

# Partitioning multi-dimensional sets in a small number of “uniform” parts

Noga Alon<sup>\*</sup>    Ilan Newman    Alexander Shen  
Gábor Tardos<sup>†</sup>    Nikolai Vereshchagin<sup>‡</sup>

## Abstract

Our main result implies the following easily formulated statement. The set of edges  $E$  of every finite bipartite graph can be split into  $\text{poly}(\log |E|)$  subsets so that all the resulting bipartite graphs are almost regular. The latter means that the ratio between the maximal and minimal non-zero degree of the left nodes is bounded by a constant and the same condition holds for the right nodes. Stated differently, every finite 2-dimensional set  $S \subset \mathbb{N}^2$  can be partitioned into  $\text{poly}(\log |S|)$  parts so that in every part the ratio between the maximal size and the minimal size of non-empty horizontal section is bounded by a constant and the same condition holds for vertical sections.

We prove a similar statement for  $n$ -dimensional sets for any  $n$  and show how it can be used to relate information inequalities for Shannon entropy of random variables to inequalities between sizes of sections and their projections of multi-dimensional finite sets.

Let  $S$  be a finite  $n$ -dimensional set, that is, a subset of  $X_1 \times X_2 \times \dots \times X_n$  for some  $X_1, X_2, \dots, X_n$ . For every set of indices  $A \subset \{1, 2, \dots, n\} = [n]$

---

<sup>\*</sup>Schools of Mathematics and Computer Science, Tel Aviv University, Tel Aviv 69978, Israel and Institute for Advanced Study, Princeton, NJ 08540, USA. Email: noga@tau.ac.il. Research supported in part by the Israel Science Foundation, by the Hermann Minkowski Minerva Center for Geometry, and by the Von Neumann Fund.

<sup>†</sup>Rényi Institute, Budapest, Hungary. Partially supported by the Hungarian National Research Grants OTKA-T-037846 and OTKA-T-048826.

<sup>‡</sup>Work done in part while visiting LIF, University of Provence. Moscow Lomonossov University, Leninskie Gory, Moscow 119992. Work was in part supported by RFBR grants 03-01-00475, 02-01-22001, 358.2003.1

consider the projection of  $S$  on coordinates in  $A$ . Let  $m_S(A)$  denote the cardinality of this projection. We will consider also sections of projections of  $S$ . Let  $A$  and  $B$  be disjoint sets of indices. Choose any point  $s$  in  $S$  and consider the set of  $A$ -coordinates of all the points in  $S$  having the same  $B$ -coordinates as  $s$ . Sets of this form are called  $A|B$ -sections of  $S$ . Let  $\max_S(A|B)$  stand for the largest cardinality of an  $A|B$ -section and  $\min_S(A|B)$  for the smallest of them.

It is natural to define  $\max(A|\emptyset) = \min(A|\emptyset) = m(A)$  and  $m(\emptyset) = \max(\emptyset|B) = \min(\emptyset|B) = 1$ .

For example let  $S \subset \mathbb{N}^2$  (Fig. 1).

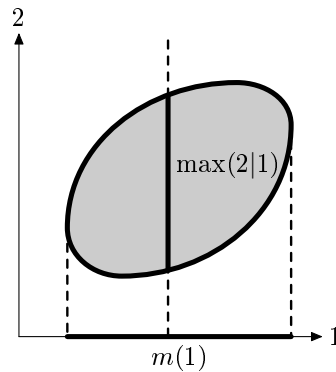


Figure 1: A 2-dimensional set and its characteristics

Then  $m_S(\{1\})$  is the number of elements in the projection of  $S$  on the horizontal axis,  $m_S(\{2\})$  is the number of elements in the projection on the vertical axis,  $\max_S(\{2\}|\{1\})$  is the maximal number of elements in vertical sections, and  $\max_S(\{1\}|\{2\})$  is the maximal number of elements in horizontal sections. The total number of elements in  $S$  is  $m_S(\{1, 2\})$ .

We have the following trivial inequality:

$$m(1, 2) \leq m(1) \cdot \max(2|1)$$

(we drop the subscript  $S$  and the brackets). Indeed, there are  $m(1)$  vertical sections and each of them has at most  $\max(2|1)$  elements. For  $n$ -dimensional sets and disjoint sets  $A$  and  $B$  of indices we have a similar inequality:

$$m(A \cup B) \leq m(B) \cdot \max(A|B).$$

Call a set  $S$  *uniform* if for all disjoint  $A, B$  this inequality specializes to equality. In other words, all  $A|B$ -sections have the same cardinality, that is,  $\max(A|B) = \min(A|B)$  (for all  $A, B$ ). Note that it is enough to require the equality

$$m(A \cup B) = m(B) \cdot \max(A|B)$$

to be true only for  $B = \bar{A}$  (the complement of  $A$ ). The simplest example of a uniform set is a “parallelepiped”—a product of  $n$  sets  $S_i$ . There are other uniform sets, for instance, the 6-element set shown on Fig 2 is uniform: all its vertical and horizontal sections have 2 elements.

