SOME REMARKS ABOUT ADDITIVE AND MULTIPLICATIVE FUNCTIONS

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The present paper contains some results about the classical multiplicative functions $\phi(n)$, $\sigma(n)$ and also about general additive and multiplicative functions.

(1) It is well known that $n/\phi(n)$ and $\sigma(n)/n$ have a distribution function.[1] Denote these functions by $f_1(x)$ and $f_2(x)$. ($f_1(x)$ denotes the density of integers for which $n/\phi(n) \leqq x$.) It is known that both $f_1(x)$ and $f_2(x)$ are strictly incrtasing and purely singular.[1] We propose to investigate $f_1(x)$ and $f_2(x)$; we shall give details only in case of $f_1(x)$. First we prove the following theorem.

THEOREM 1. *We have for every* ϵ *and sufficiently large* x

$$(1) \quad \exp\left(-\exp\left[(1+\epsilon)ax\right]\right) < 1 - f_1(x) < \exp\left(-\exp\left[(1-\epsilon)ax\right]\right)$$

where $a=\exp(-\gamma)$, γ *Euler's constant.*

We shall prove a stronger result. Put $A_r = \prod_{i=1}^{r} p_i$, p_i consecutive primes. Define A_k by $A_k/\phi(A_k) \geqq x > A_{k-1}/\phi(A_{k-1})$. Then we have

$$(2) \qquad 1/A_k < 1 - f_1(x) < 1/A_k^{1-\epsilon}.$$

First of all it is easy to see that Theorem 1 follows from (2), since from the prime number theorem we easily obtain that $\log\log A_k = (1+o(1))ax$, which shows that (1) follows from (2).

(2) means that the density of integers with $\phi(n) \leqq (1/x)n$ is between $1/A_k$ and $1/A_k^{1-\epsilon}$.

We evidently have for every $n \equiv 0 \pmod{A_k}$, $n/\phi(n) \geqq x$, which proves

$$1/A_k \leqq 1 - f_1(x).$$

To get rid of the equality sign, it will be sufficient to observe that there exist integers u with $u/\phi(u) \geqq x$, $(u, A_k)=1$, and that the density of the integers $n \equiv 0 \pmod{u}$, $n \not\equiv 0 \pmod{A_k}$ is positive. This proves the first part of (2). The proof of the second part will be much harder. We split the integers satisfying $n/\phi(n) \geqq x$ into two classes. In the first class are the integers which have more than $[(1-\epsilon_1)k] = r$ prime factors not greater than Bp_k, where $B = B(\epsilon_1)$ is a large number. In

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[1] These results are due to Schönberg and Davenport. For a more general result see P. Erdös, J. London Math. Soc. vol. 13 (1938) pp. 119–127.

the second class are the other integers satisfying $n/\phi(n) \geqq x$. It is easy to see that the number of integers of the first class does not exceed

$$2^{\pi(Bp_k)}/A_r = 2^{o(p_k)}/A_r < 1/A_k^{1-\epsilon} \tag{3}$$

since $\pi(Bp_k) = o(p_k)$ ($\pi(x)$ denotes the number of primes not greater than x), and from the prime number theorem $\log A_r > (1-\epsilon)p_k$ if ϵ_1 is small.

Let now n be any integer of the second class. A simple argument shows that

$$\prod_{p|n}{}'\left(1 - \frac{1}{p}\right) < \prod_{t=r+1}^{k-1}\left(1 - \frac{1}{p_t}\right) < 1 - \frac{c_1\epsilon_1}{\log p_k}.$$

