# Local chromatic number, Ky Fan's theorem, and circular colorings

**Gábor Simonyi**[1] **Gábor Tardos**[2]

Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences,
1364 Budapest, POB 127, Hungary

simonyi@renyi.hu tardos@renyi.hu

July 15, 2005

[1]Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. T037846, T046376, and AT048826.

[2]Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. T037846, T046234, and AT048826.

**Abstract**

The local chromatic number of a graph was introduced in [14]. It is in between the chromatic and fractional chromatic numbers. This motivates the study of the local chromatic number of graphs for which these quantities are far apart. Such graphs include Kneser graphs, their vertex color-critical subgraphs, the Schrijver (or stable Kneser) graphs; Mycielski graphs, and their generalizations; and Borsuk graphs. We give more or less tight bounds for the local chromatic number of many of these graphs.

We use an old topological result of Ky Fan [17] which generalizes the Borsuk-Ulam theorem. It implies the existence of a multicolored copy of the complete bipartite graph $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ in every proper coloring of many graphs whose chromatic number $t$ is determined via a topological argument. (This was in particular noted for Kneser graphs by Ky Fan [18].) This yields a lower bound of $\lceil t/2\rceil+1$ for the local chromatic number of these graphs. We show this bound to be tight or almost tight in many cases.

As another consequence of the above we prove that the graphs considered here have equal circular and ordinary chromatic numbers if the latter is even. This partially proves a conjecture of Johnson, Holroyd, and Stahl and was independently attained by F. Meunier [42]. We also show that odd chromatic Schrijver graphs behave differently, their circular chromatic number can be arbitrarily close to the other extreme.

# 1 Introduction

The local chromatic number of a graph is defined in [14] as the minimum number of colors that must appear within distance 1 of a vertex. For the formal definition let $N(v) = N_G(v)$ denote the *neighborhood* of a vertex $v$ in a graph $G$, that is, $N(v)$ is the set of vertices $v$ is connected to.

**Definition 1** ([14]) *The* local chromatic number $\psi(G)$ *of a graph $G$ is*

$$\psi(G) := \min_{c} \max_{v \in V(G)} |\{c(u) : u \in N(v)\}| + 1,$$

*where the minimum is taken over all proper colorings $c$ of $G$.*

The $+1$ term comes traditionally from considering "closed neighborhoods" $N(v) \cup \{v\}$ and results in a simpler form of the relations with other coloring parameters.

It is obvious that the local chromatic number of a graph $G$ cannot be more than the chromatic number $\chi(G)$. If $G$ is properly colored with $\chi(G)$ colors then each color class must contain a vertex, whose neighborhood contains all other colors. Thus a value $\psi(G) < \chi(G)$ can only be attained with a coloring in which more than $\chi(G)$ colors are used. Therefore it is somewhat surprising, that the local chromatic number can be arbitrarily less than the chromatic number, cf. [14], [19].

On the other hand, it was shown in [31] that

$$\psi(G) \geq \chi_f(G)$$

holds for any graph $G$, where $\chi_f(G)$ denotes the fractional chromatic number of $G$. For the definition and basic properties of the fractional chromatic number we refer to the books [45, 21].

This suggests to investigate the local chromatic number of graphs for which the chromatic number and the fractional chromatic number are far apart. This is our main goal in this paper.

Prime examples of graphs with a large gap between the chromatic and the fractional chromatic numbers are Kneser graphs and Mycielski graphs, cf. [45]. Other, closely related examples are provided by Schrijver graphs, that are vertex color-critical induced subgraphs of Kneser graphs, and many of the so-called generalized Mycielski graphs. In this introductory section we focus on Kneser graphs and Schrijver graphs, Mycielski graphs and generalized Mycielski graphs will be treated in detail in Subsection 4.3.

We recall that the Kneser graph $KG(n,k)$ is defined for parameters $n \geq 2k$ as the graph with all $k$-subsets of an $n$-set as vertices where two such vertices are connected if they represent disjoint $k$-sets. It is a celebrated result of Lovász [36] (see also [5, 22]) proving the earlier conjecture of Kneser, that $\chi(KG(n,k)) = n - 2k + 2$. For the fractional chromatic number one has $\chi_f(KG(n,k)) = n/k$ as easily follows from the vertex-transitivity of $KG(n,k)$ via the Erdős-Ko-Rado theorem, see [45, 21].

Bárány's proof [5] of the Lovász-Kneser theorem was generalized by Schrijver [46] who found a fascinating family of subgraphs of Kneser graphs that are vertex-critical with respect to the chromatic number.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

**Definition 2** ([46]) *The stable Kneser graph or* Schrijver graph $SG(n, k)$ *is defined as follows.*

$$\begin{aligned} V(SG(n,k)) &= \{A \subseteq [n] : |A| = k, \forall i : \ \{i, i+1\} \nsubseteq A \ \text{ and } \ \{1, n\} \nsubseteq A\}, \\ E(SG(n,k)) &= \{\{A, B\} : A \cap B = \emptyset\}. \end{aligned}$$

Thus $SG(n, k)$ is the subgraph induced by those vertices of $KG(n, k)$ that contain no neighboring elements in the cyclically arranged basic set $\{1, 2, \ldots, n\}$. These are sometimes called *stable $k$-subsets*. The result of Schrijver in [46] is that $\chi(SG(n, k)) = n - 2k + 2 (= \chi(KG(n, k))$, but deleting any vertex of $SG(n, k)$ the chromatic number drops, i.e., $SG(n, k)$ is vertex-critical with respect to the chromatic number. Recently Talbot [49] proved an Erdős-Ko-Rado type result, conjectured by Holroyd and Johnson [27], which implies that the ratio of the number of vertices and the independence number in $SG(n, k)$ is $n/k$. This gives $n/k \leq \chi_f(SG(n, k))$ and equality follows by $\chi_f(SG(n, k)) \leq \chi_f(KG(n, k)) = n/k$. Notice that $SG(n, k)$ is not vertex-transitive in general. See more on Schrijver graphs in [8, 35, 39, 54].

Concerning the local chromatic number it was observed by several people [20, 30], that $\psi(KG(n, k)) \geq n - 3k + 3$ holds, since the neighborhood of any vertex in $KG(n, k)$ induces a $KG(n - k, k)$ with chromatic number $n - 3k + 2$. Thus for $n/k$ fixed but larger than 3, $\psi(G)$ goes to infinity with $n$ and $k$. In fact, the results of [14] have a similar implication also for $2 < n/k \leq 3$. Namely, it follows from those results, that if a series of graphs $G_1, \ldots, G_i, \ldots$ is such that $\psi(G_i)$ is bounded, while $\chi(G_i)$ goes to infinity, then the number of colors to be used in colorings attaining the local chromatic number grows at least doubly exponentially in the chromatic number. However, Kneser graphs with $n/k$ fixed and $n$ (therefore also the chromatic number $n - 2k + 2$) going to infinity cannot satisfy this, since the total number of vertices grows simply exponentially in the chromatic number.

The estimates mentioned in the previous paragraph are elementary. On the other hand, all known proofs for $\chi(KG(n, k)) \geq n - 2k + 2$ use topology or at least have a topological flavor (see [36, 5, 22, 40] to mention just a few such proofs). They use (or at least, are inspired by) the Borsuk-Ulam theorem.

In this paper we use a stronger topological result due to Ky Fan [17] to establish that all proper colorings of a $t$-chromatic Kneser, Schrijver or generalized Mycielski graph contain a multicolored copy of a balanced complete bipartite graph. This was noticed by Ky Fan for Kneser graphs [18]. We also show that the implied lower bound of $\lceil t/2 \rceil + 1$ on the local chromatic number is tight or almost tight for many Schrijver graphs and generalized Mycielski graphs.

In the following section we summarize our main results in more detail.

## 2 Results

In this section we summarize our results without introducing the topological notions needed to state the results in their full generality. We will introduce the phrase that a graph $G$ is *topologically $t$-chromatic* meaning that $\chi(G) \geq t$ and this fact can be shown by a specific topological method, see Subsection 3.2. Here we use this phrase only to emphasize the generality of the corresponding statements, but the reader can always substitute the phrase "a topologically $t$-chromatic graph" by "a $t$-chromatic Kneser graph" or "a $t$-chromatic Schrijver graph" or by "a generalized Mycielski graph of chromatic number $t$".

Our general lower bound for the local chromatic number proven in Section 3 is the following.

**Theorem 1** *If $G$ is topologically $t$-chromatic for some $t \geq 2$, then*

$$\psi(G) \geq \left\lceil \frac{t}{2} \right\rceil + 1.$$

This result on the local chromatic number is the immediate consequence of the Zig-zag theorem in Subsection 3.3 that we state here in a somewhat weaker form:

**Theorem 2** *Let $G$ be a topologically $t$-chromatic graph and let $c$ be a proper coloring of $G$ with an arbitrary number of colors. Then there exists a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ of $G$ all vertices of which receive a different color in $c$.*

We use Ky Fan's generalization of the Borsuk-Ulam theorem [17] for the proof. The Zig-zag theorem was previously established for Kneser graphs by Ky Fan [18].

We remark that János Körner [30] suggested to introduce a graph invariant $b(G)$ which is the size (number of points) of the largest completely multicolored complete bipartite graph that should appear in any proper coloring of graph $G$. It is obvious from the definition that this parameter is bounded from above by $\chi(G)$ and bounded from below by the local chromatic number $\psi(G)$. An obvious consequence of Theorem 2 is that if $G$ is topologically $t$-chromatic, then $b(G) \geq t$.

In Section 4 we show that Theorem 1 is essentially tight for several Schrijver and generalized Mycielski graphs. In particular, this is always the case for a topologically $t$-chromatic graph that has a *wide* $t$-coloring as defined in Definition 4 in Subsection 4.1.

As the first application of our result on wide colorings we show, that if the chromatic number is fixed and odd, and the size of the Schrijver graph is large enough, then Theorem 1 is exactly tight:

**Theorem 3** *If $t = n - 2k + 2 > 2$ is odd and $n \geq 4t^2 - 7t$ then*

$$\psi(SG(n,k)) = \left\lceil \frac{t}{2} \right\rceil + 1.$$

See Remark 4 in Subsection 4.2 for a relaxed bound on $n$. The proof of Theorem 3 is combinatorial. It will also show that the claimed value of $\psi(SG(n,k))$ can be attained with a coloring using $t+1$ colors and avoiding the appearance of a totally multicolored $K_{\lceil \frac{t}{2} \rceil, \lceil \frac{t}{2} \rceil}$. To appreciate the latter property, cf. Theorem 2.

Since $SG(n,k)$ is an induced subgraph of $SG(n+1,k)$ Theorem 3 immediately implies that for every fixed even $t = n - 2k + 2$ and $n, k$ large enough

$$\psi(SG(n,k)) \in \left\{ \frac{t}{2} + 1, \frac{t}{2} + 2 \right\}.$$

The lower bound for the local chromatic number in Theorem 1 is smaller than $t$ whenever $t \geq 4$ but Theorem 3 claims the existence of Schrijver graphs with smaller local than ordinary chromatic number only with chromatic number 5 and up. In [47] we prove that the local chromatic number of all 4-chromatic Kneser, Schrijver, or generalized Mycielski graphs is 4. The reason is that all these graphs satisfy a somewhat stronger property, they are *strongly* topologically 4-chromatic (see Definition 3). On the other hand, we also show in [47] that topologically 4-chromatic graphs of local chromatic number 3 do exist.

To demonstrate that requiring large $n$ and $k$ in Theorem 3 is crucial we prove the following statement.

**Proposition 4** $\psi(SG(n,2)) = n - 2 = \chi(SG(n,2))$ *for every* $n \geq 4$.

As a second application of wide colorings we prove in Subsection 4.3 that Theorem 1 is also tight for several generalized Mycielski graphs. These graphs will be denoted by $M_{\boldsymbol{r}}^{(d)}(K_2)$ where $\boldsymbol{r} = (r_1, \ldots, r_d)$ is a vector of positive integers. See Subsection 4.3 for the definition. Informally, $d$ is the number of iterations and $r_i$ is the number of "levels" in iteration $i$ of the generalized Mycielski construction. $M_{\boldsymbol{r}}^{(d)}(K_2)$ is proven to be $(d+2)$-chromatic "because of a topological reason" by Stiebitz [48]. This topological reason implies that these graphs are strongly topologically $(d+2)$-chromatic. Thus Theorem 1 applies and gives the lower bound part of the following result.

**Theorem 5** *If* $\boldsymbol{r} = (r_1, \ldots, r_d)$*,* $d$ *is odd, and* $r_i \geq 7$ *for all* $i$*, then*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) = \left\lceil \frac{d}{2} \right\rceil + 2.$$

It will be shown in Theorem 13 that relaxing the $r_i \geq 7$ condition to $r_i \geq 4$ an only slightly weaker upper bound is still valid. As a counterpart we also show (see Proposition 10 in Subsection 4.3) that for the ordinary Mycielski construction, which is the special case of $\boldsymbol{r} = (2, \ldots, 2)$, the local chromatic number behaves just like the chromatic number.

The Borsuk-Ulam Theorem in topology is known to be equivalent (see Lovász [37]) to the validity of a tight lower bound on the chromatic number of graphs defined on the $n$-dimensional sphere, called Borsuk graphs. In Subsection 4.4 we prove that the local chromatic number of Borsuk graphs behaves similarly as that of the graphs already mentioned above. In this subsection we also formulate a topological consequence of our results on the tightness of Ky Fan's theorem [17]. We also give a direct proof for the same tightness result.

The circular chromatic number $\chi_c(G)$ of a graph $G$ was introduced by Vince [52], see Definition 7 in Section 5. It satisfies $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. In Section 5 we prove the following result using the Zig-zag theorem.

**Theorem 6** *If $G$ is topologically $t$-chromatic and $t$ is even, then $\chi_c(G) \geq t$.*

This theorem implies that $\chi_c(G) = \chi(G)$ if the chromatic number is even for Kneser graphs, Schrijver graphs, generalized Mycielski graphs, and certain Borsuk graphs. The result on Kneser and Schrijver graphs gives a partial solution of a conjecture by Johnson, Holroyd, and Stahl [28] and a partial answer to a question of Hajiabolhassan and Zhu [24]. These results were independently obtained by Meunier [42]. The result on generalized Mycielski graphs answers a question of Chang, Huang, and Zhu [10].