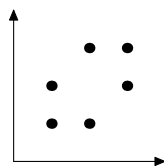


Figure 2: A uniform set

Uniform sets were used in [3] to provide a combinatorial interpretation to inequalities for Shannon entropies of random variables, called also *information inequalities*. Let  $\xi_1, \dots, \xi_n$  be random variables with finite domains having a joint distribution. Consider linear inequalities of the form

$$\sum_A \lambda_A H(\xi_A) \leq 0. \quad (1)$$

Here  $A$  ranges over non-empty subsets of the set of indices  $\{1, \dots, n\}$  and  $\xi_A$  stands for the random variable consisting of all  $\xi_i$  for  $i \in A$ , that is,  $\xi_A$  is the  $A$ -projection of the vector  $\langle \xi_1, \dots, \xi_n \rangle$ . Here are two examples of such inequalities

$$\begin{aligned} H(\xi_1) + H(\xi_2) &\geq H(\langle \xi_1, \xi_2 \rangle), \\ H(\xi_1) + H(\langle \xi_1, \xi_2, \xi_3 \rangle) &\leq H(\langle \xi_1, \xi_2 \rangle) + H(\langle \xi_1, \xi_3 \rangle). \end{aligned} \quad (2)$$

Both inequalities are true for all  $\xi_1, \xi_2, \xi_3$ . These two inequalities correspond to the following two combinatorial inequalities

$$m(1)m(2) \geq m(1, 2), \quad m(1)m(1, 2, 3) \leq m(1, 2)m(1, 3)$$

that are true for all uniform sets. Actually the first one is obviously true for all sets. However the second one is false for some sets: consider, for instance, the disjoint union of a parallelepiped  $U \times V \times W$  with large  $U, V, W$  with another parallelepiped  $P \times \{1\} \times \{1\}$  where  $|P|$  is much greater than  $|U|$  and much less than  $|U \times V|$  and  $|U \times W|$ . The first parallelepiped (as well as the second one) satisfies the equality  $m(1)m(1, 2, 3) = m(1, 2)m(1, 3)$ . However when we join it with the second parallelepiped, all the terms  $m(1, 2, 3), m(1, 2), m(1, 3)$  increase only a little but the term  $m(1)$  increases much; the inequality becomes false.

For uniform sets the second inequality can be proved as follows. Replace the term  $m(1, 2, 3)$  in the left hand side by  $m(1) \max(\{2, 3\}|1)$  and make similar replacements in the right hand side:  $m(1, 2) = m(1) \max(2|1)$  and  $m(1, 3) = m(1) \max(3|1)$ . Then the inequality becomes trivial:

$$\max(\{2, 3\}|1) \leq \max(2|1) \max(3|1)$$

(the size of every 2-dimensional section is upper bounded by the product of its linear projections).

Most known information inequalities are consequences of inequalities of type (2), such inequalities are called Shannon type inequalities. However, there are some exceptions found recently, see [4, 1].

For every inequality for Shannon entropies of random variables  $\xi_1, \dots, \xi_n$  of the form (1) we can consider the corresponding inequality for the size of  $n$ -dimensional finite sets and its projections. It is obtained by formal substitution of  $\log m(A)$  for  $H(\xi_A)$  in the formula (1):

$$\sum_A \lambda_A \log m(A) \leq 0 \tag{3}$$

or, in equivalent form,

$$\prod_A m(A)^{\lambda_A} \leq 1. \tag{4}$$

In the paper [3], it is shown that if the inequality (1) for Shannon entropy is true for all random variables then the corresponding combinatorial inequality (4) is true for all uniform sets and vice versa.