The prime indicates that the product is extended over the $p > Bp_k$. The first inequality follows from the definition of A_k, and from the fact that n is of the second class, the second inequality follows from the prime number theorem. Thus we have

$$\sum_{p|n}{}' \frac{1}{p} > \frac{c_1\epsilon_1}{\log p_k}. \tag{4}$$

Denote now by J_t the interval $(B^t p_k, B^{t+1} p_k)$, $t = 1, 2, \cdots$. It follows from (4) that for every integer of the second class there exists some t such that

$$\sum_{p|n}{}_t \frac{1}{p} > c_1 \frac{\epsilon_1}{2^t \log p_k} \tag{5}$$

where in $\sum_t$ the summation is extended over the primes in J_t. Thus for some t, n must divide more than

$$c_1\epsilon_1(B^t/2^t)(p_k/\log p_k) = B_t \tag{6}$$

primes in J_t. The density of the integers satisfying (6), that is, the density of the integers of the second class, is less than

$$\sum_{t=1}^{\infty}\left(\sum_{p \text{ in } J_t} \frac{1}{p}\right)^{B_t} \Big/ [B_t]! < \frac{1}{[B_t]!} < e^{-2p_k} < \frac{1}{A_k}, \tag{7}$$

that is, $\sum_{p \text{ in } J_t} 1/p < 1$ for large enough k (B is independent of k), if $B = B(\epsilon_1)$ is large enough. Theorem 1 now follows from (3) and (7).

From Theorem 1 we easily obtain that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n=1}^{x} \exp(\phi(n))$$

exists. In fact we can also prove that for $\alpha < a$

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n=1}^{x} \exp(\exp(\phi(n))$$

exists. For $\alpha > a$ the limit is infinite.

THEOREM 2.

$$1/A_k^{1+\epsilon} < 1 - f_1(x) < 1/A_k^{1-\epsilon}.$$

We omit the proof since it is very similar to that of Theorem 1.

THEOREM 3. *Let* $\epsilon \to 0$, *then*

$$f_1(1+\epsilon) = (1+o(1))a/\log \epsilon^{-1}, \qquad f_2(1+\epsilon) = (1+o(1))a/\log \epsilon^{-1}.$$

We prove only the first statement since the proof of the second is essentially the same. Let n be an integer with $n/\phi(n) \leqq 1+\epsilon$. Clearly n does not divide any prime $p<(1-(1+\epsilon)^{-1})^{-1} = \epsilon^{-1}+O(1)$. Thus

$$f_1(1+\epsilon) < (1+o(1))a/\log \epsilon^{-1}. \tag{8}$$

Denote by J_t the interval

$$(4^{t-1}(1-(1+\epsilon)^{-1})^{-1},\ 4^t(1-(1+\epsilon)^{-1})^{-1}).$$

If an integer $n \not\equiv 0 \pmod{p_i}$, $p_i<(1-(1+\epsilon)^{-1})^{-1}$, does not satisfy $n/\phi(n) \leqq 1+\epsilon$, then a simple computation shows that for some t it must have at least t prime factors in J_t. Thus the number of these integers does not exceed

$$(1+o(1)) \frac{a}{\log \epsilon^{-1}} \sum_{t=1}^{\infty} \left(\sum_{p \text{ in } J_t} \frac{1}{p} \right)^t \Big/ t! = o(a/\log \epsilon^{-1}),$$

which together with (8) proves Theorem 3.

It follows from Theorem 3 that $f_1'(1) = \infty$. It would be easy to show that $f_1'(n/\phi(n)) = \infty$ for every n.

Denote by $f_1{}^\alpha$ and $f_2{}^\alpha$ the distribution functions of

$$\prod_{p|n} \left(1 - \frac{1}{p}\right)^{-\alpha} \quad \text{and} \quad \sum_{d|n} \frac{1}{d^\alpha}, \qquad \alpha > 0.$$

THEOREM 4.

$$f_1^{(\alpha)}(1+\epsilon) = (1+o(1)) \frac{a\alpha}{\log \epsilon^{-1}}, \qquad f_2^{(\alpha)} = (1+o(1)) \frac{a\alpha}{\log \epsilon^{-1}}.$$

We omit the proof since it is very similar to that of Theorem 3.

Let us denote by $F_\alpha(x)$, $\alpha>0$, the distribution function of $\prod_{p|n}(1-1/\log p^\alpha)^{-1}$, $\alpha>0$.