We will also discuss the circular chromatic number of odd chromatic Borsuk and Schrijver graphs showing that they can be close to one less than the chromatic number. We will use a a similar result for generalized Mycielski graphs proven by Lam, Lin, Gu, and Song [33].

# 3 Lower bound

## 3.1 Topological preliminaries

The following is a brief overview of some of the topological concepts we need. We refer to [7, 26] and [39] for basic concepts and also for a more detailed discussion of the notions and facts given below.

A $\mathbb{Z}_2$-*space* (or *involution space*) is a pair $(T, \nu)$ of a topological space $T$ and the involution $\nu : T \to T$, which is continuous and satisfies that $\nu^2$ is the identity map. The points $x \in T$ and $\nu(x)$ are called *antipodal*. The involution $\nu$ and the $\mathbb{Z}_2$-space $(T, \nu)$ are *free* if $\nu(x) \neq x$ for all points $x$ of $T$. If the involution is understood from the context we speak about $T$ rather than the pair $(T, \nu)$. This is the case, in particular, for the unit sphere $S^d$ in $\mathbb{R}^{d+1}$ with the involution given by the central reflection $\boldsymbol{x} \mapsto -\boldsymbol{x}$. A continuous map $f : S \to T$ between $\mathbb{Z}_2$-spaces $(S, \nu)$ and $(T, \pi)$ is a $\mathbb{Z}_2$-*map* (or an *equivariant map*) if it respects the respective involutions, that is $f \circ \nu = \pi \circ f$. If such a map exists we write $(S, \nu) \to (T, \pi)$. If $(S, \nu) \to (T, \pi)$ does not hold we write $(S, \nu) \not\to (T, \pi)$. If both $S \to T$ and $T \to S$ we call the $\mathbb{Z}_2$-spaces $S$ and $T$ $\mathbb{Z}_2$-equivalent and write $S \leftrightarrow T$.

We try to avoid using homotopy equivalence and $\mathbb{Z}_2$-homotopy equivalence (i.e., homotopy equivalence given by $\mathbb{Z}_2$-maps), but we will have to use two simple observations.

First, if the $\mathbb{Z}_2$-spaces $S$ and $T$ are $\mathbb{Z}_2$-homotopy equivalent, then $S \leftrightarrow T$. Second, if the space $S$ is homotopy equivalent to a sphere $S^h$ (this relation is between topological spaces, not $\mathbb{Z}_2$-spaces), then for any involution $\nu$ we have $S^h \to (S, \nu)$.

The $\mathbb{Z}_2$-index of a $\mathbb{Z}_2$-space $(T, \nu)$ is defined (see e.g. [41, 39]) as

$$\mathrm{ind}(T, \nu) := \min\{d \geq 0 : (T, \nu) \to S^d\},$$

where $\mathrm{ind}(T, \nu)$ is set to be $\infty$ if $(T, \nu) \not\to S^d$ for all $d$.

The $\mathbb{Z}_2$-coindex of a $\mathbb{Z}_2$-space $(T, \nu)$ is defined as

$$\mathrm{coind}(T, \nu) := \max\{d \geq 0 : S^d \to (T, \nu)\}.$$

If such a map exists for all $d$, then we set $\mathrm{coind}(T, \nu) = \infty$. Notice that if $(T, \nu)$ is not free, we have $\mathrm{ind}(T, \nu) = \mathrm{coind}(T, \nu) = \infty$.

Note that $S \to T$ implies $\mathrm{ind}(S) \leq \mathrm{ind}(T)$ and $\mathrm{coind}(S) \leq \mathrm{coind}(T)$. In particular, $\mathbb{Z}_2$-equivalent spaces have equal index and also equal coindex.

The celebrated Borsuk-Ulam Theorem can be stated in many equivalent forms. Here we state three of them. For more equivalent versions and several proofs we refer to [39]. Here (i) and (ii) are standard forms of the Borsuk-Ulam Theorem, while (iii) is clearly equivalent to (ii).

**Borsuk-Ulam Theorem.**

**(i)** *(Lyusternik-Schnirel'man version) Let $d \geq 0$ and let $\mathcal{H}$ be a collection of open (or closed) sets covering $S^d$ with no $H \in \mathcal{H}$ containing a pair of antipodal points. Then $|\mathcal{H}| \geq d + 2$.*

**(ii)** *$S^{d+1} \not\to S^d$ for any $d \geq 0$.*

**(iii)** *For a $\mathbb{Z}_2$-space $T$ we have* $\mathrm{ind}(T) \geq \mathrm{coind}(T)$.

The suspension $\mathrm{susp}(S)$ of a topological space $S$ is defined as the factor of the space $S \times [-1, 1]$ that identifies all the points in $S \times \{-1\}$ and identifies also the points in $S \times \{1\}$. If $S$ is a $\mathbb{Z}_2$-space with the involution $\nu$, then the suspension $\mathrm{susp}(S)$ is also a $\mathbb{Z}_2$-space with the involution $(x, t) \mapsto (\nu(x), -t)$. Any $\mathbb{Z}_2$-map $f : S \to T$ naturally extends to a $\mathbb{Z}_2$-map $\mathrm{susp}(f) : \mathrm{susp}(S) \to \mathrm{susp}(T)$ given by $(x, t) \mapsto (f(x), t)$. We have $\mathrm{susp}(S^n) \cong S^{n+1}$ with a $\mathbb{Z}_2$-homeomorphism. These observations show the well known inequalities below.

**Lemma 3.1** *For any $\mathbb{Z}_2$-space $S$* $\mathrm{ind}(\mathrm{susp}(S)) \leq \mathrm{ind}(S) + 1$ *and* $\mathrm{coind}(\mathrm{susp}(S)) \geq \mathrm{coind}(S) + 1$.

A(n abstract) simplicial complex $K$ is a non-empty, hereditary set system. That is, $F \in K$, $F' \subseteq F$ implies $F' \in K$ and we have $\emptyset \in K$. In this paper we consider only finite simplicial complexes. The non-empty sets in $K$ are called *simplices.* We call the set $V(K)=\{x:\{x\}\in K\}$ the set of *vertices* of $K$. In a *geometric realization* of $K$ a vertex $x$ corresponds to a point $||x||$ in a Euclidean space, a simplex $\sigma$ corresponds to its *body*, the convex hull of its vertices: $||\sigma|| = \mathrm{conv}(\{||x|| : x\in\sigma\})$. We assume that the points $||x||$ for $x\in\sigma$ are affine independent, and so $||\sigma||$ is a geometric simplex. We also assume that disjoint simplices have disjoint bodies. The body of the complex $K$ is $||K|| = \cup_{\sigma\in K}||\sigma||$, it is determined up to homeomorphism by $K$. Any point in $p\in ||K||$ has a unique representation as a convex combination $p=\sum_{x\in V(K)}\alpha_x||x||$ such that $\{x:\alpha_x>0\}\in K$.

A map $f : V(K)\to V(L)$ is called simplicial if it maps simplices to simplices, that is $\sigma\in K$ implies $f(\sigma)\in L$. In this case we define $||f|| : ||K||\to ||L||$ by setting $||f||(||x||) = ||f(x)||$ for vertices $x\in V(K)$ and taking an affine extension of this function to the bodies of each of the simplices in $K$. If $||K||$ and $||L||$ are $\mathbb{Z}_2$-spaces (usually with an involution also given by simplicial maps), then we say that $f$ is a $\mathbb{Z}_2$*-map* if $||f||$ is a $\mathbb{Z}_2$-map. If $||K||$ is a $\mathbb{Z}_2$-space we use $\mathrm{ind}(K)$ and $\mathrm{coind}(K)$ for $\mathrm{ind}(||K||)$ and $\mathrm{coind}(||K||)$, respectively.

Following the papers [1, 32, 41] we introduce the *box complex* $B_0(G)$ for any finite graph $G$. See [41] for several similar complexes. We define $B_0(G)$ to be a simplicial complex on the vertices $V(G)\times\{1,2\}$. For subsets $S,T\subseteq V(G)$ we denote the set $S\times\{1\}\cup T\times\{2\}$ by $S\uplus T$, following the convention of [39, 41]. For $v\in V(G)$ we denote by $+v$ the vertex $(v,1)\in\{v\}\uplus\emptyset$ and $-v$ denotes the vertex $(v,2)\in\emptyset\uplus\{v\}$. We set $S\uplus T\in B_0(G)$ if $S\cap T=\emptyset$ and the complete bipartite graph with sides $S$ and $T$ is a subgraph of $G$. Note that $V(G)\uplus\emptyset$ and $\emptyset\uplus V(G)$ are simplices of $B_0(G)$.

The $\mathbb{Z}_2$-map $S\uplus T\mapsto T\uplus S$ acts simplicially on $B_0(G)$. It makes the body of the complex a free $\mathbb{Z}_2$-space.

We define the *hom space* $H(G)$ of $G$ to be the subspace consisting of those points $p\in ||B_0(G)||$ that, when written as a convex combination $p=\sum_{x\in V(B_0(G))}\alpha_x||x||$ with $\{x:\alpha_x>0\}\in B_0(G)$ give $\sum_{x\in V(G)\uplus\emptyset}\alpha_x = 1/2$.

Notice that $H(G)$ can also be obtained as the body of a *cell complex* $Hom(K_2,G)$, see [3], or of a simplicial complex $B_{chain}(G)$, see [41].

A useful connection between $B_0(G)$ and $H(G)$ follows from a combination of results of Csorba [11] and Matoušek and Ziegler [41].

**Proposition 7** $||B_0(G)|| \leftrightarrow \mathrm{susp}(H(G))$

**Proof.** Csorba [11] proves the $\mathbb{Z}_2$-homotopy equivalence of $||B_0(G)||$ and the suspension of the body of yet another box complex $B(G)$ of $G$. As we mentioned, $\mathbb{Z}_2$-homotopy equivalence implies $\mathbb{Z}_2$-equivalence. Matoušek and Ziegler [41] prove the $\mathbb{Z}_2$-equivalence of $||B(G)||$ and $H(G)$. Finally for $\mathbb{Z}_2$-spaces $S$ and $T$ if $S\to T$, then $\mathrm{susp}(S)\to\mathrm{susp}(T)$, therefore $||B(G)||\leftrightarrow H(G)$ implies $\mathrm{susp}(||B(G)||)\leftrightarrow\mathrm{susp}(H(G))$. $\square$

Note that Csorba [11] proves, cf. also Živaljević [55], the $\mathbb{Z}_2$-homotopy equivalence of $||B(G)||$ and $H(G)$, and therefore we could also claim $\mathbb{Z}_2$-homotopy equivalence in Proposition 7.

## 3.2 Some earlier topological bounds

A graph homomorphism is an edge preserving map from the vertex set of a graph $F$ to the vertex set of another graph $G$. If there is a homomorphism $f$ from $F$ to $G$, then it generates a simplicial map from $B_0(F)$ to $B_0(G)$ in the natural way. This map is a $\mathbb{Z}_2$-map and thus it shows $||B_0(F)|| \to ||B_0(G)||$. One can often prove $||B_0(F)|| \not\to ||B_0(G)||$ using the indexes or coindexes of these complexes and this relation implies the non-existence of a homomorphism from $F$ to $G$. A similar argument applies with the spaces $H(\cdot)$ in place of $||B_0(\cdot)||$.

Coloring a graph $G$ with $m$ colors can be considered as a graph homomorphism from $G$ to the complete graph $K_m$. The box complex $B_0(K_m)$ is the boundary complex of the $m$-dimensional *cross-polytope* (i.e., the convex hull of the basis vectors and their negatives in $\mathbb{R}^m$), thus $||B_0(K_m)|| \cong S^{m-1}$ with a $\mathbb{Z}_2$-homeomorphism and $\mathrm{coind}(B_0(G)) \leq \mathrm{ind}(B_0(G)) \leq m-1$ is necessary for $G$ being $m$-colorable. Similarly, $\mathrm{coind}(H(G)) \leq \mathrm{ind}(H(G)) \leq m-2$ is also necessary for $\chi(G) \leq m$ since $H(K_m)$ can be obtained from intersecting the boundary of the $m$-dimensional cross-polytope with the hyperplane $\sum x_i = 0$, and therefore $H(K_m) \cong S^{m-2}$ with a $\mathbb{Z}_2$-homeomorphism. These four lower bounds on $\chi(G)$ can be arranged in a single line of inequalities using Lemma 3.1 and Proposition 7:

$$\chi(G) \geq \mathrm{ind}(H(G)) + 2 \geq \mathrm{ind}(B_0(G)) + 1 \geq \mathrm{coind}(B_0(G)) + 1 \geq \mathrm{coind}(H(G)) + 2 \quad (1)$$

In fact, many of the known proofs of Kneser's conjecture can be interpreted as a proof of an appropriate lower bound on the (co)index of one of the above complexes. In particular, Bárány's simple proof [5] exhibits a map showing $S^{n-2k} \to H(KG(n,k))$ to conclude that $\mathrm{coind}(H(KG(n,k))) \geq n-2k$ and thus $\chi(KG(n,k)) \geq n-2k+2$. The even simpler proof of Greene [22] exhibits a map showing $S^{n-2k+1} \to B_0(KG(n,k))$ to conclude that $\mathrm{coind}(B_0(KG(n,k))) \geq n-2k+1$ and thus $\chi(KG(n,k)) \geq n-2k+2$. Schrijver's proof [46] of $\chi(SG(n,k)) \geq n-2k+2$ is a generalization of Bárány's and it also can be interpreted as a proof of $S^{n-2k} \to H(SG(n,k))$. We remark that the same kind of technique is used with other complexes related to graphs, too. In particular, Lovász's original proof [36] can also be considered as exhibiting a $\mathbb{Z}_2$-map from $S^{n-2k}$ to such a complex, different from the ones we consider here. For a detailed discussion of several such complexes and their usefulness in bounding the chromatic number we refer the reader to [41].

The above discussion gives several possible "topological reasons" that can force a graph to be at least $t$-chromatic. Here we single out two such reasons. We would like to stress that these two reasons are just two out of many and refer to the paper [2] for some that are not even mentioned above. In this sense, our terminology is somewhat arbitrary. The statement of our results in Section 2 becomes precise by applying the conventions given by the following definition.