The goal of our paper is to go further: to every linear inequality for Shannon entropy, we provide a combinatorial interpretation that is true for *every* finite set (another interpretation of this kind was presented in [2]). Namely we show that if the inequality (1) is true for all random variables

then every finite set can be partitioned into a small number of parts so that every part “almost” satisfies the corresponding inequality (4). The number of parts is bounded by a polynomial of the logarithm of the cardinality of the set and “almost” means that the constant 1 in the right hand side of (4) is replaced by some constant depending only on  $n$ , the number of variables:

**Theorem 1.** *For every  $n$  there is a constant  $d$  and a polynomial  $p$  such that the following holds. Every finite set  $S \subset \mathbb{N}^n$  can be partitioned into  $p(\log |S|)$  parts so that for every part we have*

$$\prod_A m(A)^{\lambda_A} \leq d \tag{5}$$

whenever the parameters  $\lambda_A$  satisfy  $\sum_A |\lambda_A| \leq 1$  and the inequality (1) is true for all random variables.

The proof of the theorem consists of two parts: we first prove that every set  $S \subset \mathbb{N}^n$  can be partitioned into  $\text{poly}(\log |S|)$  almost uniform parts and then we prove that every almost uniform set satisfies the inequality (5).

Let us give the definition of an almost uniform set. Fix a constant  $c$  and call a set  $S$  *c-uniform* if

$$c \cdot m(A \cup B) \geq m(B) \cdot \max(A|B)$$

for all disjoint sets of indices  $A, B$ . In other words, the cardinality of the largest  $A|B$ -section exceeds the average cardinality of  $A|B$ -sections, that is equal to  $m(A \cup B)/m(B)$ , by at most a factor of  $c$ . Uniform sets are 1-uniform sets. Call a set  $S$  *weakly c-uniform* if

$$c \cdot m(\bar{A} \cup A) \geq m(A) \cdot \max(\bar{A}|A)$$

for every set of indices  $A$ . One can show that weak 2-uniformity does not imply  $c$ -uniformity: consider the 3-dimensional set  $\{(0, i, i) : 1 \leq i \leq n\} \cup \{(i, i, j) : 1 \leq i, j \leq n\}$ . This set is weakly 2-uniform, but it is not  $n/2$ -uniform.

Almost uniform sets have the following simple property.

**Lemma 1.** *If the inequality (1) is true for all random variables and  $\sum_A |\lambda_A| \leq 1$  then for every weakly  $c$ -uniform set  $S \subset \mathbb{N}^n$  we have*

$$\sum_A \lambda_A \log m(A) \leq \log c.$$

*Proof.* Let  $\xi = \langle \xi_1, \dots, \xi_n \rangle$  be the random variable that is uniformly distributed in  $S$ . Then the Shannon entropy of its projection  $\xi_A$  on any set of coordinates  $A$  is at most  $\log m(A)$  and at least  $\log m(A) - \log c$ . Indeed, as  $\xi_A$  has at most  $m(A)$  different outcomes, its entropy is upper bounded by  $\log m(A)$ . Every outcome of  $\xi_A$  has probability at most  $\max(\bar{A}|A)/|S|$ . As  $S$  is  $c$ -uniform, this is less than  $c/m(A)$ . Hence the entropy of  $\xi_A$  is greater than the minus logarithm of this ratio.

By assumption the inequality (1) is true for  $\xi_1, \dots, \xi_n$ . Replace each term  $H(\xi_A)$  in it by the term  $\log m(A)$ , which differs from it by at most  $\log c$ .  $\square$

Thus to prove Theorem 1 it is enough to show the following combinatorial statement.

**Theorem 2.** *For all  $n$  there are a constant  $c$  and a polynomial  $p$  such that every finite set  $S \subset \mathbb{N}^n$  can be partitioned into  $p(\log |S|)$   $c$ -uniform parts.*

*Proof.* We associate a *weight* with every partition of  $S$  into subsets. We will show that the partition with the smallest weight (existing, as the number of partitions is finite) satisfies the statement of the theorem.

We first define a weight for every element  $s \in S$ . Let  $X$  be a part from the partition. The weight of every element  $s \in X$  is defined by the formula

$$w(s) = \sum_{A,B} \log \max_X(B|A) - d \log |X|, \quad (6)$$

where the sum is over all disjoint pairs of indices  $A, B \subseteq \{1, 2, \dots, n\}$  and  $d$  is a constant (depending on  $n$ , to be chosen later). Note that the sum includes the terms  $\log |B|$  for all  $B \subseteq \{1, 2, \dots, n\}$  as we can let  $A = \emptyset$ . Let us stress that the weights of elements in the same part coincide.

We then define the weight of a partition as the sum of the weight of all the elements.

