list two of them below. The first one is by Florian Lehner.

Theorem 1 *The Infinite Motion Conjecture is true for graphs of growth*

$$O(2^{(1-\epsilon)\frac{\sqrt{n}}{2}})$$

The second sample is from by Johannes Cuno, Wilfried Imrich and Florian Lehner.

Theorem 2 *Let G be an infinite, connected graph with infinite motion. If $m(G) \geq |Aut(G)|$, then $D(G) \leq 2$.*

For countable G this is easily shown by induction, for graphs G with larger cardinality by transfinite induction. We have the following corollary.

Corollary 1 *Let G be an uncountably infinite, connected graph with infinite motion, and suppose that $2^{m(G)} > |Aut(G)|$. Then, under the assumption of the general continuum hypothesis, $D(G) \leq 2$.*

For the analogous result for countably infinite, connected graphs one assumes the CH. However, if the graphs are locally finite, then the CH is not needed. This was shown by Simon Mark Smith (see Imrich, Smith, Tucker, Watkins, in preparation) and follows from results of either Halin, Trofimov or Evans.