**Definition 3** *We say that a graph $G$ is* topologically $t$-chromatic *if*

$$\operatorname{coind}(B_0(G)) \geq t-1.$$

*We say that a graph $G$ is* strongly topologically $t$-chromatic *if*

$$\operatorname{coind}(H(G)) \geq t-2.$$

By inequality (1) if a graph is strongly topologically $t$-chromatic, then it is topologically $t$-chromatic, and if $G$ is topologically $t$-chromatic, then $\chi(G) \geq t$. In [47] we show the existence of a graph for any $t \geq 4$ that is topologically $t$-chromatic but not strongly topologically $t$-chromatic. We also show there that the two notions have different consequences in terms of the local chromatic number for $t = 4$.

The notion that a graph is (strongly) topologically $t$-chromatic is useful, as it applies to many widely studied classes of graphs. As we mentioned above, Bárány [5] and Schrijver [46] establish this for $t$-chromatic Kneser and Schrijver graphs. For the reader's convenience we recall the proof here. See the analogous statement for generalized Mycielski graphs and (certain finite subgraphs of the) Borsuk graphs after we introduce those graphs.

**Proposition 8** (Bárány; Schrijver) *The $t$-chromatic Kneser and Schrijver graphs are strongly topologically $t$-chromatic.*

**Proof.** We need to prove that $SG(n,k)$ is strongly topologically $(n-2k+2)$-chromatic, i.e., that $\operatorname{coind}(H(SG(n,k))) \geq n-2k$. The statement for Kneser graphs follows. For $\boldsymbol{x} \in S^{n-2k}$ let $H_{\boldsymbol{x}}$ denote the open hemisphere in $S^{n-2k}$ around $\boldsymbol{x}$. Consider an arrangement of the elements of $[n]$ on $S^{n-2k}$ so that each open hemisphere contains a stable $k$-subset, i.e., a vertex of $SG(n,k)$. It is not hard to check that identifying $i \in [n]$ with $\boldsymbol{v}_i/|\boldsymbol{v}_i|$ for $\boldsymbol{v}_i = (-1)^i(1, i, i^2, \ldots, i^{n-2k}) \in \mathbb{R}^{n-2k+1}$ provides such an arrangement. (See [46] or [39] for details of this.) For each vertex $v$ of $SG(n,k)$ and $\boldsymbol{x} \in S^{n-2k}$ let $D_v(\boldsymbol{x})$ denote the smallest distance of a point in $v$ from the set $S^{n-2k} \setminus H_{\boldsymbol{x}}$ and let $D(\boldsymbol{x}) = \sum_{v \in V(SG(n,k))} D_v(\boldsymbol{x})$. Note that $D_v(\boldsymbol{x}) > 0$ if $v$ is contained in $H_{\boldsymbol{x}}$ and therefore $D(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$. Let $f(\boldsymbol{x}) := \frac{1}{2D(\boldsymbol{x})} \sum_{v \in V(SG(n,k))} D_v(\boldsymbol{x})||+v|| + \frac{1}{2D(-\boldsymbol{x})} \sum_{v \in V(SG(n,k))} D_v(-\boldsymbol{x})||-v||$. This $f$ is a $\mathbb{Z}_2$-map $S^{n-2k} \to H(SG(n,k))$ proving the proposition. □

## 3.3 Ky Fan's result on covers of spheres and the Zig-Zag theorem

The following result of Ky Fan [17] implies the Lyusternik-Schnirel'man version of the Borsuk-Ulam theorem. Here we state two equivalent versions of the result, both in terms of sets covering the sphere. See the original paper for another version generalizing another standard form of the Borsuk-Ulam theorem.

**Ky Fan's Theorem.**

**(i)** *Let $\mathcal{A}$ be a system of open (or a finite system of closed) subsets of $S^k$ covering the entire sphere. Assume a linear order $<$ is given on $\mathcal{A}$ and all sets $A \in \mathcal{A}$ satisfy $A \cap -A = \emptyset$. Then there are sets $A_1 < A_2 < \ldots < A_{k+2}$ of $\mathcal{A}$ and a point $\boldsymbol{x} \in S^k$ such that $(-1)^i \boldsymbol{x} \in A_i$ for all $i = 1, \ldots, k+2$.*

**(ii)** *Let $\mathcal{A}$ be a system of open (or a finite system of closed) subsets of $S^k$ such that $\cup_{A \in \mathcal{A}}(A \cup -A) = S^k$. Assume a linear order $<$ is given on $\mathcal{A}$ and all sets $A \in \mathcal{A}$ satisfy $A \cap -A = \emptyset$. Then there are sets $A_1 < A_2 < \ldots < A_{k+1}$ of $\mathcal{A}$ and a point $\boldsymbol{x} \in S^k$ such that $(-1)^i \boldsymbol{x} \in A_i$ for all $i = 1, \ldots, k+1$.*

The Borsuk-Ulam theorem is easily seen to be implied by version (i), that shows in particular, that $|\mathcal{A}| \geq k+2$. We remark that [17] contains the above statements only about closed sets. The statements on open sets can be deduced by a standard argument using the compactness of the sphere. We also remark that version (ii) is formulated a little differently in [17]. A place where one finds exactly the above formulation (for closed sets, but for any $\mathbb{Z}_2$-space) is Bacon's paper [4].

**Zig-zag Theorem** *Let $G$ be a topologically $t$-chromatic finite graph and let $c$ be an arbitrary proper coloring of $G$ by an arbitrary number of colors. We assume the colors are linearly ordered. Then $G$ contains a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ such that $c$ assigns distinct colors to all $t$ vertices of this subgraph and these colors appear alternating on the two sides of the bipartite subgraph with respect to their order.*

**Proof.** We have $\text{coind}(B_0(G)) \geq t-1$, so there exists a $\mathbb{Z}_2$-map $f : S^{t-1} \to B_0(G)$. For any color $i$ we define a set $A_i \subset S^{t-1}$ letting $\boldsymbol{x} \in A_i$ if and only if for the minimal simplex $U_{\boldsymbol{x}} \uplus V_{\boldsymbol{x}}$ containing $f(\boldsymbol{x})$ there exists a vertex $z \in U_{\boldsymbol{x}}$ with $c(z) = i$. These sets are open, but they do not necessarily cover the entire sphere $S^{t-1}$. Notice that $-A_i$ consists of the points $\boldsymbol{x} \in S^{t-1}$ with $-\boldsymbol{x} \in A_i$, which happens if and only if there exists a vertex $z \in U_{-\boldsymbol{x}}$ with $c(z) = i$. Here $U_{-\boldsymbol{x}} = V_{\boldsymbol{x}}$. For every $\boldsymbol{x} \in S^{t-1}$ either $U_{\boldsymbol{x}}$ or $V_{\boldsymbol{x}}$ is not empty, therefore we have $\cup_i (A_i \cup -A_i) = S^{t-1}$. Assume for a contradiction that for a color $i$ we have $A_i \cap -A_i \neq \emptyset$ and let $\boldsymbol{x}$ be a point in the intersection. We have a vertex $z \in U_{\boldsymbol{x}}$ and a vertex $z' \in V_{\boldsymbol{x}}$ with $c(z) = c(z') = i$. By the definition of $B_0(G)$ the vertices $z$ and $z'$ are connected in $G$. This contradicts the choice of $c$ as a proper coloring. The contradiction shows that $A_i \cap -A_i = \emptyset$ for all colors $i$.

Applying version (ii) of Ky Fan's theorem we get that for some colors $i_1 < i_2 < \ldots < i_t$ and a point $\boldsymbol{x} \in S^{t-1}$ we have $(-1)^j \boldsymbol{x} \in A_{i_j}$ for $j = 1, 2, \ldots t$. This implies the existence of vertices $z_j \in U_{(-1)^j \boldsymbol{x}}$ with $c(z_j) = i_j$. Now $U_{(-1)^j \boldsymbol{x}} = U_{\boldsymbol{x}}$ for even $j$ and $U_{(-1)^j \boldsymbol{x}} = V_{\boldsymbol{x}}$ for odd $j$. Therefore the complete bipartite graph with sides $\{z_j | j \text{ is even}\}$ and $\{z_j | j \text{ is odd}\}$ is a subgraph of $G$ with the required properties. □

This result was previously established for Kneser graphs in [18].

*Remark 1.* Since for any fixed coloring we are allowed to order the colors in an arbitrary manner, the Zig-zag Theorem implies the existence of several totally multicolored copies of $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$. For a uniform random order any fixed totally multicolored $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ satisfies the zig-zag rule with probability $1/\binom{t}{\lfloor t/2\rfloor}$ if $t$ is odd and with probability $2/\binom{t}{t/2}$ if $t$ is even. Thus the Zig-zag Theorem implies the existence of many differently colored totally multicolored subgraphs $K_{\lceil \frac{t}{2}\rceil,\lfloor \frac{t}{2}\rfloor}$ in $G$: $\binom{t}{\lfloor t/2\rfloor}$ copies for odd $t$ and $\binom{t}{t/2}/2$ copies for even $t$.

If the coloring uses only $t$ colors we get a totally multicolored $K_{\lceil \frac{t}{2}\rceil,\lfloor \frac{t}{2}\rfloor}$ subgraph with all possible colorings, and the number of these different subgraphs is exactly the lower bound stated. $\Diamond$

**Proof of Theorems 1 and 2.**

Theorems 1 and 2 are direct consequences of the Zig-zag theorem. For Theorem 2 this is obvious. To prove Theorem 1 consider any vertex of the $\lfloor t/2\rfloor$ side of a multicolored complete bipartite graph. It has $\lceil t/2\rceil$ differently colored neighbors on the other side, thus at least $\lceil t/2\rceil$ different colors in its neighborhood. $\Box$

*Remark 2.* Theorem 1 gives tight lower bounds for the local chromatic number of topologically $t$-chromatic graphs for odd $t$ as several examples of the next section will show. In [47] we present examples that show that the situation is similar for even values of $t$. However, the graphs establishing this fact are *not* strongly topologically $t$-chromatic, whereas the graphs showing tightness of Theorem 1 for odd $t$ are. This leaves open the question whether $\psi(G) \geq t/2 + 2$ holds for all strongly topologically $t$-chromatic graphs $G$ and even $t \geq 4$. While we prove this statement in [47] for $t = 4$ we do not know the answer for higher values of $t$. $\Diamond$

# 4 Upper bound

In this section we present the combinatorial constructions that prove Theorems 3 and 5. In both cases general observations on wide colorings (to be defined below) prove useful. The upper bound in either of Theorems 3 or 5 implies the existence of certain open covers of spheres. These topological consequences and the local chromatic number of Borsuk graphs are discussed in the last subsection of this section.

## 4.1 Wide colorings

We start here with a general method to alter a $t$-coloring and get a $(t+1)$-coloring showing that $\psi \leq t/2 + 2$. It works if the original coloring was wide as defined below.

**Definition 4** *A vertex coloring of a graph is called* wide *if the end vertices of all walks of length* 5 *receive different colors.*

Note that any wide coloring is proper, furthermore any pair of vertices of distance 3 or 5 receive distinct colors. Moreover, if a graph has a wide coloring it does not contain a cycle of length 3 or 5. For graphs that do not have cycles of length 3, 5, 7, or 9 any coloring is wide that assigns different colors to vertices of distance 1, 3 or 5 apart.

Another equivalent definition (considered in [23]) is that a proper coloring is wide if the neighborhood of any color class is an independent set and so is the second neighborhood.

**Lemma 4.1** *If a graph $G$ has a wide coloring using $t$ colors, then $\psi(G) \le \lfloor t/2 \rfloor + 2$.*

**Proof.** Let $c_0$ be the wide $t$-coloring of $G$. We alter this coloring by switching the color of the neighbors of the troublesome vertices to a new color. We define a vertex $x$ to be *troublesome* if $|c_0(N(x))| > t/2$. Assume the color $\beta$ is not used in the coloring $c_0$. For $x \in V(G)$ we let

$$c(x) = \begin{cases} \beta & \text{if } x \text{ has a troublesome neighbor} \\ c_0(x) & \text{otherwise.} \end{cases}$$

The color class $\beta$ in $c$ is the union of the neighborhoods of troublesome vertices. To see that this is an independent set consider any two vertices $z$ and $z'$ of color $\beta$. Let $y$ be a troublesome neighbor of $z$ and let $y'$ be a troublesome neighbor of $z'$. Both $c_0(N(y))$ and $c_0(N(y'))$ contain more than half of the $t$ colors in $c_0$, therefore these sets are not disjoint. We have a neighbor $x$ of $y$ and a neighbor $x'$ of $y'$ satisfying $c_0(x) = c_0(x')$. This shows that $z$ and $z'$ are not connected, as otherwise the walk $xyzz'y'x'$ of length 5 would have two end vertices in the same color class.

All other color classes of $c$ are subsets of the corresponding color classes in $c_0$, and are therefore independent. Thus $c$ is a proper coloring.

Any troublesome vertex $x$ has now all its neighbors recolored, therefore $c(N(x)) = \{\beta\}$. For the vertices of $G$ that are not troublesome one has $|c_0(N(x))| \le t/2$ and $c(N(x)) \subseteq c_0(N(x)) \cup \{\beta\}$, therefore $|c(N(x))| \le t/2 + 1$. Thus the coloring $c$ shows $\psi(G) \le t/2 + 2$ as claimed. □

We note that the coloring $c$ found in the proof uses $t + 1$ colors and any vertex that sees the maximal number $\lfloor t/2 \rfloor + 1$ of the colors in its neighborhood must have a neighbor of color $\beta$. In particular, for odd $t$ one will always find two vertices of the same color in any $K_{(t+1)/2,(t+1)/2}$ subgraph.

## 4.2 Schrijver graphs

In this subsection we prove Theorem 3 which shows that the local chromatic number of Schrijver graphs with certain parameters are as low as allowed by Theorem 1. We also prove Proposition 4 to show that for some other Schrijver graphs the local chromatic number agrees with the chromatic number.

For the proof of Theorem 3 we will use the following simple lemma.

**Lemma 4.2** *Let $u, v \subseteq [n]$ be two vertices of $SG(n,k)$. If there is a walk of length $2s$ between $u$ and $v$ in $SG(n,k)$ then $|v \setminus u| \le s(t-2)$, where $t = n - 2k + 2 = \chi(SG(n,k))$.*

**Proof.** Let $xyz$ be a length two walk in $SG(n,k)$. Since $y$ is disjoint from $x$, it contains all but $n-2k=t-2$ elements of $[n]\setminus x$. As $z$ is disjoint from $y$ it can contain at most $t-2$ elements not contained in $x$. This proves the statement for $s=1$.