Let us prove first that if  $d$  is sufficiently large then the number of parts in every partition of the smallest weight is small. Namely, we will prove that if we glue together any two parts for which the terms  $\log \max(B|A)$  are close enough (differ by at most 1) for every  $B, A$ , then the weight of the partition decreases. Indeed, let  $X, Y$  be two parts for which  $\log \max_X(B|A)$  differs from  $\log \max_Y(B|A)$  by at most 1, for every disjoint  $B, A$ . This assumption on  $X, Y$  implies that  $|X \cup Y| \geq 1.5 \max(|X|, |Y|)$  by choosing  $B = [n]$ ,  $A = \emptyset$ . Similarly,  $\max_{X \cup Y}(B|A) \leq 3 \max_X(B|A)$  and  $\max_{X \cup Y}(B|A) \leq 3 \max_Y(B|A)$ .

Thus fixing  $B, A$  and summing up the contribution of all elements in  $X \cup Y$  to the term  $\log \max_{X \cup Y}(B|A)$  gives:

$$\begin{aligned} & |X \cup Y| \cdot \log \max_{X \cup Y}(B|A) \\ & \leq |X| \log \max_X(B|A) + |Y| \log \max_Y(B|A) + (|X \cup Y|) \log 3. \end{aligned}$$

Hence, there is an increase of at most  $\log 3(|X \cup Y|)$  with respect to the contribution of the corresponding term before the ‘glue’. On the other hand the term  $d \log |X \cup Y|$  contributes (summing up for all elements in  $X \cup Y$ ) at least  $d(|X \cup Y|) \log(|X \cup Y|)$  plugging in that  $|X \cup Y| \geq 1.5 \max(|X|, |Y|)$  we get that this is at least  $d|X| \log |X| + d|Y| \log |Y| + d|X \cup Y| \log(1.5)$ . Thus if we choose  $d \geq 3^n \log 3 / \log(1.5)$  (to compensate the increase for all  $A, B \subseteq [n]$ ), the value of the partition will certainly decrease.

Let  $d$  be chosen as described. Let us classify the parts in the partition according to the integer part of  $\log \max(B|A)$  for all  $A$  and  $B$ . As shown above, no two parts fall into the same class. Thus the number of parts is bounded by a polynomial of the logarithm of the cardinality of the partitioned set (recall that  $n = O(1)$  here). This implies the upper bound on the number of parts.

It remains to show that in every partition of the smallest weight all the parts are almost uniform. We will show that every part that is considerably non-uniform can be split into two parts so that the weight of the partition decreases. Splitting a part does not affect weights of points in other parts so we may consider only the change of the weight in the split part. As the result of such a splitting, all the terms in the formula given in equation (6) decrease. We need to split the part so that the total decrease of the sum  $\sum_{A,B} \log m_X(B|A)$  is greater than the total decrease of the term  $d \log |X|$ . The decrease of the term  $d \log |X|$  can be expressed by a simple formula: if  $X$  is split in two parts of cardinalities  $p|X|$  and  $q|X|$ , respectively, (thus  $p + q = 1$ ) then the total decrease of  $d \log |X|$  is equal to  $d \cdot |X| \cdot H(p, q)$ , where

$$H(p, q) = p(-\log p) + q(-\log q) \leq 1$$

is the binary entropy function. Therefore the average decrease per element of the term  $d \log |X|$  is at most  $d$ . Hence it suffices to find a splitting such that the first term in the weight of every point in  $X$  decreases by more than  $d$ .

Assume that  $X$  is not  $c$ -uniform, that is for some disjoint sets of indices  $A$  and  $B$  we have  $\max_X(B|A) \geq cd_X(B|A)$  where  $d_X(B|A)$  is the average size

of the  $(B|A)$  sections. We split  $X$  into two parts. The first part contains all small  $(B|A)$  sections and the second contains all the remaining ones. As the threshold take the geometric mean of the size of the maximal section and the size of the average section. In the first part, the size of the maximal section decreases by a factor of at least  $\sqrt{c}$  (compared to  $X$ ). In the other part, all the sections exceed  $\sqrt{c}$  times the average section of  $X$ , hence the number of sections in the second part is  $\sqrt{c}$  smaller than that in  $X$ . That is, the size of the  $A$ -projection of the second part is  $\sqrt{c}$  times smaller than that of  $X$ .

As the result of the splitting, in both parts at least one term of the sum in equation (6) decreases by  $\log \sqrt{c}$  (and all the other do not increase). Therefore if  $c$  is large, that is,  $\log \sqrt{c} > d$ , the decrease in the contribution of the first term in (6) dominates the contribution of the  $d \log |X|$  term and hence the total weight of elements of  $X$  decreases. This means that in every partition of the minimal weight all parts are  $c$ -uniform for a constant  $c$  depending only on  $n$ .  $\square$

Theorem 2 can be strengthened by requiring that in all the parts the ratio between the largest section and the smallest section is bounded by a constant. We do not need this for Theorem 1. However we think this is interesting in its own right.

