Probability Theory

Solutions #6

Problems 6.4, 6.7, 6.10 deal with situations when we are counting rare events. Although we might be able to use the binomial distribution to represent the number of occurrences, in these cases the Poisson approximation is more fitting. (Large number of experiments, small probability of success, but in average the expected number of successes is of order 1.) The parameter of the distribution will be the expected number of events occurring.

6.4 If λ is the average number of raisins then we need $0.01 > p(0, \lambda) = e^{-\lambda}$ and from that $\lambda \approx 4.6$.

6.7 If we don't hit a tree then there were no tree centers in a rectangle of .2 m×120 m. The number of trees in this rectangle may be approximated by a Poisson distributed random variable with parameter $\lambda = .2 \times 120 \times \frac{6}{100} = 1.44$. Thus the probability that we hit a tree may be approximated by $1 - p(0, 1.44) \approx 0.76$.

6.8 Let X and Y denote the number of the two types of stars in the given volume. We need to prove that

$$\mathbf{P}(X + Y = n) = p(n, a + b) = e^{-(a+b)} \frac{(a + b)^n}{n!}.$$

Since $\{X + Y = n\} = \bigcup_{i=0}^{n} \{X = i, Y = n - i\}$ and the events on the right are disjoint, we have

$$\mathbf{P}(X + Y = n) = \sum_{i=0}^{n} \mathbf{P}(X = i, Y = n - i) = \sum_{i=0}^{n} \mathbf{P}(X = i)\mathbf{P}(Y = n - i)$$

where we also used the independence of X, Y. We know the distributions of X and Y, therefore

$$\sum_{i=0}^{n} \mathbf{P}(X = i)\mathbf{P}(Y = n - i) = \sum_{i=0}^{n} e^{-a} \frac{a^i}{i!} e^{-b} \frac{b^{n-i}}{(n - i)!} = e^{-(a+b)} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

The statement follows from the binomial theorem.

Since the probability that we see a star in a given *small* volume of space is highly unlikely, it is reasonable to approximate the number of stars in a given volume by a Poisson random variable. The statement we just proved gives that the sum of two independent Poisson random variables is also Poisson and its parameter is just the sum of the parameters.

6.10 The probability of getting at least three hits is $p \approx 0.00082$ (one can calculate this using the hypergeometric distribution). The number of times we get at least three hits during n weeks of playing may be approximated by a Poisson random variable with parameter $\lambda = np$. We need $0.5 > p(0, \lambda) = e^{-np}$, from that $n \geq 844$.

6.12 We proved the second equality in problem 6.8. For the first one observe, that the right hand side gives the probability of getting k successes out of $n_1 + n_2$ independent tries where each try has a probability of p to be successful. The left hand side gives the probability of the same event: but we just break up the event into the union of disjoint events by the number of successes in the first n_1 tries.

The interpretation is similar to that of problem 6.8: if we have two independent binomial random variables with parameters (p, n_1) and (p, n_2) then the sum is also binomially distributed with parameters $(p, n_1 + n_2)$.