Now let $x_0x_1\dots x_{2s}$ be a $2s$-length walk between $u=x_0$ and $v=x_{2s}$ and assume the statement is true for $s-1$. Since $|v\setminus u|\le |v\setminus x_{2s-2}|+|x_{2s-2}\setminus u|\le (t-2)+(s-1)(t-2)$ we can complete the proof by induction. □

We remark that Lemma 4.2 remains true for $KG(n,k)$ with literally the same proof, but we will need it for $SG(n,k)$, this is why it is stated that way.

**Theorem 3** (restated) *If $t=n-2k+2>2$ is odd and $n\ge 4t^2-7t$, then*

$$\psi(SG(n,k))=\left\lceil\frac{t}{2}\right\rceil+1.$$

**Proof.** We need to show that $\psi(SG(n,k))=(t+3)/2$. Note that the $t=3$ case is trivial as all 3-chromatic graphs have local chromatic number 3. The lower bound for the local chromatic number follows from Theorem 1 and Proposition 8.

We define a wide coloring $c_0$ of $SG(n,k)$ using $t$ colors. From this Lemma 4.1 gives the upper bound on $\psi(SG(n,k))$.

Let $[n]=\{1,\dots,n\}$ be partitioned into $t$ sets, each containing an odd number of consecutive elements of $[n]$. More formally, $[n]$ is partitioned into disjoint sets $A_1,\dots,A_t$, where each $A_i$ contains consecutive elements and $|A_i|=2p_i-1$. We need $p_i\ge 2t-3$ for the proof, this is possible as long as $n\ge t(4t-7)$ as assumed.

Notice, that $\sum_{i=1}^{t}(p_i-1)=k-1$, and therefore any $k$-element subset $x$ of $[n]$ must contain more than half (i.e., at least $p_i$) of the elements in some $A_i$. We define our coloring $c_0$ by arbitrarily choosing such an index $i$ as the color $c_0(x)$. This is a proper coloring even for the graph $KG(n,k)$ since if two sets $x$ and $y$ both contain more than half of the elements of $A_i$, then they are not disjoint.

As a coloring of $KG(n,k)$ the coloring $c_0$ is not wide. We need to show that the coloring $c_0$ becomes wide if we restrict it to the subgraph $SG(n,k)$.

The main observation is the following: $A_i$ contains a single subset of cardinality $p_i$ that does not contain two consecutive elements. Let $C_i$ be this set consisting of the first, third, etc. elements of $A_i$. A vertex of $SG(n,k)$ has no two consecutive elements, thus a vertex $x$ of $SG(n,k)$ of color $i$ must contain $C_i$.

Consider a walk $x_0x_1\dots x_5$ of length 5 in $SG(n,k)$ and let $i=c_0(x_0)$. Thus the set $x_0$ contains $C_i$. By Lemma 4.2 $|x_4\setminus x_0|\le 2(t-2)$. In particular, $x_4$ contains all but at most $2t-4$ elements of $C_i$. As $p_i=|C_i|\ge 2t-3$, this means $x_4\cap C_i\ne\emptyset$. Thus the set $x_5$, which is disjoint from $x_4$, cannot contain all elements of $C_i$, showing $c_0(x_5)\ne i$. This proves that the coloring $c_0$ is wide, thus Lemma 4.1 completes the proof of the theorem. □

Note that the smallest Schrijver graph for which the above proof gives $\psi(SG(n,k))<\chi(SG(n,k))$ is $G=SG(65,31)$ with $\chi(G)=5$ and $\psi(G)=4$. In Remark 4 below we show how the lower bound on $n$ can be lowered somewhat. After that we show that some lower bound is needed as $\psi(SG(n,2)) = \chi(SG(n,2))$ for every $n$.

*Remark 3.* In [14] universal graphs $U(m,r)$ are defined for which it is shown that a graph $G$ can be colored with $m$ colors such that the neighborhood of every vertex contains fewer than $r$ colors if and only if a homomorphism from $G$ to $U(m,r)$ exists. The proof of Theorem 3 gives, for odd $t$, a $(t+1)$-coloring of $SG(n,k)$ (for appropriately large $n$ and $k$ that give chromatic number $t$) for which no neighborhood contains more than $(t+1)/2$ colors, thus establishing the existence of a homomorphism from $SG(n,k)$ to $U(t+1,(t+3)/2)$. This, in particular, proves that $\chi(U(t+1,(t+3)/2)) \geq t$, which is a special case of Theorem 2.6 in [14]. It is not hard to see that this inequality is actually an equality. Further, by the composition of the appropriate maps, the existence of this homomorphism also proves that $U(t+1,(t+3)/2)$ is strongly topologically $t$-chromatic. $\Diamond$

*Remark 4.* For the price of letting the proof be a bit more complicated one can improve upon the bound given on $n$ in Theorem 3. In particular, one can show that the same conclusion holds for odd $t$ and $n \geq 2t^2-4t+3$. More generally, we can show $\psi(SG(n,k)) \leq \chi(SG(n,k)) - m = n - 2k + 2 - m$ provided that $\chi(SG(n,k)) \geq 2m+3$ and $n \geq 8m^2+16m+9$ or $\chi(SG(n,k)) \geq 4m+3$ and $n \geq 20m+9$. The smallest Schrijver graph for which we can prove that the local chromatic number is smaller than the ordinary chromatic number is $SG(33,15)$ with 1496 vertices and $\chi = 5$ but $\psi = 4$. (In general, one has $|V(SG(n,k))| = \frac{n}{k}\binom{n-k-1}{k-1}$, cf. Lemma 1 in [49].) The smallest $n$ and $k$ for which we can prove $\psi(SG(n,k)) < \chi(SG(n,k))$ is for the graph $SG(29,12)$ for which $\chi = 7$ but $\psi \leq 6$.

We only sketch the proof. For a similar and more detailed proof see Theorem 13. The idea is again to take a basic coloring $c_0$ of $SG(n,k)$ and obtain a new coloring $c$ by recoloring to a new color some neighbors of those vertices $v$ for which $|c_0(N(v))|$ is too large. The novelty is that now we do not recolor all such neighbors, just enough of them, and also the definition of the basic coloring $c_0$ is a bit different. Partition $[n]$ into $t = n-2k+2$ intervals $A_1,\dots,A_t$, each of odd length as in the proof of Theorem 3 and also define $C_i$ similarly to be the unique largest subset of $A_i$ not containing consecutive elements. For a vertex $x$ we define $c_0(x)$ to be the *smallest* $i$ for which $C_i \subseteq x$. Note that such an $i$ must exist. Now we define when to recolor a vertex to the new color $\beta$ if our goal is to prove $\psi(SG(n,k)) \leq b := t-m$, where $m > 0$. We let $c(y) = \beta$ iff $y$ is the neighbor of a vertex $x$ having at least $b-2$ different colors *smaller* than $c_0(y)$ in its neighborhood. Otherwise, $c(y) = c_0(y)$. It is clear that $|c(N(x))| \leq b-1$ is satisfied, the only problem we face is that $c$ may not be a proper coloring. To avoid this problem we only need that the recolored vertices form an independent set. For each vertex $v$ define the index set $I(v) := \{j : v \cap C_j = \emptyset\}$. If $y$ and $y'$ are recolored vertices then they are neighbors of some $x$ and $x'$, respectively, where $I(x)$ contains $c_0(y)$ and at least $b-2$ indices smaller than $c_0(y)$ and $I(x')$ contains $c_0(y')$ and at least $b-2$ indices smaller than $c_0(y')$. Since $|[n] \setminus (x \cup y)| = t-2$, there are at most $t-2$ elements in $\cup_{j\in I(x)} C_j$ not contained in $y$.

The definition of $c_0$ also implies that at least one element of $C_j$ is missing from $y$ for every $j < c_0(y)$. Similarly, there are at most $t-2$ elements in $\cup_{j\in I(x')}C_j$ not contained in $y'$ and at least one element of $C_j$ is missing from $y'$ for every $j < c_0(y')$. These conditions lead to $y \cap y' \neq \emptyset$ if the sizes $|A_i| = 2|C_i|-1$ are appropriately chosen. In particular, if $t \geq 2m+3$ and $|A_t| \geq 1$, $|A_{t-1}| \geq 2m+3$, $|A_{t-2}| \geq \ldots \geq |A_{t-(2m+2)}| \geq 4m+5$, or $t \geq 4m+3$ and $|A_t| \geq 1$, $|A_{t-1}| \geq 3$, $|A_{t-2}| \geq \ldots \geq |A_{t-(4m+2)}| \geq 5$, then the above argument leads to a proof of $\psi(SG(n,k)) \leq t-m$. (It takes some further but simple argument why the last two intervals $A_i$ can be chosen smaller than the previous ones.) These two possible choices of the interval sizes give the two general bounds on $n$ we claimed sufficient for attaining $\psi(SG(n,k)) \leq t-m$. The strengthening of Theorem 3 is obtained by the $m = (t-3)/2$ special case of the first bound. $\Diamond$

**Proposition 4** (restated) $\psi(SG(n,2)) = n-2 = \chi(SG(n,2))$ *for every* $n \geq 4$.

**Proof.** In the $n=4$ case $SG(n,2)$ consists of a single edge and the statement of the proposition is trivial. Assume for a contradiction that $\psi(SG(n,2)) \leq n-3$ for some $n \geq 5$ and let $c$ be a proper coloring of $SG(n,2)$ showing this with the minimal number of colors. As $\chi(SG(n,2)) = n-2$ and a coloring of a graph $G$ with exactly $\chi(G)$ colors cannot show $\psi(G) < \chi(G)$ the coloring $c$ uses at least $n-1$ colors.

It is worth visualizing the vertices of $SG(n,2)$ as diagonals of an $n$-gon (cf. [8]). In other words, $SG(n,2)$ is the complement of the line graph of $D_n$, where $D_n$ is the complement of the cycle $C_n$. The color classes are independent sets in $SG(n,2)$, so they are either stars or triangles in $D_n$.

We say that a vertex $x$ *sees* the color classes of its neighbors. By our assumption every vertex sees at most $n-4$ color classes.

Assume a color class consists of a single vertex $x$. As $x$ sees at most $n-4$ of the at least $n-1$ color classes we can choose a different color for $x$. The resulting coloring attains the same local chromatic number with fewer colors. This contradicts the choice of $c$ and shows that no color class is a singleton.

A triangle color class is seen by all other edges of $D_n$. A star color class with center $i$ and at least three elements is seen by all vertices that, as edges of $D_n$, are not incident to $i$. For star color classes of two edges there can be one additional vertex not seeing the class. So every color class is seen by all but at most $n-2$ vertices. We double count the pairs of a vertex $x$ and a color class $C$ seen by $x$. On one hand every vertex sees at most $n-4$ classes. On the other hand all the color classes are seen by at least $\left(\binom{n}{2}-n\right)-(n-2)$ vertices. We have

$$(n-1)\left(\binom{n}{2}-2n+2\right) \leq \left(\binom{n}{2}-n\right)(n-4),$$

and this contradicts our $n \geq 5$ assumption. The contradiction proves the statement. $\square$

## 4.3 Generalized Mycielski graphs

Another class of graphs for which the chromatic number is known only via the topological method is formed by generalized Mycielski graphs, see [23, 39, 48]. They are interesting for us also for another reason: there is a big gap between their fractional and ordinary chromatic numbers (see [34, 50]), therefore the local chromatic number can take its value from a large interval.

Recall that the Mycielskian $M(G)$ of a graph $G$ is the graph defined on $(\{0,1\}\times V(G))\cup\{z\}$ with edge set $E(M(G)) = \{\{(0,v),(i,w)\} : \{v,w\} \in E(G), i \in \{0,1\}\} \cup \{\{(1,v),z\} : v \in V(G)\}$. Mycielski [43] used this construction to increase the chromatic number of a graph while keeping the clique number fixed: $\chi(M(G)) = \chi(G)+1$ and $\omega(M(G)) = \omega(G)$.

Following Tardif [50], the same construction can also be described as the direct (also called categorical) product of $G$ with a path on three vertices having a loop at one end and then identifying all vertices that have the other end of the path as their first coordinate. Recall that the direct product of $F$ and $G$ is a graph on $V(F)\times V(G)$ with an edge between $(u,v)$ and $(u',v')$ if and only if $\{u,u'\} \in E(F)$ and $\{v,v'\} \in E(G)$. The generalized Mycielskian of $G$ (called a cone over $G$ by Tardif [50]) $M_r(G)$ is then defined by taking the direct product of $P$ and $G$, where $P$ is a path on $r+1$ vertices having a loop at one end, and then identifying all the vertices in the product with the loopless end of the path as their first coordinate. With this notation $M(G) = M_2(G)$. These graphs were considered by Stiebitz [48], who proved that if $G$ is $k$-chromatic "for a topological reason" then $M_r(G)$ is $(k+1)$-chromatic for a similar reason. (Gyárfás, Jensen, and Stiebitz [23] also consider these graphs and quote Stiebitz's argument a special case of which is also presented in [39].) The topological reason of Stiebitz is in different terms than those we use in this paper but using results of [3] they imply strong topological $(t+d)$-chromaticity for graphs obtained by $d$ iterations of the generalized Mychielski construction starting, e.g, from $K_t$ or from a $t$-chromatic Schrijver graph. More precisely, Stiebitz proved that the body of the so-called neighborhood complex $\mathcal{N}(M_r(G))$ of $M_r(G)$, introduced in [36] by Lovász, is homotopy equivalent to the suspension of $||\mathcal{N}(G)||$. Since $\mathrm{susp}(S^n) \cong S^{n+1}$ this implies that whenever $||\mathcal{N}(G)||$ is homotopy equivalent to an $n$-dimensional sphere, then $||\mathcal{N}(M_r(G))||$ is homotopy equivalent to the $(n+1)$-dimensional sphere. This happens, for example, if $G$ is a complete graph, or an odd cycle. By a recent result of Björner and de Longueville [8] we also have a similar situation if $G$ is isomorphic to any Schrijver graph $SG(n,k)$. Notice that the latter include complete graphs and odd cycles.

