Hasse principle for G-trace forms

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This is a joint work with R. Parimala and J-P. Serre. Let k be a field of characteristic $\neq 2$, and let L be a Galois extension of k with group G. Let us denote by $q_L : L \times L \to k$ the *trace form*, defined by $q_L(x, y) = \mathrm{Tr}_{L/k}(xy)$. Let $(gx)_{g \in G}$ be a normal basis of L over k. We say that this is a *self–dual normal basis* if $q_L(gx, hx) = \delta_{g,h}$. If the order of G is odd, then L always has a self–dual normal basis over k (cf. [1]). This is no longer true in general if the order of G is even; some partial results are given in [2].

If k is an algebraic number field, then it is natural to ask whether a local–global principle holds for this problem. In order to make this question precise, we have to consider G–Galois algebras and not only field extensions. Moreover, it is useful to note that q_L is a *G–quadratic form*, in other words $q_L(gx, gy) = q_L(x, y)$ for all $x, y \in L$ and $g \in G$. The G–Galois algebra has a self–dual normal basis if and only if the G–form q_L is isomorphic to the unit G–form. The following results are proved in [3] under an additional hypothesis on G, and in [4] in general :

Theorem. *Suppose that k is a global field of characteristic $\neq 2$. Let G be a finite group, and let L and L' be two G–Galois algebras such that for all places v of k, the G–forms q_{L_v} and $q_{L'_v}$ are isomorphic over k_v. Then the G–forms q_L and $q_{L'}$ are isomorphic over k.*

Corollary. *A G–Galois algebra has a self–dual normal basis over k if and only if such a basis exists over all the completions of k.*

Bibliography

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