THEOREM 5.

$$F_1(1+\epsilon) = (1+o(1))b\epsilon,$$

that is, $F_1'(1) = b$. *Also* $F_\alpha'(1) = 0$ *for* $\alpha < 1$ *and* $F_1'(1) = \infty$ *for* $\alpha > 1$.

We do not give the details of the proof since it would be long and similar to that of Theorem 3. We just make the following remarks: If n satisfies

$$\sum_{p|n} \frac{1}{\log p} \leqq 1 + \epsilon$$

then n does not divide any prime $p \leqq \exp(1/\epsilon)$. Thus $F_1'(1+\epsilon) \leqq (1+o(1))a\epsilon$. But here (unlike in Theorem 3) we have $F_1(1+\epsilon) = (1+o(1))b$, $b < a$. We obtain analogous results if we consider the additive function $\sum_{p|n} 1/\log p$. It is possible that $F_1'(x)$ exists for every $1 \leqq x$, but this we can not prove.

(2) The following results are well known:

$$\sum_{m=1}^{x} \frac{\phi(m)}{m} = (1+o(1))\frac{6}{\pi^2}x, \qquad \sum_{m=1}^{x} \frac{\sigma(m)}{m} = (1+o(1))\frac{\pi^2}{6}x.$$

The density of integers for which $\sigma(n+1)/(n+1) > \sigma(n)/n$ is $1/2$, also the density of integers for which $\phi(n+1)/(n+1) > \phi(n)/n$ is $1/2$.[2] Now we prove the following theorem.

THEOREM 6. *Let* $g(n)/\log\log\log n \to \infty$. *Then we have*

(i) $$\sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} = (1+o(1))\frac{6}{\pi^2}g(n).$$

(ii) *The number of integers* m *in* $(n, n+g(n))$ *which satisfy* $\phi(m+1)/(m+1) > \phi(m)/m$ *equals* $(1+o(1))g(n)/2$.

(iii) *The number of integers* m *in* $(n, n+g(n))$ *which satisfy* $m/\phi(m) \leqq c$ *equals* $(1+o(1))g(n)f_1(c)$. *In other words the distribution function of* $\phi(m)/m$ *in* $(n, n+g(n))$ *is the same as the distribution function of* $\phi(m)/m$.

All these results are best possible; they become false if for infinitely many n, $g(n) < c \log\log\log n$.

We prove only (i); the proof of (ii) and (iii) are similar. Let $A = A(n)$ tend to infinity sufficiently slowly. Put

$$\frac{\phi(m)}{m} = D_1(m)D_2(m),$$

[2] P. Erdös, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 530–540.

where

$$D_1(m) = \prod_{p|m}{}' \left(1 - \frac{1}{p}\right), \qquad D_2(m) = \prod_{p|m}{}'' \left(1 - \frac{1}{p}\right).$$

The prime indicates that $p \leqq A$, the two primes that $p > A$. We evidently have

$$\begin{aligned} \sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} &< \sum_{m=n}^{n+g(n)} D_1(m) = \sum_d{}''' \left(\frac{g(n)}{d}\right)\frac{\mu(d)}{d} \\ &= (1 + o(1))g(n) \prod_{p \leqq A} \left(1 - \frac{1}{p^2}\right) = (1 + o(1)) \frac{\pi^2}{6} g(n) \end{aligned} \tag{9}$$

where the three primes indicate that the prime factors of d are not greater than A, and $(g(n)/d)$ denotes the number of multiples of d in $(n, n+g(n))$. Now we show that for sufficiently large A the number of integers in $(n, n+g(n))$ which satisfy

$$D_2(m) < 1 - \epsilon \tag{10}$$

is $o(g(n))$. It will be sufficient to show that

$$\prod_m D_2(m) > (1 - \eta)^{g(n)} \tag{11}$$

for every $\eta > 0$, the product over m runs in $(n, n+g(n))$. We evidently have

$$\prod_m D_2(m) > \prod{}_1 \left(1 - \frac{1}{p}\right)^{2g(n)/p-1} \prod{}_2 \left(1 - \frac{1}{p}\right)$$

where, in $\prod_1$, $A < p \leqq g(n)$, and in $\prod_2$, p runs through the prime factors greater than $g(n)$ of $n(n+1) \cdots (n+g(n))$. Clearly

$$\prod{}_1 > \prod_{p>A} \left(1 - \frac{c}{p^2}\right)^{g(n)} > (1 - \eta_1)^{g(n)}.$$