It is known, that $||\mathcal{N}(F)||$ is homotopy equivalent to $H(F)$ for every graph $F$, see Proposition 4.2 in [3]. All this implies that $\mathrm{coind}(H(M_r(G))) = \mathrm{coind}(H(G))+1$ whenever $H(G)$ is homotopy equivalent to a sphere, in particular, whenever $G$ is a complete graph or an odd cycle, or, more generally, a Schrijver graph. In the first version of this paper we wrote that it is very likely that Stiebitz's proof can be generalized to show that $H(M_r(G)) \leftrightarrow \mathrm{susp}(H(G))$ and therefore $\mathrm{coind}(H(M_r(G))) \geq \mathrm{coind}(H(G)) + 1$ holds always. Since then Csorba [12] succeeded to prove this generalization. In fact, he proved $\mathbb{Z}_2$-homotopy equivalence of $H(M_r(G))$ and $\mathrm{susp}(H(G))$. Nevertheless, here we restrict attention to graphs $G$ with $H(G)$ homotopy equivalent to a sphere.

For an integer vector $\boldsymbol{r} = (r_1, \ldots, r_d)$ with $r_i \geq 1$ for all $i$ we let $M_{\boldsymbol{r}}^{(d)}(G) = M_{r_d}(M_{r_{d-1}}(\ldots M_{r_1}(G)\ldots))$ denote the graph obtained by a $d$-fold application of the generalized Mycielski construction with respective parameters $r_1, \ldots, r_d$.

**Proposition 9** (Stiebitz) *If $G$ is a graph for which $H(G)$ is homotopy equivalent to a sphere $S^h$ with $h = \chi(G) - 2$ (in particular, $G$ is a complete graph or an odd cycle, or, more generally, a Schrijver graph) and $\boldsymbol{r} = (r_1, \ldots, r_d)$ is arbitrary, then $M_{\boldsymbol{r}}^{(d)}(G)$ is strongly topologically $t$-chromatic for $t = \chi(M_{\boldsymbol{r}}^{(d)}(G)) = \chi(G) + d$.* □

It is interesting to remark that $\chi(M_r(G)) > \chi(G)$ does not hold in general if $r \geq 3$, e.g., for $\overline{C}_7$, the complement of the 7-cycle, one has $\chi(M_3(\overline{C}_7)) = \chi(\overline{C}_7) = 4$ (cf. [50]). Still, the result of Stiebitz implies that the sequence $\{\chi(M_{\boldsymbol{r}}^{(d)}(G))\}_{d=1}^{\infty}$ may avoid to increase only a finite number of times.

The fractional chromatic number of Mycielski graphs were determined by Larsen, Propp, and Ullman [34], who proved that $\chi_f(M(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$ holds for every $G$. This already shows that there is a large gap between the chromatic and the fractional chromatic numbers of $M_{\boldsymbol{r}}^{(d)}(G)$ if $d$ is large enough and $r_i \geq 2$ for all $i$, since obviously, $\chi_f(M_r(F)) \leq \chi_f(M(F))$ holds if $r \geq 2$. The previous result was generalized by Tardif [50] who showed that $\chi_f(M_r(G))$ can also be expressed by $\chi_f(G)$ as $\chi_f(G) + \frac{1}{\sum_{i=0}^{r-1}(\chi_f(G)-1)^i}$ whenever $G$ has at least one edge.

First we show that for the original Mycielski construction the local chromatic number behaves similarly to the chromatic number.

**Proposition 10** *For any graph $G$ we have*

$$\psi(M(G)) = \psi(G) + 1.$$

**Proof.** We proceed similarly as one does in the proof of $\chi(M(G)) = \chi(G) + 1$. Recall that $V(M(G)) = \{0,1\} \times V(G) \cup \{z\}$.

For the upper bound consider a coloring $c'$ of $G$ establishing its local chromatic number and let $\alpha$ and $\beta$ be two colors not used by $c'$. We define $c((0,x)) = c'(x)$, $c((1,x)) = \alpha$ and $c(z) = \beta$. This proper coloring shows $\psi(M(G)) \leq \psi(G) + 1$.

For the lower bound consider an arbitrary proper coloring $c$ of $M(G)$. We have to show that some vertex must see at least $\psi(G)$ different colors in its neighborhood.

We define the coloring $c'$ of $G$ as follows:

$$c'(x) = \begin{cases} c((0,x)) & \text{if } c((0,x)) \neq c(z) \\ c((1,x)) & \text{otherwise.} \end{cases}$$

It follows from the construction that $c'$ is a proper coloring of $G$. Note that $c'$ does not use the color $c(z)$.

By the definition of $\psi(G)$, there is some vertex $x$ of $G$ that has at least $\psi(G)-1$ different colors in its neighborhood $N_G(x)$. If $c'(y)=c(0,y)$ for all vertices $y\in N_G(x)$, then the vertex $(1,x)$ has all these colors in its neighborhood, and also the additional color $c(z)$. If however $c'(y)\neq c(0,y)$ for a neighbor $y$ of $x$, then the vertex $(0,x)$ sees all the colors $c'(N_G(x))$ in its neighborhood $N_{M(G)}(0,x)$, and also the additional color $c(0,y)=c(z)$. In both cases a vertex has $\psi(G)$ different colors in its neighborhood as claimed. □

We remark that $M_1(G)$ is simply the graph $G$ with a new vertex connected to every vertex of $G$, therefore the following trivially holds.

**Proposition 11** *For any graph $G$ we have*

$$\psi(M_1(G))=\chi(G)+1.$$

□

For our first upper bound we apply Lemma 4.1. We use the following result of Gyárfás, Jensen, and Stiebitz [23]. The lemma below is an immediate generalization of the $l=2$ special case of Theorem 4.1 in [23]. We reproduce the simple proof from [23] for the sake of completeness.

**Lemma 4.3** ([23]) *If $G$ has a wide coloring with $t$ colors and $r\geq 7$, then $M_r(G)$ has a wide coloring with $t+1$ colors.*

**Proof.** As there is a homomorphism from $M_r(G)$ to $M_7(G)$ if $r>7$ it is enough to give the coloring for $r=7$. We fix a wide $t$-coloring $c_0$ of $G$ and use the additional color $\gamma$. The coloring of $M_7(G)$ is given as

$$c((v,x))=\begin{cases}\gamma & \text{if } v \text{ is the vertex at distance 3, 5 or 7 from the loop}\\ c_0(x) & \text{otherwise.}\end{cases}$$

It is straightforward to check that $c$ is a wide coloring. □

We can apply the results of Stiebitz and Gyárfás et al. recursively to give tight or almost tight bounds for the local chromatic number of the graphs $M_{\boldsymbol{r}}^{(d)}(G)$ in many cases:

**Corollary 12** *If $G$ has a wide $t$-coloring and $\boldsymbol{r}=(r_1,\dots,r_d)$ with $r_i\geq 7$ for all $i$, then $\psi(M_{\boldsymbol{r}}^{(d)}(G))\leq\frac{t+d}{2}+2$.*

*If $H(G)$ is homotopy equivalent to a sphere $S^h$, then $\psi(M_{\boldsymbol{r}}^{(d)}(G))\geq\frac{h+d}{2}+2$.*

**Proof.** For the first statement we apply Lemma 4.3 recursively to show that $M_{\boldsymbol{r}}^{(d)}(G)$ has a wide $(t+d)$-coloring and then apply Lemma 4.1.

For the second statement we apply the result of Stiebitz recursively to show that $H(M_{\boldsymbol{r}}^{(d)}(G))$ is homotopy equivalent to $S^{h+d}$. As noted in the preliminaries of the present subsection this implies $\mathrm{coind}(H(M_{\boldsymbol{r}}^{(d)}(G))) \geq h+d$. By Theorem 1 the statement follows. □

**Theorem 5** (restated) *If $\boldsymbol{r} = (r_1, \ldots, r_d)$, $d$ is odd, and $r_i \geq 7$ for all $i$, then*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) = \left\lceil \frac{d}{2} \right\rceil + 2.$$

**Proof.** Notice that for $\boldsymbol{r} = (r_1, \ldots, r_d)$ with $d$ odd and $r_i \geq 7$ for all $i$ the lower and upper bounds of Corollary 12 give the exact value for the local chromatic number $\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) = (d+5)/2$. This proves the theorem. □

Notice that a similar argument gives the exact value of $\psi(G)$ for the more complicated graph $G = M_{\boldsymbol{r}}^{(d)}(SG(n,k))$ whenever $n+d$ is odd, $r_i \geq 7$ for all $i$, and $n \geq 4t^2 - 7t$ for $t = n - 2k + 2$. This follows from Corollary 12 via the wide colorability of $SG(n,k)$ for $n \geq 4t^2 - 7t$ shown in the proof of Theorem 3 and Björner and de Longueville's result [8] about the homotopy equivalence of $H(SG(n,k))$ to $S^{n-2k}$. (Instead of the latter we can also use Csorba's result [12] mentioned above and refer to the strong topological $t$-chromaticity of $SG(n,k)$.)

We summarize our knowledge on $\psi(M_{\boldsymbol{r}}^{(d)}(K_2))$ after proving the following theorem, which shows that almost the same upper bound as in Corollary 12 is implied from the relaxed condition $r_i \geq 4$.

**Theorem 13** *For $\boldsymbol{r} = (r_1, \ldots, r_d)$ with $r_i \geq 4$ for all $i$ one has*

$$\psi(M_{\boldsymbol{r}}^{(d)}(G)) \leq \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor + 2.$$

*Moreover, for $G \cong K_2$, the following slightly sharper bound holds:*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) \leq \left\lceil \frac{d}{2} \right\rceil + 3.$$

**Proof.** We denote the vertices of $Y := M_{\boldsymbol{r}}^{(d)}(G)$ in accordance to the description of the generalized Mycielski construction via graph products. That is, a vertex of $Y$ is a sequence $a_1 a_2 \ldots a_d u$ of length $(d+1)$, where $\forall i: \; a_i \in \{0, 1, \ldots, r_i\} \cup \{*\}$, $u \in V(G) \cup \{*\}$ and if $a_i = r_i$ for some $i$ then necessarily $u = *$ and $a_j = *$ for every $j > i$, and this is the only way $*$ can appear in a sequence. To define adjacency we denote by $\hat{P}_{r_i+1}$ the path on $\{0, 1, \ldots, r_i\}$ where the edges are of the form $\{i-1, i\}, i \in \{1, \ldots, r_i\}$ and there is a loop at vertex 0. Two vertices $a_1 a_2 \ldots a_d u$ and $a'_1 a'_2 \ldots a'_d u'$ are adjacent in $Y$ if and only if

$$u = * \text{ or } u' = * \text{ or } \{u, u'\} \in E(G) \text{ and}$$

$$\forall i: \quad a_i = * \text{ or } a'_i = * \text{ or } \{a_i, a'_i\} \in E(\hat{P}_{r_i+1}).$$

Our strategy is similar to that used in Remark 4. Namely, we give an original coloring $c_0$, identify the set of "troublesome" vertices for this coloring, and recolor most of the neighbors of these vertices to a new color.

Let us fix a coloring $c_G$ of $G$ with at most $\psi(G) - 1$ colors in the neighborhood of a vertex. Let the colors we use in this coloring be called $0, -1, -2$, etc. Now we define $c_0$ as follows.

$$c_0(a_1 \dots a_d u) = \begin{cases} c_G(u) & \text{if } \forall i : a_i \leq 2 \\ i & \text{if } a_i \geq 3 \text{ is odd and } a_j \leq 2 \text{ for all } j < i \\ 0 & \text{if } \exists i : a_i \geq 4 \text{ is even and } a_j \leq 2 \text{ for all } j < i \end{cases}$$

It is clear that vertices having the same color form independent sets, i.e., $c_0$ is a proper coloring. Notice that if a vertex has neighbors of many different "positive" colors, then it must have many coordinates that are equal to 2. Now we recolor most of the neighbors of these vertices.

Let $\beta$ be a color not used by $c_0$ and set $c(a_1 \dots a_d u) = \beta$ if $|\{i : a_i \text{ is odd}\}| > d/2$. (In fact, it would be enough to give color $\beta$ only to those of the above vertices, for which the first $\lfloor \frac{d}{2} \rfloor$ odd coordinates are equal to 1. We recolor more vertices for the sake of simplicity.) Otherwise, let $c(a_1 \dots a_d u) = c_0(a_1 \dots a_d u)$.

First, we have to show that $c$ is proper. To this end we only have to show that no pair of vertices getting color $\beta$ can be adjacent. If two vertices, $\boldsymbol{x} = x_1 \dots x_d v_x$ and $\boldsymbol{y} = y_1 \dots y_d v_y$ are colored $\beta$ then both have more than $d/2$ odd coordinates (among their first $d$ coordinates). Thus there is some common coordinate $i$ for which $x_i$ and $y_i$ are both odd. This implies that $\boldsymbol{x}$ and $\boldsymbol{y}$ are not adjacent.

Now we show that for any vertex $\boldsymbol{a}$ we have $|c(N(\boldsymbol{a})) \cap \{1, \dots, d\}| \leq d/2$. Indeed, if $|c_0(N(\boldsymbol{a})) \cap \{1, \dots, d\}| > d/2$ then we have $\boldsymbol{a} = a_1 \dots a_d u$ with more than $d/2$ coordinates $a_i$ that are even and positive. Furthermore, the first $\lfloor d/2 \rfloor$ of these coordinates should be 2. Let $I$ be the set of indices of these first $\lfloor d/2 \rfloor$ even and positive coordinates. We claim that $c(N(\boldsymbol{a})) \cap \{1, \dots, d\} \subseteq I$. This is so, since if a neighbor has an odd coordinate somewhere outside $I$, then it cannot have $*$ at the positions of $I$, therefore it has more than $d/2$ odd coordinates and it is recolored by $c$ to the color $\beta$.

It is also clear that no vertex can see more than $\psi(G) - 1$ "negative" colors in its neighborhood in either coloring $c_0$ or $c$. Thus the neighborhood of any vertex can contain at most $\lfloor d/2 \rfloor + (\psi(G) - 1) + 2$ colors, where the last 2 is added because of the possible appearance of colors $\beta$ and 0 in the neighborhood. This proves $\psi(Y) \leq d/2 + \psi(G) + 2$ proving the first statement in the theorem.