Call a set *strongly  $d$ -uniform* if for every disjoint sets  $A, B$  of indices

$$\max(A|B) / \min(A|B) \leq d.$$

Every uniform set is strongly 1-uniform.

**Theorem 3.** *For all  $n$  there are a constant  $d$  and a polynomial  $p(k)$  such that every finite set  $S$  can be partitioned into  $p(\log |S|)$  strongly  $d$ -uniform parts.*

*Proof.* First let us note that it is enough to prove that for some polynomial  $q$  and a constant  $d$  every set  $S$  has a strongly  $d$ -uniform subset  $T$  of size at least  $|S|/q(\log |S|)$ . Indeed, remove a large strongly  $d$ -uniform part  $T$  from  $S$ . We obtain a set  $S' \subset S$  of cardinality at most  $|S|(1 - 1/q(\log |S|))$ . Then remove from  $S'$  another  $d$ -uniform subset  $T'$  getting a set  $S''$  of cardinality at most  $|S|(1 - 1/q(\log |S|))^2$ . Repeating this  $O(q(\log |S|) \log |S|)$  times we get the empty set and obtain the partition satisfying the theorem.

To find a large strongly  $d$ -uniform subset  $T$  of a given set  $S$  apply Theorem 2 to  $S$  and take the largest part  $T$  in the partition existing by the



theorem (the part with the largest cardinality). That part  $T$  is  $c$ -uniform and has at least  $1/p(\log |S|)$  fraction of the elements of  $S$ . However for some  $A, B$  it may have  $B|A$ -sections that are much smaller (say,  $d$ -times smaller) than the largest  $B|A$ -section. If this is the case then pick any small section and remove it from  $T$ , that is, remove all the elements of  $T$  whose projection on coordinates in  $A \cup B$  belongs to that section. Repeat such removals in any order until either  $T$  becomes empty or strongly  $d$ -uniform. We claim that if the constant  $d$  is chosen appropriately then  $T$  cannot become empty and moreover it loses at most half of its elements. Indeed, fix a pair of set indices  $A, B$  and count the total number of removed elements in  $B|A$ -sections. After removing any small  $B|A$ -section the set  $T$  loses at most

$$\frac{\max_T(B|A) \max_T(C|A \cup B)}{d}$$

elements, where  $C$  stands for the complement of  $A \cup B$ . The total number of  $B|A$ -sections in  $T$  is equal to  $m_T(A)$  so the total number of removed elements is bounded by

$$\frac{m_T(A) \max_T(B|A) \max_T(C|A \cup B)}{d},$$

As  $T$  is  $c$ -uniform, the product of the first two terms in the numerator is at most  $c \cdot m_T(A \cup B)$ , and again by  $c$ -uniformity, the product of all three terms does not exceed  $c^2 \cdot |T|$ . Hence we can let  $d$  be equal to the number of pairs  $(A, B)$  times  $2c^2$ .  $\square$

We conclude by a simple observation that the converse to Theorem 1 is true, even in a stronger form.

**Theorem 4.** *Let us be given  $n$  and coefficients  $\lambda_A$  for  $A \subseteq \{1, \dots, n\}$ . Assume that every finite set  $S \subseteq \mathbb{N}^n$  can be partitioned into  $O(|S|^{o(1)})$  parts so that for every part  $T$  we have*

$$\sum_A \lambda_A \log m_T(A) = o(\log |S|).$$

*Then the inequality (1) holds for all random variables  $\xi_1, \dots, \xi_n$ .*

*Proof.* We will use a result of [3]: for every tuple  $\xi_1, \dots, \xi_n$  of jointly distributed random variables and for every natural  $N$  we can find a uniform set  $S \subseteq \mathbb{N}^n$  such that  $\log m_S(A) = N \cdot H(\xi_A) + o(N)$  for every set  $A$  of

indices (recall that here  $n = O(1)$  and  $N$  is tending to infinity). In particular,  $\log |S| = N \cdot H(\xi) + o(N) = O(N)$ . Choose a large  $N$  and let  $S$  be the uniform set as above. By the assumption the set  $S$  can be split into  $c = |S|^{o(1)} = 2^{o(N)}$  parts so that for every part  $T$  it holds

$$\sum_A \lambda_A \log m_T(A) = o(\log |S|) = o(N). \quad (7)$$

Pick the largest part  $T$  in the partition. We claim that  $\log m_T(A)$  is close to  $\log m_S(A)$  and hence close to  $N \cdot H(\xi_A)$  for all  $A$ . More specifically,

$$m_S(A)/c \leq m_T(A) \leq m_S(A).$$

The second inequality is obvious, as  $T$  is a subset of  $S$ . To prove the first one let  $B$  be the complement of  $A$ . Compare the inequality (which is always true),  $m_T(A) \max_T(B|A) \geq |T|$ , with the equality  $m_S(A) \max_S(B|A) = |S|$ , which is true as  $S$  is uniform. Using  $|T| \geq |S|/c$  the bound follows.