From the prime number theorem we have $\prod_{p \leqq x} p < e^{2x}$. Thus

$$\prod{}_2 > \prod_{p \leqq 2y} \left(1 - \frac{1}{p}\right) > \frac{c_1}{\log y}$$

where $y = \log [n(n+1) \cdots (n+g(n))]$. Hence using $g(n)/\log \log \log n \to \infty$, we obtain by a simple calculation that

$$\prod{}_2 > (1 - \eta_2)^{g(n)}$$

which proves (11) and therefore (10). From (9) and (10) we obtain by a simple argument that

$$(12)\qquad \sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} > (1 - o(1)) \sum_{m=n}^{n+g(n)} D_1(m) = (1 + o(1))g(n)\frac{\pi^2}{6}\cdot$$

(i) now follows from 9 and (12).[3]

Now we are going to prove that (i) is best possible. Put $g(N) = c \log\log\log N$, $n/2 < N < n$. Further let $A_1, A_2, \cdots, A_r$, $r = [2^{-1} \log\log\log n]$ be relatively prime integers all of whose prime factors are less than $2^{-1}\log n$ and for which

$$1/4 < \phi(A_i)/A_i < 1/2, \qquad i = 1, 2, \cdots, r.$$

This is obviously possible since

$$\prod_{p<(\log n)/2}\left(1 - \frac{1}{p}\right) < \frac{c}{\log\log n} < \left(\frac{1}{4}\right)^{(\log\log\log n)/2}.$$

Now choose $n/2 < N < n$ so that $N + j \equiv 0 \pmod{A_j}$, $j \leqq r$. This is possible since by the prime number theorem $A_1 \cdot A_2 \cdots A_r < n/2$. (In all cases where we refer to the prime number theorem a more elementary result would be sufficient.) Clearly

$$\sum_{m=N+1}^{N+(\log\log\log n)/2} \frac{\phi(m)}{m} < \frac{\log\log\log n}{4}.$$

From (9) we have

$$(13)\qquad \sum_{N+(\log\log\log n)/2}^{N+g(N)} \frac{\phi(m)}{m} < (1 + o(1))\frac{6}{\pi^2}\left(g(N) - \frac{\log\log\log n}{2}\right).$$

Thus finally from (10) and (11) we obtain by a simple calculation

$$\sum_{m=N}^{N+g(N)} \frac{\phi(m)}{m} < (1 - c)\frac{6}{\pi^2} g(N),$$

which shows that (i) is best possible.[4]

THEOREM 7. *Let $g_1(n)/\log\log n \to \infty$. Then we have*

$$\text{(i)}\qquad \sum_{m=n}^{n+g_1(n)} \frac{\sigma(m)}{m} = (1 + o(1))\frac{\pi^2}{6} g_1(n).$$

(ii) *Let $g_2(n)/\log\log\log n \to \infty$. The number of integers m in $(n, n+g_2(n))$ which satisfy $\sigma(n+1)/(n+1) > \sigma(n)/n$ equals* $(1+o(1))$ $\cdot g(n)/2$.

[3] This proof is similar to a proof in P. Erdös, J. London Math. Soc. vol. 10 (1935) pp. 128–131.

[4] This proof is similar to a proof of Chowla and Pillai, J. London Math. Soc. vol. 5 (1930) pp. 95–101.