For $G \cong K_2$ the above gives $\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) \leq \lfloor d/2 \rfloor + 4$ which implies the second statement for odd $d$. For even $d$ the bound of the second statement is 1 less. We can gain 1 as follows. When defining $c$ let us recolor to $\beta$ those vertices $\boldsymbol{a} = a_1 \dots a_d u$, too, for which the number of odd coordinates $a_i$ is exactly $\frac{d}{2}$ and $c_G(u) = -1$. The proof proceeds similarly as before but we gain 1 by observing that those vertices who see $-1$ can see only $\frac{d}{2}-1$ "positive" colors. $\square$

We collect the implications of Theorems 5, 13 and Propositions 10 and 11. It would be interesting to estimate the value $\psi(M_{\boldsymbol{r}}^{(d)}(K_2))$ for the missing case $\boldsymbol{r}=(3,\dots,3)$. What we know then is $\lceil d/2\rceil+2\le\psi\le d+2$.

**Corollary 14** *For $\boldsymbol{r}=(r_1,\dots,r_d)$ we have*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2))=\begin{cases}(d+5)/2 & \text{if } d \text{ is odd and } \forall i: r_i\ge 7\\ \lceil d/2\rceil+2 \text{ or } \lceil d/2\rceil+3 & \text{if } \forall i: r_i\ge 4\\ d+2 & \text{if } r_d=1 \text{ or } \forall i: r_i=2.\end{cases}$$

$\square$

*Remark 5.* The improvement for even $d$ given in the last paragraph of the proof of Theorem 13 can also be obtained in a different way we explain here. Instead of changing the rule for recoloring, we can enforce that a vertex can see only $\psi(G)-2$ negative colors. This can be achieved by setting the starting graph $G$ to be $M_4(K_2)\cong C_9$ instead of $K_2$ itself and coloring this $C_9$ with the pattern $-1,0,-1,-2,0,-2,-3,0,-3$ along the cycle. One can readily check that every vertex can see only one non-0 color in its neighborhood.

The same trick can be used also if the starting graph is not $K_2$ or $C_9$, but some large enough Schrijver graph of odd chromatic number. Coloring it as in the proof of Lemma 4.1 (using the wide coloring as given in the proof of Theorem 3), we arrive to the same phenomenon if we let the new color (of the proof of Lemma 4.1) be 0. $\Diamond$

*Remark 6.* Gyárfás, Jensen, and Stiebitz [23] use generalized Mycielski graphs to show that another graph they denote by $G_k$ is $k$-chromatic. The way they prove it is that they exhibit a homomorphism from $M_{\boldsymbol{r}}^{(k-2)}(K_2)$ to $G_k$ for $\boldsymbol{r}=(4,\dots,4)$. The existence of this homomorphism implies that $G_k$ is strongly topologically $k$-chromatic, thus its local chromatic number is at least $k/2+1$. We do not know any non-trivial upper bound for $\psi(G_k)$. Also note that [23] gives universal graphs for the property of having a wide $t$-coloring. By Lemma 4.1 this graph has $\psi\le t/2+2$. On the other hand, since any graph with a wide $t$-coloring admits a homomorphism to this graph, and we have seen the wide $t$-colorability of some strongly topologically $t$-chromatic graphs, it is strongly topologically $t$-chromatic, as well. This gives $\psi\ge t/2+1$. $\Diamond$

## 4.4 Borsuk graphs and the tightness of Ky Fan's theorem

The following definition goes back to Erdős and Hajnal [15], see also [37].

**Definition 5** *The Borsuk graph $B(n,\alpha)$ of parameters $n$ and $0<\alpha<2$ is the infinite graph whose vertices are the points of the unit sphere in $\mathbb{R}^n$ (i.e., $S^{n-1}$) and its edges connect the pairs of points with distance at least $\alpha$.*

The Borsuk-Ulam theorem implies that $\chi(B(n,\alpha)) \geq n+1$, and, as Lovász [37] remarks, these two statements are in fact equivalent. For $\alpha$ large enough (depending on $n$) this lower bound on the chromatic number is sharp as shown by the standard $(n+1)$-coloring of the sphere $S^{n-1}$ (see [37, 39] or cf. the proof of Corollary 15 below).

The local chromatic number of Borsuk graphs for large enough $\alpha$ can also be determined by our methods. First we want to argue that Theorem 1 is applicable for this infinite graph. Lovász gives in [37] for any $n$ and $\alpha$ a finite graph $G_P = G_P(n,\alpha) \subseteq B(n,\alpha)$ which has the property that its neighborhood complex $\mathcal{N}(G_P)$ is homotopy equivalent to $S^{n-1}$. Now we can continue the argument the same way as in the previous subsection: Proposition 4.2 in [3] states that $\mathcal{N}(F)$ is homotopy equivalent to $H(F)$ for every graph $F$, thus $\mathrm{coind}(H(G_P)) \geq n-1$, i.e., $G_P$ is strongly topologically $(n+1)$-chromatic. As $G_P \subseteq B(n,\alpha)$ we have $\lceil \frac{n+3}{2} \rceil \leq \psi(G_P) \leq \psi(B(n,\alpha))$ by Theorem 1.

The following lemma shows the special role of Borsuk graphs among strongly topologically t-chromatic graphs. It will also show that our earlier upper bounds on the local chromatic number have direct implications for Borsuk graphs.

**Lemma 4.4** *A finite graph $G$ is strongly topologically $(n+1)$-chromatic if and only if for some $\alpha < 2$ there is a graph homomorphism from $B(n,\alpha)$ to $G$.*

**Proof.** For the if part consider the finite graph $G_P \subseteq B(n,\alpha)$ given by Lovász [37] satisfying $\mathrm{coind}(H(G_P)) \geq n-1$. If there is a homomorphism from $B(n,\alpha)$ to $G$, it clearly gives a homomorphism also from $G_P$ to $G$ which further generates a $\mathbb{Z}_2$-map from $H(G_P)$ to $H(G)$. This proves $\mathrm{coind}(H(G)) \geq n-1$.

For the only if part, let $f : S^{n-1} \to H(G)$ be a $\mathbb{Z}_2$-map. For a point $\boldsymbol{x} \in S^{n-1}$ write $f(\boldsymbol{x}) \in H(G)$ as the convex combination $f(\boldsymbol{x}) = \sum \alpha_v(\boldsymbol{x})||{+v}|| + \sum \beta_v(\boldsymbol{x})||{-v}||$ of the vertices of $||B_0(G)||$. Here the summations are for the vertices $v$ of $G$, $\sum \alpha_v(\boldsymbol{x}) = \sum \beta_v(\boldsymbol{x}) = 1/2$, and $\{v : \alpha_v(\boldsymbol{x}) > 0\} \uplus \{v : \beta_v(\boldsymbol{x}) > 0\} \in B_0(G)$. Note that $\alpha_v$ and $\beta_v$ are continuous as $f$ is continuous and $\beta_v(\boldsymbol{x}) = \alpha_v(-\boldsymbol{x})$ by the equivariance of $f$. Set $\varepsilon = 1/(2|V(G)|)$. For $\boldsymbol{x} \in S^{n-1}$ select an arbitrary vertex $v = g(\boldsymbol{x})$ of $G$ with $\alpha_v \geq \varepsilon$. We claim that $g$ is a graph homomorphism from $B(n,\alpha)$ to $G$ if $\alpha$ is close enough to 2. By compactness it is enough to prove that if we have vertices $v$ and $w$ of $G$ and sequences $\boldsymbol{x}_i \to \boldsymbol{x}$ and $\boldsymbol{y}_i \to -\boldsymbol{x}$ of points in $S^{n-1}$ with $g(\boldsymbol{x}_i) = v$ and $g(\boldsymbol{y}_i) = w$ for all $i$, then $v$ and $w$ are connected in $G$. But since $\alpha_v$ is continuous we have $\alpha_v(\boldsymbol{x}) \geq \varepsilon$ and similarly $\beta_w(\boldsymbol{x}) = \alpha_w(-\boldsymbol{x}) \geq \varepsilon$ and so $+v$ and $-w$ are contained in the smallest simplex of $B_0(G)$ containing $f(\boldsymbol{x})$ proving that $v$ and $w$ are connected. □

By Lemma 4.4 either of Theorems 3 or 5 implies that the above given lower bound on $\psi(B(n,\alpha))$ is tight whenever $\chi(B(n,\alpha))$ is odd, that is, $n$ is even, and $\alpha < 2$ is close enough to 2. In the following corollary we give an explicit bound on $\alpha$ by proving for that value of $\alpha$ that the standard coloring is wide.

**Corollary 15** *If $n$ is even and $2 - \frac{1}{25n+50} \leq \alpha < 2$, then*

$$\psi(B(n,\alpha)) = \frac{n}{2} + 2.$$

**Proof.** The lower bound on $\psi(B(n,\alpha))$ follows from the discussion preceding Lemma 4.4. The upper bound follows from Lemma 4.1 as long as we can give a wide $(n+1)$-coloring of the graph $B(n,\alpha)$.

To this end we use the standard $(n+1)$-coloring of $B(n,\alpha)$ (see, e.g., [37, 39]). Consider a regular simplex $R$ inscribed into the unit sphere $S^{n-1}$ and color a point $\boldsymbol{x} \in S^{n-1}$ by the facet of $R$ intersected by the segment from the origin to $\boldsymbol{x}$. If this segment meets a lower dimensional face then we arbitrarily choose a facet containing this face. To see for what $\alpha$ gives this a proper coloring we have to find the maximal distance $\alpha_0$ between pairs of points that we can color the same. Calculation shows that projections from the origin of the middle points of two disjoint $(n/2-1)$-dimensional faces of $R$ are farthest apart, thus $\alpha_0 = 2\sqrt{1-1/(n+2)}$. (Notice that [37] gives a different treshold value for $\alpha$. We were informed by László Lovász [38], however, that it was noticed by several researchers that the correct value is larger than the one given in [37].)

We let $\varphi = 2\arccos(\alpha/2)$. Clearly, $\boldsymbol{x}$ and $\boldsymbol{y}$ is connected if and only if the length of the shortest arc on $S^{n-1}$ connecting $-\boldsymbol{x}$ and $\boldsymbol{y}$ is at most $\varphi$. Therefore $\boldsymbol{x}$ and $\boldsymbol{y}$ are connected by a walk of length 5 if and only if the length of this same minimal arc is at most $5\varphi$. For the standard coloring the length of the shortest arc between $-\boldsymbol{x}$ and $\boldsymbol{y}$ for two vertices $\boldsymbol{x}$ and $\boldsymbol{y}$ colored with the same color is at least $2\arccos(\alpha_0/2) = 2\arcsin(n+2)^{-1/2}$. Therefore the standard coloring is wide as long as $\alpha > 2\cos\left(\frac{\arcsin(n+2)^{-1/2}}{5}\right)$. Here easy calculation gives that the right hand side is less than $2 - \frac{1}{25n+50}$. □

Our investigations of the local chromatic number led us to consider the following function $Q(h)$. The question of its values was independently asked by Micha Perles motivated by a related question of Matatyahu Rubin[1].

**Definition 6** *For a nonnegative integer parameter $h$ let $Q(h)$ denote the minimum $l$ for which $S^h$ can be covered by open sets in such a way that no point of the sphere is contained in more than $l$ of these sets and none of the covering sets contains an antipodal pair of points.*

Ky Fan's theorem implies $Q(h) \geq \frac{h}{2}+1$. Either of Theorems 3 or 5 implies the upper bound $Q(h) \leq \frac{h}{2}+2$. Using the concepts of Corollary 15 and Lemma 4.1 one can give an explicit covering of the sphere $S^{2l-3}$ by open subsets where no point is contained in more than $l$ of the sets and no set contains an antipodal pair of points. In fact, the covering we give satisfies a stronger requirement and proves that version (ii) of Ky Fan's theorem is tight, while version (i) is almost tight.

**Corollary 16** *There is a configuration $\mathcal{A}$ of $k+2$ open (closed) sets such that $\cup_{A\in\mathcal{A}}(A \cup -A) = S^k$, all sets $A \in \mathcal{A}$ satisfy $A \cap -A = \emptyset$, and no $\boldsymbol{x} \in S^k$ is contained in more than $\lceil \frac{k+1}{2} \rceil$ of these sets.*

*Furthermore, for every $\boldsymbol{x}$ the number of sets in $\mathcal{A}$ containing either $\boldsymbol{x}$ or $-\boldsymbol{x}$ is at most $k+1$.*

[1] We thank Imre Bárány [6] and Gil Kalai [29] for this information.

**Proof.** First we construct closed sets. Consider the unit sphere $S^k$ in $\mathbb{R}^{k+1}$. Let $R$ be a regular simplex inscribed in the sphere. Let $B_1, \ldots, B_{k+2}$ be the subsets of the sphere obtained by the central projection of the facets of $R$. These closed sets cover $S^k$. Let $C_0$ be the set of points covered by at least $\lceil\frac{k+3}{2}\rceil$ of the sets $B_i$. Notice that $C_0$ is the union of the central projections of the $\lfloor\frac{k-1}{2}\rfloor$-dimensional faces of $R$. For odd $k$ let $C = C_0$, while for even $k$ let $C = C_0 \cup C_1$, where $C_1$ is the set of points in $B_1$ covered by exactly $k/2+1$ of the sets $B_i$. Thus $C_1$ is the union of the central projections of the $\frac{k}{2}$-dimensional faces of a facet of $R$. Observe that $C \cap -C = \emptyset$. Take $0 < \delta < \mathrm{dist}(C, -C)/2$ and let $D$ be the open $\delta$-neighborhood of $C$ in $S^k$. For $1 \le i \le k+2$ let $A_i = B_i \setminus D$. These closed sets cover $S^k \setminus D$ and none of them contains a pair of antipodal points. As $D \cap -D = \emptyset$ we have $\cup_{i=1}^{k+2}(A_i \cup -A_i) = S^k$. It is clear that every point of the sphere is covered by at most $\lceil\frac{k+1}{2}\rceil$ of the sets $A_i$ proving the first statement of the corollary.

For the second statement note that if each set $B_i$ contains at least one of a pair of antipodal points, then one of these points belongs to $C$ and is therefore not covered by any of the sets $A_i$. Note also, that for odd $k$ the second statement follows also from the first.