Thus if we replace  $\log m_T(A)$  in the inequality (7) by  $\log m_S(A)$  the left hand side can increase by at most  $O(\log c) = o(N)$ . Replacing  $\log m_S(A)$  by  $N \cdot H(\xi_A)$  changes it also by at most  $o(N)$ . Thus we obtain the inequality

$$\sum_A \lambda_A \cdot N \cdot H(\xi_A) \leq o(N).$$

Divide it by  $N$  and take the limit. □

It is interesting to estimate the minimal degree of a polynomial  $p$  in Theorem 2. We can find good estimates for its degree in the case of *weak*  $c$ -uniform sets (note that these sets are enough for the proof of Theorem 1).

**Theorem 5.** *Let us fix  $n$  and let  $k = 2^n - 2$ . There exists a  $c > 0$  such that every finite  $n$ -dimensional set  $S$  has a weakly  $c$ -uniform subset of cardinality at least  $\frac{|S|}{(\log |S|)^k}$ . On the other hand for all  $m$  and  $c$  there is a  $n$ -dimensional set  $S$  of cardinality  $\Omega(m^k 2^m)$  all whose  $c$ -uniform subsets have cardinality at most  $O(2^m (\log m + \log c)^k)$ . The constants in the  $O$ - and  $\Omega$ -notations depend on  $n$ .*

This implies that the minimal degree of a polynomial  $p$  such that for some  $c$  every  $n$ -dimensional set  $S$  can be partitioned into  $p(\log |S|)$  weakly  $c$ -uniform subsets is in the range  $[2^n - 2; 2^n - 1]$ .

*Proof.* We first prove the upper bound. To this end we show the following

**Lemma 2.** *For  $k = 2^n - 2$  every weakly  $\alpha$ -uniform  $n$ -dimensional set  $S$  has a weakly  $2^k \sqrt{\alpha}$ -uniform subset of cardinality at least  $|S|/2^k$ .*

*Proof.* Consider a subset  $A$  of  $\{1, \dots, n\}$  and let  $d$  denote the average cardinality of  $\bar{A}|A$ -sections of  $S$ . Let  $S_0 \subseteq S$  contain all  $(\bar{A}|A)$  sections of cardinality  $d\sqrt{\alpha/2}$  or less. Let  $S_1$  contain all the remaining points. Let  $T$  denote the largest set among  $S_0, S_1$ . We claim that  $T$  is  $\sqrt{2\alpha}$ -uniform with respect to  $\bar{A}|A$ -sections. Indeed, if  $T = S_0$  then the cardinality of all  $\bar{A}|A$ -sections in  $T$  is at most  $d\sqrt{\alpha/2}$ , and the average size of a  $\bar{A}|A$ -section is at least  $d/2$  (removing points does not increase the  $A$ -projection of  $S$ , and the cardinality of  $T$  is at least  $|S|/2$ ). If  $T = S_1$  then the cardinality of  $\bar{A}|A$ -sections in  $T$  is at least  $d\sqrt{\alpha/2}$  and at most  $d\alpha$  hence  $T$  is  $\sqrt{2\alpha}$ -uniform with respect to  $\bar{A}|A$ -sections. In both cases of  $T = T_1$  or  $T = T_0$ ,  $T$  is certainly  $2\alpha$ -uniform with respect to  $\bar{B}|B$ -sections for  $B \neq A$ , as section size cannot increase.