(iii) *The number of integers* m *in* $(n, n+g(n))$ *which satisfy* $\sigma(m)/m < c$ *equals* $(1+o(1))\, g(n)\, f_2(c)$.[5] *All these results are best possible.*

We omit the proof of Theorem 7, since it is similar to that of Theorem 6. We must allow $g_1(n)/\log\log n\to\infty$, since it is well known that for some $m\leqq n$, $\sigma(m)>c\log\log n$ (for example, $m=\prod_{p<(\log n)/2} p$).

Let $f(n)\leqq 1$ and $F(n)\geqq 1$ be multiplicative functions with

$$\sum_p \frac{1-f(p)}{p} < \infty \quad\text{and}\quad \sum_p \frac{F(p)-1}{p} < \infty.$$

Then we have:

THEOREM 8. *Let* $A=A(n)$ *tend to infinity arbitrarily slowly, then*

$$\frac{1}{A}\sum_{m=n}^{n+A} f(m) < (1+(o(1))\frac{1}{n}\sum_{m=1}^{n} f(m)$$

and

$$\frac{1}{A}\sum_{m=n}^{n+A} F(m) > (1+o(1))\frac{1}{n}\sum_{m=1}^{n} F(m).$$

The proof is quite trivial; it is similar to that of (9). It can be shown that $\lim (1/n)\sum_{m=1}^{n} f(m)$ and $\lim (1/n)\sum_{m=1}^{n} F(m)$ exist.

Denote by $V(n)$ the number of prime factors of n and by $d(n)$ the number of divisors of n. We can prove analogs to Theorem 6 for these functions. But the results are very unsatisfactory since for $v(n)$ we have to choose $g(n)=n^{\epsilon/\log\log n}$ and for $d(n)$, $g(n)=n^c$ for some suitable c. These results are probably very far from best possible.

(3) Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, $p_1^{\alpha_1}<p_2^{\alpha_2}<\cdots<p_k^{\alpha_k}$. Put $(p_i^{\alpha_i})^{b_i}=p_i^{\alpha_{i+1}}$. We prove the following theorem.

THEOREM 9. *Let* $1<x$, *then for almost all* n *the number of* b*'s greater than* x *equals*

$$x^{-1}\log\log n + o(\log\log n).$$

REMARK. We immediately obtain that every interval $(x, x+\epsilon)$ contains $(1+o(1))\,(\epsilon/x(x+\epsilon))\log\log n$ b's.

We are going to give only an outline of the proof. First of all we can assume that all the α's are 1, since for large r the number of integers not greater than n for which r or more of the α's is greater than 1 is less than ϵn, since the number of these integers is clearly less than

$$\left(\sum_p \frac{1}{p^2}\right)^r\Big/ r! < \epsilon n.$$

[5] This result has been stated previously, see footnote 4.

Denote by $F(n)$ the number of prime factors p of n such that no prime q in (p, p^x) divides n. $F(n)$ is thus the number of b's not less than x. We have

$$\sum_{m=1}^{n} F(m) = \frac{1}{x} \log \log n + o(\log \log n). \tag{14}$$

We now give a sketch of the proof. Clearly

$$\sum_{m=1}^{n} F(m) = \sum_{p} f_p(n)$$

where $f_p(n)$ denotes the number of integers $m \leqq n$, with $m \equiv 0 \pmod p$ and $m \not\equiv 0 \pmod q$, $p < q < p^x$. It is easy to see that for $p < n^\epsilon$

$$f_p(n) = (1 + o(1))n/px \qquad (p \text{ large}).$$

Also for all p

$$f_p(n) \leqq n/p.$$

Thus

$$\sum_{m=1}^{n} F(m) = \sum_{p \leqq n^\epsilon} \frac{n}{px} + O \sum_{n^\epsilon < p < n} \frac{n}{p} + o(\log \log n)$$
$$= (1 + o(1)) \frac{\log \log n}{x},$$

which proves (14). Now we have to show that

$$F(m) = (1 + o(1))(\log \log n)/x$$

for almost all $m \leqq n$. We use Turán's method.[6] We have

$$\sum_{m=1}^{n} \left(F(m) - \frac{1}{x} \log \log n\right)^2$$
$$= \sum_{m=1}^{n} F^2(m) - \frac{2}{x} \log \log n \sum_{m=1}^{n} F(m) + n \left(\frac{\log \log n}{x}\right)^2.$$