To construct open sets as required we can simply take the open $\varepsilon$-neighborhoods of $A_i$. For small enough $\varepsilon > 0$ they maintain the properties required in the corollary. □

**Corollary 17** *There is a configuration of $k+3$ open (closed) sets covering $S^k$ none of which contains a pair of antipodal points, such that no $\boldsymbol{x} \in S^k$ is contained in more than $\lceil\frac{k+3}{2}\rceil$ of these sets and for every $\boldsymbol{x} \in S^k$ the number of sets that contain one of $\boldsymbol{x}$ and $-\boldsymbol{x}$ is at most $k+2$.*

**Proof.** For closed sets consider the sets $A_i$ in the proof of Corollary 16 together with the closure of $D$. For open sets consider the open $\varepsilon$-neighborhoods of these sets for suitably small $\varepsilon > 0$. □

Note that covering with $k+3$ sets is optimal in Corollary 17 if $k \ge 3$. By the Borsuk-Ulam Theorem (form (i)) fewer than $k+2$ open (or closed) sets not containing antipodal pairs of points is not enough to cover $S^k$. If we cover with $k+2$ sets (open or closed), then it gives rise to a proper coloring of $B(k+1, \alpha)$ for large enough $\alpha$ in a natural way. This coloring uses the optimal number $k+2$ of colors, therefore it has a vertex with $k+1$ different colors in its neighborhood. A compactness argument establishes from this that there is a point in $S^k$ covered by $k+1$ sets. A similar argument gives that $k+2$ in Corollary 16 is also optimal if $k \ge 3$.

**Corollary 18**

$$\frac{h}{2} + 1 \le Q(h) \le \frac{h}{2} + 2.$$

**Proof.** The lower bound is implied by Ky Fan's theorem. The upper bound follows from Corollary 17. □

Notice that for odd $h$ Corollary 18 gives the exact value $Q(h) = \frac{h+3}{2}$. For $h$ even we either have $Q(h) = \frac{h}{2} + 1$ or $Q(h) = \frac{h}{2} + 2$. It is trivial that $Q(0) = 1$. In [47] we show $Q(2) = 3$. This was independently proved by Imre Bárány [6]. For $h > 2$ even it remains open whether the lower or the upper bound of Corollary 18 is exact. We also refer to [47] for a more complete discussion of the connections between local colorings and the problem of $Q(h)$.

# 5 Circular colorings

In this section we show an application of the Zig-zag Theorem for the circular chromatic number of graphs. This will result in the partial solution of a conjecture by Johnson, Holroyd, and Stahl [28] and in a partial answer to a question of Hajiabolhassan and Zhu [24] concerning the circular chromatic number of Kneser graphs and Schrijver graphs, respectively. We also answer a question of Chang, Huang, and Zhu [10] concerning the circular chromatic number of iterated Mycielskians of complete graphs.

The circular chromatic number of a graph was introduced by Vince [52] under the name star chromatic number as follows.

**Definition 7** *For positive integers $p$ and $q$ a coloring $c : V(G) \to [p]$ of a graph $G$ is called a* $(p, q)$-coloring *if for all adjacent vertices $u$ and $v$ one has $q \le |c(u) - c(v)| \le p - q$. The* circular chromatic number *of $G$ is defined as*

$$\chi_c(G) = \inf \left\{ \frac{p}{q} : \text{ there is a } (p,q)\text{-coloring of } G \right\}.$$

It is known that the above infimum is always attained for finite graphs. An alternative description of $\chi_c(G)$, explaining its name, is that it is the minimum length of the perimeter of a circle on which we can represent the vertices of $G$ by arcs of length 1 in such a way that arcs belonging to adjacent vertices do not overlap. For a proof of this equivalence and for an extensive bibliography on the circular chromatic number we refer to Zhu's survey article [53].

It is known that for every graph $G$ one has $\chi(G) - 1 < \chi_c(G) \le \chi(G)$. Thus $\chi_c(G)$ determines the value of $\chi(G)$ while this is not true the other way round. Therefore the circular chromatic number can be considered as a refinement of the chromatic number.

Our main result on the circular chromatic number is Theorem 6. Here we restate the theorem with the explicit meaning of being topologically $t$-chromatic.

**Theorem 6** (restated) *For a finite graph $G$ we have $\chi_c(G) \ge \mathrm{coind}(B_0(G)) + 1$ if $\mathrm{coind}(B_0(G))$ is odd.*

**Proof.** Let $t = \mathrm{coind}(B_0(G)) + 1$ be an even number and let $c$ be a $(p, q)$-coloring of $G$. By the Zig-zag Theorem there is a $K_{\frac{t}{2},\frac{t}{2}}$ in $G$ which is completely multicolored by colors appearing in an alternating manner in its two sides. Let these colors be $c_1 < c_2 < \ldots < c_t$.

Since the vertex colored $c_i$ is adjacent to that colored $c_{i+1}$, we have $c_{i+1} \geq c_i + q$ and $c_t \geq c_1 + (t-1)q$. Since $t$ is even, the vertices colored $c_1$ and $c_t$ are also adjacent, therefore we must have $c_t - c_1 \leq p - q$. The last two inequalities give $p/q \geq t$ as needed. □

This result has been independently obtained by Meunier [42] for Schrijver graphs.

## 5.1 Circular chromatic number of even chromatic Kneser and Schrijver graphs

Johnson, Holroyd, and Stahl [28] considered the circular chromatic number of Kneser graphs and formulated the following conjecture. (See also as Conjecture 7.1 and Question 8.27 in [53].)

**Conjecture** (Johnson, Holroyd, Stahl [28]): For any $n \geq 2k$

$$\chi_c(KG(n,k)) = \chi(KG(n,k)).$$

It is proven in [28] that the above conjecture holds if $k = 2$ or $n = 2k+1$ or $n = 2k+2$.

Lih and Liu [35] investigated the circular chromatic number of Schrijver graphs and proved that $\chi_c(SG(n,2)) = n - 2 = \chi(SG(n,2))$ whenever $n \neq 5$. (For $n = 2k+1$ one always has $\chi_c(SG(2k+1,k)) = 2 + \frac{1}{k}$.) It was conjectured in [35] and proved in [24] that for every fixed $k$ there is a threshold $l(k)$ for which $n \geq l(k)$ implies $\chi_c(SG(n,k)) = \chi(SG(n,k))$. This clearly implies the analogous statement for Kneser graphs, for which the explicit threshold $l(k) = 2k^2(k-1)$ is given in [24]. At the end of their paper [24] Hajiabolhassan and Zhu ask what is the minimum $l(k)$ for which $n \geq l(k)$ implies $\chi_c(SG(n,k)) = \chi(SG(n,k))$. We show that no such threshold is needed if $n$ is even.

**Corollary 19** *The Johnson-Holroyd-Stahl conjecture holds for every even $n$. Moreover, if $n$ is even, then the stronger equality*

$$\chi_c(SG(n,k)) = \chi(SG(n,k))$$

*also holds.*

**Proof.** As $t$-chromatic Kneser graphs and Schrijver graphs are topologically $t$-chromatic, Theorem 6 implies the statement of the corollary. □

As mentioned above this result has been obtained independently by Meunier [42].

We show in Subsection 5.3 that for odd $n$ the situation is different.

## 5.2 Circular chromatic number of Mycielski graphs and Borsuk graphs

The circular chromatic number of Mycielski graphs was also studied extensively, cf. [10, 16, 25, 53]. Chang, Huang, and Zhu [10] formulated the conjecture that $\chi_c(M^d(K_n)) =$

$\chi(M^d(K_n)) = n+d$ whenever $n \geq d+2$. Here $M^d(G)$ denotes the $d$-fold iterated Mycielskian of graph $G$, i.e., using the notation of Subsection 4.3 we have $M^d(G) = M_{\boldsymbol{r}}^{(d)}(G)$ with $\boldsymbol{r} = (2,\dots,2)$. The above conjecture was verified for the special cases $d = 1, 2$ in [10], where it was also shown that $\chi_c(M^d(G)) \leq \chi(M^d(G)) - 1/2$ if $\chi(G) = d+1$. A simpler proof for the above special cases of the conjecture was given (for $d = 2$ with the extra condition $n \geq 5$) in [16]. Recently Hajiabolhassan and Zhu [25] proved that $n \geq 2^d + 2$ implies $\chi_c(M^d(K_n)) = \chi(M^d(K_n)) = n+d$. Our results show that $\chi_c(M^d(K_n)) = \chi(M^d(K_n)) = n+d$ always holds if $n+d$ is even. This also answers the question of Chang, Huang, and Zhu asking the value of $\chi_c(M^n(K_n))$ (Question 2 in [10]). The stated equality is given by the following immediate consequence of Theorem 6.

**Corollary 20** *If $H(G)$ is homotopy equivalent to the sphere $S^h$, $\boldsymbol{r}$ is a vector of positive integers, and $h+d$ is even, then $\chi_c(M_{\boldsymbol{r}}^{(d)}(G)) \geq d+h+2$.*
*In particular, $\chi_c(M_{\boldsymbol{r}}^{(d)}(K_n)) = n+d$ whenever $n+d$ is even.*

**Proof.** The condition on $G$ implies $\operatorname{coind}(H(M_{\boldsymbol{r}}^{(d)}(G))) = h+d$ by Stiebitz's result [48] (cf. the discussion and Proposition 9 in Subsection 4.3), which further implies $\operatorname{coind}(B_0(M_{\boldsymbol{r}}^{(d)}(G))) = h+d+1$. This gives the conclusion by Theorem 6.

The second statement follows by the homotopy equivalence of $H(K_n)$ with $S^{n-2}$ and the chromatic number of $M_{\boldsymbol{r}}^{(d)}(K_n)$ being $n+d$. $\square$

The above mentioned conjecture of Chang, Huang, and Zhu for $n+d$ even is a special case with $\boldsymbol{r} = (2,2,\dots,2)$ and $n \geq d+2$. Since $n+n$ is always even, the answer $\chi_c(M^n(K_n)) = 2n$ to their question also follows.

Corollary 20 also implies a recent result of Lam, Lin, Gu, and Song [33] who proved that for the generalized Mycielskian of odd order complete graphs $\chi_c(M_r(K_{2m-1})) = 2m$.

Lam, Lin, Gu, and Song [33] also determined the circular chromatic number of the generalized Mycielskian of even order complete graphs. They proved $\chi_c(M_r(K_{2m})) = 2m + 1/(\lfloor (r-1)/m \rfloor + 1)$. This result can be used to bound the circular chromatic number of the Borsuk graph $B(2s,\alpha)$ from above.

**Theorem 21** *For the Borsuk graph $B(n,\alpha)$ we have*

**(i)** $\chi_c(B(n,\alpha)) = n+1$ *if $n$ is odd and $\alpha$ is large enough;*

**(ii)** $\chi_c(B(n,\alpha)) \to n$ *as $\alpha \to 2$ if $n$ is even.*

**Proof.** The lower bound of part (i) immediately follows from Theorem 6 considering again the finite subgraph $G_P$ of $B(n,\alpha)$ defined in [37] and already mentioned in the proof of Lemma 4.4. The matching upper bound is provided by $\chi(B(n,\alpha)) = n+1$ for large enough $\alpha$, see [37] and Subsection 4.4.

For (ii) we have $\chi_c(B(n,\alpha)) > \chi(B(n,\alpha)) - 1 \geq n$. For an upper bound we use that $\chi_c(M_r(K_n)) \to n$ if $r$ goes to infinity by the result of Lam, Lin, Gu, and Song [33] quoted above. By the result of Stiebitz [48] and Lemma 4.4 we have a graph homomorphism from $B(n,\alpha)$ to $M_r(K_n)$ for any $r$ and large enough $\alpha$. As $(p,q)$-colorings can be defined in terms of graph homomorphisms (see [9]), we have $\chi_c(G) \le \chi_c(H)$ if there exists a graph homomorphism from $G$ to $H$. This finishes the proof of part (ii) of the theorem. □

*Remark 7.* By Theorem 21 (ii) we have a sequence of $(p_i,q_i)$-colorings of the graphs $B(n,\alpha_i)$ where $n$ is even such that $\alpha_i \to 2$ and $p_i/q_i \to n$. By a direct construction we can show that a single function $g : S^{n-1} \to C$ is enough. Here $C$ is a circle of unit perimeter. We need

$$\inf\{\mathrm{dist}_C(g(\boldsymbol{x}), g(\boldsymbol{y})) : \{\boldsymbol{x},\boldsymbol{y}\} \in E(B(n,\alpha))\} \to 1/n \text{ as } \alpha < 2 \text{ goes to } 2. \qquad (2)$$

The distance $\mathrm{dist}_C(\cdot,\cdot)$ is measured along the circle $C$. Clearly, if $p/q > n$ and we split $C$ into $p$ arcs $a_1,\dots,a_p$ of equal length and color the point $\boldsymbol{x}$ with $i$ if $g(\boldsymbol{x}) \in a_i$, then this is a $(p,q)$-coloring of $B(n,\alpha)$ for $\alpha$ close enough to 2.

For $n = 2$ any $\mathbb{Z}_2$-map $g : S^1 \to C$ satisfies expression (2). Let $n > 2$. The map $g$ to be constructed must not be continuous by the Borsuk-Ulam theorem. Let us choose a set $H$ of $n-1$ equidistant points in $C$ and for $b \in C$ let $T(b)$ denote the unique set of $n/2$ equidistant points in $C$ containing $b$.

We consider $S^{n-1}$ as the *join* of the sphere $S^{n-3}$ and the circle $S^1$. All points in $S^{n-1}$ are now either in $S^{n-3}$, or in $S^1$, or in the interval connecting a point in $S^{n-3}$ to a point in $S^1$. We define $g$ on $S^{n-3}$ such that it takes values only from $H$ and it is a proper coloring of $B(n-2,\alpha)$ for large enough $\alpha$. We define $g$ on $S^1$ such that if $\boldsymbol{y}$ goes a full circle around $S^1$ with uniform velocity, then its image $g(\boldsymbol{y})$ covers an arc of length $2/n$ of $C$ and it also moves with uniform velocity. Notice that although $g$ is not continuous on $S^1$, the set $T(g(\boldsymbol{y}))$ depends on $\boldsymbol{y} \in S^1$ in a continuous manner. Also note that for a point $\boldsymbol{x} \in S^1$ the images $g(\boldsymbol{x})$ and $g(-\boldsymbol{x})$ are $1/n$ apart on $C$ and $T(g(\boldsymbol{x})) \cup T(g(-\boldsymbol{x}))$ is a set of $n$ equidistant points.