Apply this procedure to all non-empty proper subsets  $A$  of  $\{1, \dots, n\}$ . Each application decreases the cardinality of the set by a factor of at most 2, thus the cardinality of the resulting set is at least  $|S|/2^k$ . For each  $A$  the ratio between the cardinalities of the maximum and the average  $(\bar{A}|A)$  sections has been multiplied by a factor of 2 for at most  $k - 1$  times, while at least once it has decreased from some  $r$  to  $\sqrt{2r}$ . Hence the resulting set is weakly  $2^k \sqrt{\alpha}$ -uniform.  $\square$

To end the proof of the upper bound in Theorem 5, apply the lemma  $N = \lfloor \log \log |S| \rfloor$  times to the given set  $S$ . As  $S$  is certainly  $|S|$ -uniform, we obtain that it contains a subset of cardinality at least  $\frac{|S|}{2^{kN}} \geq \frac{|S|}{(\log |S|)^k}$  that is weakly  $c$ -uniform with

$$c = (2^k)^{1+1/2+\dots+2^{1-N}} \cdot |S|^{2^{-N}} < 2^{2k+2}.$$

It remains to prove the lower bound. To this end we first establish the following

**Lemma 3.** *For  $k = 2^n - 2$  and for all  $m$  there is a family of  $\Omega(m^k)$  uniform  $n$ -dimensional sets of cardinality  $2^m$  each, such that the following holds. For every set  $S$  in the family and for every set  $A \subset [n]$ , the cardinality of  $A|\bar{A}$ -sections of  $S$  is equal to  $2^i$  for some natural  $i$ . In addition, for every different  $S_1, S_2$  in the family there is  $A \subseteq [n]$  such that the cardinality of  $\bar{A}|A$ -sections of  $S_1$  differs from the cardinality of  $\bar{A}|A$ -sections of  $S_2$ .*

We first finish the proof of Theorem 5 using this lemma. Let  $\mathcal{F}$  be the family claimed by the lemma and assume without loss of generality that  $|\mathcal{F}| \leq m^k$  and the sets  $S \in \mathcal{F}$  are pairwise disjoint. We claim that the union  $\cup \mathcal{F}$  satisfies the statement of the theorem. Indeed, let  $T$  be a weakly  $c$ -uniform subset. We need to prove that  $T$  has at most  $O(2^m(\log m + \log c)^k)$  points. Let  $d_T(\bar{A}|A)$  be the average cardinality of an  $\bar{A}|A$ -section of  $T$ .

Every point in  $T$  belongs to a set  $S \in \mathcal{F}$  in the family. Divide all points in  $T$  into three groups.

(1) Let  $T_1$  consist of those points in  $T$  that are in some  $S \in \mathcal{F}$  such that for some set  $A_S \subseteq [n]$

$$\max_S(\bar{A}_S|A_S) \leq \frac{d_T(\bar{A}_S|A_S)}{2 \cdot m^k}$$

We claim that  $|T_1| \leq |T|/2$ . Indeed,

$$|T_1| \leq \sum_S m_{T_1}(A_S) \cdot \max_S(\bar{A}_S|A_S) \leq \sum_S m_T(A_S) \cdot \frac{d_T(\bar{A}_S|A_S)}{2 \cdot m^k} \leq |T|/2.$$

(2) Let  $T_2$  consist of those points of  $T$  that are in some  $S \in \mathcal{F}$  for which there is a set  $A_S$  of indices such that

$$\max_S(\bar{A}|A) \geq c \cdot m^k \cdot d_T(\bar{A}|A).$$

For every such  $S$ ,  $|T_2 \cap S| \leq m_S(A_S) \cdot \max_{T \cap S}(\bar{A}_S|A_S) \leq m_S(A_S) \cdot c \cdot d_T(\bar{A}_S|A_S)$  where the last inequality is by the assumption that  $T$  is weakly  $c$ -uniform.

Plugging the bound on  $d_T(\bar{A}_S|A_S)$  implied by the assumption on  $T_2$ , we get:  $|S \cap T_2| \leq m_S(A_S) \cdot \max_S(\bar{A}_S|A_S)/m^k \leq 2^m/m^k$ . Summing up for all  $S$  we get that  $|T_2| \leq 2^m$ .

(3) The remaining points  $T \setminus (T_1 \cup T_2)$ . These points belong to sets  $S \in \mathcal{F}$  such that for all  $A$  the cardinality of the  $A$ -section of  $S$  is in the range

$$d_T(\bar{A}|A)/(2m^k) \leq \max_S(\bar{A}|A) \leq c \cdot m^k \cdot d_T(\bar{A}|A),$$

and hence may take at most  $\log(4c \cdot m^{2k})$  different values. Thus the vector  $(\max_S(\bar{A}|A) : A \subset [n])$  may take at most  $\log^k(4c \cdot m^{2k})$  different values for any  $S$  as above. However, by assumption on the family  $S$ , no two  $S$ 's can have the same such vector. We conclude that the number of such  $S$  is at most  $\log^k(4c \cdot m^{2k})$ . As every such  $S$  has  $2^m$  points we obtain  $|T_3| \leq 2^m \log^k(4c \cdot m^{2k})$ .