Now

$$\sum_{m=1}^{n} F^2(m) = (1 + o(1))n \left(\frac{\log \log n}{x}\right)^2. \tag{15}$$

We omit the proof of (15), it is similar to the proof of (14). Thus

$$\sum_{m=1}^{n} \left(F(m) - \frac{1}{x} \log \log n\right)^2 = o(n(\log \log n)^2)$$

which proves Theorem 9.

[6] P. Turán, J. London Math. Soc. vol. 9 (1934) pp. 274–276.

THEOREM 10. *For almost all n we have*

$$\sum_{p_i \mid n} b_i = (1 + o(1)) \log \log n \log \log \log n.$$

THEOREM 11. *Let $1 < x$ be any number. For almost all n there exist intervals (m, m^x), $m^x \leqq n$, such that for every $m \leqq y \leqq m^x$, $n \not\equiv 0 \pmod{y}$.*

We omit the proofs of Theorems 10 and 11. They are similar to that of Theorem 9.

For some time I have not been able to decide the following question: Is it true that almost all integers n have divisors d_1 and d_2, such that $d_1 < d_2 < 2d_1$.

(4) Let $f(n)$ be an additive function which has a distribution function. Then it is well known that[7]

$$\sum_p \frac{f(p)'}{p} < \infty, \qquad \sum_p \frac{(f(p)')^2}{p} < \infty, \tag{16}$$

$f(p)' = f(p)$ if $|f(p)| \leqq 1$ and $f(p)' = 1$ if $|f(p)| > 1$. Assume now that $|f(p^\alpha)| \leqq C$ ($f(n)$ is assumed to be real valued). We prove the following theorem.

THEOREM 12. *Let $|f(p^\alpha)| \leqq c$. Denote by $F(x)$ the distribution function of $f(x)$. We have*

$$F(x) > 1 - \exp(-cx),$$

for every c and sufficiently large x. In other words the density of integers with $f(n) \geqq x$ is less than $\exp(-cx)$.

Put $g(n) = \exp(2cf(n))$, $g(n)$ is multiplicative and clearly has a distribution function. Define

$$f_k(n) = \sum_{p \mid n, p \leqq k} f(p), \qquad g_k(n) = \exp(2cf_k(n)).$$

For sake of simplicity we assume that $f(p^\alpha) = f(p)$. It is well known that the distribution function $F_k(x)$ of $f_k(n)$ converges to $F(x)$, thus the distribution function $G_k(x)$ of $g_k(x)$ converges to $G(x)$ ($G(x)$ is the distribution function of $g(x)$). Suppose now that Theorem 12 is false, then there exists a constant c and infinitely many x_r with $x_r \to \infty$ and

$$F(x_r) > 1 - \exp(-cx_r).$$

Therefore for any r there exists a k so large that

$$F_k(x_r) > 1 - \exp(-cx_r).$$

[7] P. Erdös and A. Wintner, Amer. J. Math. vol. 61 (1939) pp. 713–721.