Summarizing, we have

$$|T| = |T_1| + |T_2| + |T_3| \leq |T|/2 + 2^m + 2^m \log^k(4c \cdot m^{2k}),$$

that is,

$$|T| \leq 2^{m+1} + 2^{m+1} \log^k(4c \cdot m^{2k}).$$

This proves Theorem 5. □

It remains to prove the lemma.

*Proof of Lemma 3.* Consider a function  $J$  that maps every  $i$  in  $\{1, \dots, n\}$  to a subset of  $[m] = \{1, \dots, m\}$  such that the union of all  $J(i)$  covers  $[m]$ . Associate with  $J$  the following  $n$  dimensional set  $S$ : For every binary string  $x$  of length  $m$  include the point  $\langle x_1, \dots, x_n \rangle$  in  $S$ , where  $x_i$  is represented in binary notation by the concatenation of all bits of  $x$  whose indices are in  $J(i)$ . (Let 0 be represented by the empty string.) Obviously different strings  $x$  give different points in  $S$ , thus  $S$  has  $2^m$  points.

Every  $(\bar{A}|A)$ -section of  $S$  consists of points associated with strings  $x$  having the same projection on the coordinates in the set  $J(A) = \bigcup_{i \in A} J(i)$ . The number of such strings is  $2^{m-|J(A)|}$ . As it depends only on  $A$ , the set  $S$  is uniform.

We need also that for different sets  $S_1, S_2$  in the family there is  $A$  such that the cardinality of  $\bar{A}|A$ -sections of  $S_1$  differs from the cardinality of  $\bar{A}|A$ -sections of  $S_2$ . For the described sets this means that we are not allowed to use different functions  $J$  for which the mappings  $A \mapsto |J(A)|$  are the same. Among all functions  $J$  having the same mapping  $A \mapsto |J(A)|$  we will choose one arbitrary function and put the corresponding set  $S$  in the family. Let us count how many sets in the family we obtain. That is, let  $\mathcal{P}([m])$  denote the power-set of  $[m]$ . We need to count the number of different mappings  $J : [n] \rightarrow \mathcal{P}([m])$ , such that  $J([n]) = [m]$  and for which the vectors  $(|J(A)| : A \subset [n])$  are distinct.

To this end, given a function  $J$ , consider the ‘atoms’ of the set system  $\{J(i) : i \in [n]\}$ , namely all sets of type,

$$K(I) = \bigcap_{i \in I} J(i) \cap \bigcap_{i \notin I} \overline{J(i)},$$

where  $I$  is a non-empty subset of  $\{1, \dots, n\}$ .

It is clear that the sizes of the atoms determine uniquely the sizes of the sets  $J(A)$ ,  $A \subseteq [n]$ . Also, the converse is true by the inclusion-exclusion formula.

Therefore the number of different mappings  $J$  as above is identical to the number of representations of  $m$  as a sum of  $2^n - 1$  non-negative integer terms, that is,  $\binom{m+2^n-2}{2^n-2} = \binom{m+k}{k}$ . This is a polynomial of  $m$  of degree  $k$ , and therefore  $\Omega(m^k)$  as claimed. □

### Questions.

1. Is it true that every 2-dimensional finite set  $S$  can be partitioned into  $\text{poly}(\log |S|)$  *uniform* parts? Is this true for higher dimensions?
2. Theorem 5 asserts that for any  $n$  there is some  $c > 1$  for which there is a big weakly  $c$ -uniform subset in every  $n$ -dimensional set. How big such set can be found for (weakly)  $(1+\epsilon)$ -uniform subsets, for  $\epsilon$  tending to 0 (or even below 1)? We can obtain some good estimates for the 2-dimensional case, but the general case seems more difficult.

## References

- [1] K. Makarychev, Yu. Makarychev, A. Romashchenko, N. Vereshchagin. “A New class of non Shannon type inequalities for entropies”, *Communications in Information and Systems*, 2:2 (2002) 147-166.
- [2] A. Romashchenko, A. Shen, N. Vereshchagin. “Combinatorial interpretation of Kolmogorov complexity”, *Theoretical Computer Science* 271 (2002) 111–123.
- [3] T.H. Chan. “A combinatorial approach to information inequalities”, *Communications in Information and Systems*, 2001, Vol. 1, No 3, pp. 241–254.
- [4] R. W. Yeung, Z. Zhang, “On Characterization of entropy function via information inequalities.” *IEEE Trans. Inform. Theory*, vol.44, pp.1440-1450, July 1998.