Thus the density of integers with $g_k(n) > \exp(2cx_r)$ is greater than $\exp(-cx_r)$ and hence

$$\sum_{m \leqq n} g_k(m) > (1 - \epsilon) \exp(cx_r) \cdot n$$

for n sufficiently large. Thus for any A there exists k and n_0, such that for all $n > n_0$

$$\sum_{m \leqq n} g_k(m) > An. \tag{17}$$

On the other hand

$$\sum_{m \leqq n} g_k(m) = \sum_{m=1}^{n} \prod_{p \mid m} g_k(p) = \sum_{m=1}^{n} \prod_{p \mid m} (1 + (g_k(p) - 1)).$$

Put $g_k(p) - 1 = h_k(p)$. Clearly

$$\sum_{m=1}^{n} g_k(m) = \sum_{m=1}^{n} \prod_{p \mid m} (1 + h_k(p)) = \sum_{d} \left[\frac{n}{d}\right] h_k(d)$$

where $h_k(d) = \prod_{p \mid d} h_k(p)$. Thus

$$\sum_{m=1}^{n} g_k(m) \leqq n \sum_{d} \frac{h_k(d)}{d} = n \prod_{p} \left(1 + \frac{h_k(p)}{p}\right).$$

From the fact that $g(n)$ has a distribution function and that $f(p^\alpha)$ is bounded, it easily follows that (we shall give the details in the proof of Theorem 13)

$$\sum_{p} \frac{h(p)}{p} < \infty, \qquad \sum_{p} \frac{(h(p))^2}{p} < \infty, \qquad h(p) = g(p) - 1.$$

Thus finally

$$\sum_{m=1}^{n} g_k(m) < c_1 n \prod_{p} \left(1 + \frac{h(p)}{p}\right) < c_2 n,$$

which contradicts (17), and this contradiction establishes the theorem.

It is easy to see that Theorem 12 is best possible. Let $\phi(x)$ tend to infinity arbitrarily slowly; then there exists an additive function $f(n)$ such that its distribution function $F(x)$ satisfies $F(x_i) < 1 - \exp(-\phi(x_i)x_i)$ for an infinite sequence x_i with $x_i \to \infty$. We omit the proof.

THEOREM 13. *Let $g(n) \geqq 0$ be multiplicative. Then the necessary and sufficient condition for the existence of a distribution function is that*

$$\sum_p \frac{(g(p)-1)'}{p} < \infty, \qquad \sum_p \frac{((g(p)-1)')^2}{p} < \infty, \tag{18}$$

where $(g(p)-1)' = g(p)-1$ *if* $|g(p)-1| \leqq 1$ *and* 1 *otherwise.*

The proof follows very easily from (16). Put $\log(g(n)) = f(n)$. $g(n)$ has a distribution function if and only if $f(n)$ has a distribution function. Thus from (16)

$$\sum_p \frac{(\log g(p))'}{p} < \infty, \qquad \sum_p \frac{((\log g(p))')^2}{p} < \infty. \tag{19}$$

Now it follows from (19) that if we neglect a sequence of primes q with $\sum 1/q < \infty$ that $|g(p)-1| < 1/2$. Thus

$$\log g(p) = \log(1 + (g(p)-1)) = g(p) - 1 + (1/2)(g(p)-1)^2 + \cdots.$$

Also simple computation shows that $(\log g(p))^1 > (1/4)(g(p)-1)^2$. Thus from (19)

$$\sum_p \frac{(g(p)-1)^2}{p} < \infty$$

and

$$\sum_p ((1/2)(g(p)-1)^2 + (g(p)-1)^3 + \cdots) < \infty.$$

Thus $\sum_p (g(p)-1)/p < \infty$, which shows that (18) is necessary.

If the two series in (18) converge, then clearly

$$\sum_p \frac{\log g(p)}{p} = \sum_p \left(\frac{(g(p)-1)}{p} + \frac{(1/2)(g(p)-1)^2}{p} + \cdots \right) < \infty$$

and

$$\sum_p \frac{(\log g(p))^2}{p} < c \sum_p \frac{(g(p)-1)^2}{p} < \infty,$$

which shows that $f(n)$, and therefore $g(n)$, has a distribution function. Thus (18) is necessary, which completes the proof of Theorem 13.

These results suggest that if $g(n)$ is multiplicative, satisfies (18), $|g(p^\alpha)| < c$, then $g(n)$ has a mean value, that is, $\lim(1/x)\sum_{n=1}^{x} f(n)$ exists. I have not yet been able to prove this.

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