Let $\boldsymbol{x} \in S^{n-3}$ and $\boldsymbol{y} \in S^1$. Assume that a point $\boldsymbol{z}$ moves with uniform velocity from $\boldsymbol{x}$ to $\boldsymbol{y}$ along the interval connecting them. We define $g$ on this interval such that $g(\boldsymbol{z})$ moves with uniform velocity along $C$ covering an arc of length at most $1/n$ from $g(\boldsymbol{x})$ to a point in $T(g(\boldsymbol{y}))$. The choice of the point in $T(g(\boldsymbol{y}))$ is uniquely determined unless $g(\boldsymbol{x}) \in T(g(-\boldsymbol{y}))$. In the latter case we make an arbitrary choice of the two possible points for the destination of the image $g(\boldsymbol{z})$.

It is not hard to prove that the function $g$ defined above satisfies expression (2). ◊

## 5.3 Circular chromatic number of odd chromatic Schrijver graphs

In this subsection we show that the parity condition on $\chi(SG(n,k))$ in Corollary 19 is relevant, for odd chromatic Schrijver graphs the circular chromatic number can be arbitrarily close to its lower bound.

**Theorem 22** *For every $\varepsilon > 0$ and every odd $t \geq 3$ if $n \geq t^3/\varepsilon$ and $t = n - 2k + 2$, then*

$$1 - \varepsilon < \chi(SG(n,k)) - \chi_c(SG(n,k)) < 1.$$

The second inequality is well-known and holds for any graph. We included it only for completeness. To prove the first inequality we need some preparation. We remark that the bound on $n$ in the theorem is not best possible. Our method proves $\chi(SG(n,k)) - \chi_c(SG(n,k)) \geq 1 - 1/i$ if $i$ is a positive integer and $n \geq 6(i-1)\binom{t}{3} + t$.

First we extend our notion of wide coloring.

**Definition 8** *For a positive integer $s$ we call a vertex coloring of a graph $s$-wide if the two end vertices of any walk of length $2s - 1$ receive different colors.*

Our original wide colorings are 3-wide, while 1-wide simply means proper. Gyárfás, Jensen, and Stiebitz [23] investigated $s$-wide colorings (in different terms) and mention (referring to a referee in the $s > 2$ case) the existence of homomorphism universal graphs for $s$-wide colorability with $t$ colors. We give a somewhat different family of such universal graphs. In the $s = 2$ case the color-criticality of the given universal graph is proven in [23] implying its minimality among graphs admitting 2-wide $t$-colorings. Later in Subsection 6.1 we generalize this result showing that the members of our family are color-critical for every $s$. Thus they must be minimal and therefore isomorphic to a retract of the corresponding graphs given in [23].

**Definition 9** *Let $H_s$ be the path on the vertices $0, 1, 2, \ldots, s$ ($i$ and $i - 1$ connected for $1 \leq i \leq s$) with a loop at $s$. We define $W(s,t)$ to be the graph with*

$$V(W(s,t)) = \{(x_1 \ldots x_t) : \forall i\ x_i \in \{0, 1, \ldots, s\}, \exists! i\ x_i = 0,\ \exists j\ x_j = 1\},$$

$$E(W(s,t)) = \{\{x_1 \ldots x_t, y_1 \ldots y_t\} : \forall i\ \{x_i, y_i\} \in E(H_s)\}.$$

Note that $W(s,t)$ is an induced subgraph of the direct power $H_s^t$ (cf. Subsection 4.3).

**Proposition 23** *A graph $G$ admits an $s$-wide coloring with $t$ colors if and only if there is a homomorphism from $G$ to $W(s,t)$.*

**Proof.** For the if part color vertex $\boldsymbol{x} = x_1 \ldots x_t$ of $W(s,t)$ with $c(\boldsymbol{x}) = i$ if $x_i = 0$. Any walk between two vertices colored $i$ either has even length or contains two vertices $\boldsymbol{y}$ and $\boldsymbol{z}$ with $y_i = z_i = s$. These $\boldsymbol{y}$ and $\boldsymbol{z}$ are both at least at distance $s$ apart from both ends of the walk, thus our coloring of $W(s,t)$ with $t$ colors is $s$-wide. Any graph admitting a homomorphism $\varphi$ to $W(s,t)$ is $s$-widely colored with $t$ colors by $c_G(v) := c(\varphi(v))$.

For the only if part assume $c$ is an $s$-wide $t$-coloring of $G$ with colors $1, \ldots, t$. Let $\varphi(v)$ be an arbitrary vertex of $W(s,t)$ if $v$ is an isolated vertex of $G$. For a non-isolated vertex $v$ of $G$ let $\varphi(v) = \boldsymbol{x} = x_1 \ldots x_t$ with $x_i = \min(s, d_i(v))$, where $d_i(v)$ is the distance of color class $i$ from $v$. It is clear that $x_i = 0$ for $i = c(v)$ and for no other $i$, while $x_i = 1$ for the colors of the neighbors of $v$ in $G$. Thus the image of $\varphi$ is indeed in $V(W(s,t))$. It takes an easy checking that $\varphi$ is a homomorphism. □

The following lemma is a straightforward extension of the argument given in the proof of Theorem 3.

**Lemma 5.1** *If $n \geq (2s-2)t^2 - (4s-5)t$ then $SG(n,k)$ admits an $s$-wide $t$-coloring.*

**Proof.** We use the notation introduced in the proof of Theorem 3.

Let $n \geq t(2(s-1)(t-2)+1)$ as in the statement and let $c_0$ be the coloring defined in the mentioned proof. The lower bound on $n$ now allows to assume that $|C_i| \geq (s-1)(t-2)+1$. We show that $c_0$ is $s$-wide.

Consider a walk $x_0x_1 \ldots x_{2s-1}$ of length $(2s-1)$ in $SG(n,k)$ and let $i = c_0(x_0)$. Then $C_i \subseteq x_0$. By Lemma 4.2 $|x_0 \setminus x_{2s-2}| \leq (s-1)(t-2) < |C_i|$. Thus $x_{2s-2}$ is not disjoint from $C_i$. As $x_{2s-1}$ is disjoint from $x_{2s-2}$, it does not contain $C_i$ and thus its color is not $i$. □

**Lemma 5.2** *$W(s,t)$ admits a homomorphism to $M_s(K_{t-1})$.*

**Proof.** Recall our notation for the (iterated) generalized Mycielskians from Subsection 4.3.

We define the following mapping from $V(W(s,t))$ to $V(M_s(K_{t-1}))$.

$$\varphi(x_1 \ldots x_t) := \begin{cases} (s-x_t, i) & \text{if } x_t \neq x_i = 0 \\ (s,*) & \text{if } x_t = 0. \end{cases}$$

One can easily check that $\varphi$ is indeed a homomorphism. □

**Proof of Theorem 22.** By Lemma 5.1, if $n \geq (2s-2)t^2 - (4s-5)t$, then $SG(n,k)$ has an $s$-wide $t$-coloring, thus by Proposition 23 it admits a homomorphism to $W(s,t)$. Composing this with the homomorphism given by Lemma 5.2 we conclude that $SG(n,k)$ admits a homomorphism to $M_s(K_{t-1})$, implying $\chi_c(SG(n,k)) \leq \chi_c(M_s(K_{t-1}))$.

We continue by using Lam, Lin, Gu, and Song's result [33], who proved, as already quoted in the previous subsection, that $\chi_c(M_s(K_{t-1})) = t - 1 + \frac{1}{\lfloor \frac{2s-2}{t-1} \rfloor + 1}$ if $t$ is odd. Thus, for odd $t$ and $i > 0$ integer we choose $s = (t-1)(i-1)/2 + 1$ and $\chi(SG(n,k)) - \chi_c(SG(n,k)) = t - \chi_c(SG(n,k)) \geq 1 - 1/i$ follows from the $n \geq 6(i-1)\binom{t}{3} + t$ bound.

To get the form of the statement claimed in the theorem we choose $i = \lfloor 1/\varepsilon \rfloor + 1$. □

*Remark 8* It is not hard to see that the graphs $M_s(K_{t-1})$ can also be interpreted as homomorphism universal graphs for a property related to wide colorings. Namely, a graph admits a homomorphism into $M_s(K_{t-1})$ if and only if it can be colored with $t$ colors so that there is no walk of length $2s-1$ connecting two (not necessarily different) points of one particular color class, say, color class $t$. Realizing this, the statement of Lemma 5.2 is immediate. ◊

# 6 Further remarks

## 6.1 Color-criticality of $W(s, t)$

In this subsection we prove the edge color-criticality of the graphs $W(s, t)$ introduced in the previous section. This generalizes Theorem 2.3 in [23], see Remark 9 after the proof.

**Theorem 24** *For every integer $s \geq 1$ and $t \geq 2$ the graph $W(s, t)$ has chromatic number $t$, but deleting any of its edges the resulting graph is $(t-1)$-chromatic.*

**Proof.** $\chi(W(s, t)) \geq t$ follows from the fact that some $t$-chromatic Schrijver graphs admit a homomorphism to $W(s, t)$ which is implied by Lemma 5.1 and Proposition 23. The coloring giving vertex $\boldsymbol{x} = x_1 \dots x_t$ of $W(s, t)$ color $i$ iff $x_i = 0$ is proper proving $\chi(W(s, t)) \leq t$.

We prove edge-criticality by induction on $t$. For $t = 2$ the statement is trivial as $W(s, t)$ is isomorphic to $K_2$. Assume that $t \geq 3$ and edge-criticality holds for $t - 1$. Let $\{x_1 \dots x_t, y_1 \dots y_t\}$ be an edge of $W(s, t)$ and $W'$ be the graph remaining after removal of this edge. We need to give a proper $(t-1)$-coloring $c$ of $W'$.

Let $i$ and $j$ be the coordinates for which $x_i = y_j = 0$. We have $x_j = y_i = 1$, in particular, $i \neq j$. Let $r$ be a coordinate different from both $i$ and $j$. We may assume without loss of generality that $r = 1$, and also that $y_1 \geq x_1$. Coordinates $i$ and $j$ make sure that $x_2 x_3 \dots x_t$ and $y_2 y_3 \dots y_t$ are vertices of $W(s, t-1)$, and in fact, they are connected by an edge $e$.

A proper $(t-2)$-coloring of the graph $W(s, t-1) \setminus e$ exists by the induction hypothesis. Let $c_0$ be such a coloring. Let $\alpha$ be a color of $c_0$ and $\beta$ a color that does not appear in $c_0$. We define the coloring $c$ of $W'$ as follows:

$$c(z_1 z_2 \dots z_t) = \begin{cases} \alpha & \text{if } z_1 < x_1,\ x_1 - z_1 \text{ is even} \\ \beta & \text{if } z_1 < x_1,\ x_1 - z_1 \text{ is odd} \\ \alpha & \text{if } z_1 = x_1 = 1,\ z_i \neq 1 \text{ for } i > 1 \\ \beta & \text{if } z_1 > x_1,\ z_i = x_i \text{ for } i > 1 \\ c_0(z_2 z_3 \dots z_t) & \text{otherwise.} \end{cases}$$

It takes a straightforward case analysis to check that $c$ is a proper $(t-1)$-coloring of $W'$. □

*Remark 9.* Gyárfás, Jensen, and Stiebitz [23] proved the $s = 2$ version of the previous theorem using a homomorphism from their universal graph with parameter $t$ to a generalized Mycielskian of the same type of graph with parameter $t - 1$. In fact, our proof is a direct generalization of theirs using very similar ideas. Behind the coloring we gave is the recognition of a homomorphism from $W(s, t)$ to $M_{3s-2}(W(s, t-1))$. ♢

## 6.2 Hadwiger's conjecture and the Zig-zag theorem

Hadwiger's conjecture, one of the most famous open problems in graph theory, states that if a graph $G$ contains no $K_{r+1}$ minor, then $\chi(G) \le r$. For detailed information on the history and status of this conjecture we refer to Toft's survey [51]. We only mention that even $\chi(G) = O(r)$ is not known to be implied by the hypothesis for general $r$.

As a fractional and linear approximation version, Reed and Seymour [44] proved that if $G$ has no $K_{r+1}$ minor then $\chi_f(G) \le 2r$. This means that graphs with $\chi_f(G)$ and $\chi(G)$ appropriately close and not containing a $K_{r+1}$ minor satisfy $\chi(G) = O(r)$.

We know that the main examples of graphs in [45] for $\chi_f(G) << \chi(G)$ (Kneser graphs, Mycielski graphs), as well as many other graphs studied in this paper, satisfy the hypothesis of the Zig-zag theorem, therefore their $t$-chromatic versions must contain $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$ subgraphs. (We mention that for strongly topologically $t$-chromatic graphs this consequence, in fact, the containment of $K_{a,b}$ for every $a, b$ satisfying $a + b = t$, was proven by Csorba, Lange, Schurr, and Wassmer [13].) However, a $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$ subgraph contains a $K_{\lfloor\frac{t}{2}\rfloor+1}$ minor (just take a matching of size $\lfloor\frac{t-2}{2}\rfloor$ plus one point from each side of the bipartite graph) proving the following statement which shows that the same kind of approximation is valid for these graphs, too.

**Corollary 25** *If a topologically $t$-chromatic graph contains no $K_{r+1}$ minor, then $t < 2r$.*

$\square$

**Acknowledgments:** We thank Imre Bárány, Péter Csorba, Gábor Elek, László Fehér, László Lovász, Jiří Matoušek, and Gábor Moussong for many fruitful conversations that helped us to better understand the topological concepts used in this paper.

# References

[1] N. Alon, P. Frankl, L. Lovász, The chromatic number of Kneser hypergraphs, *Trans. Amer. Math. Soc.*, **298** (1986), 359–370.

[2] E. Babson, D. N. Kozlov, Topological obstructions to graph colorings, *Electron. Res. Announc. Amer. Math. Soc.* **9** (2003), 61-68, arXiv:math.CO/0305300.

[3] E. Babson, D. N. Kozlov, Complexes of graph homomorphisms, to appear in *Israel J. Math.*, arXiv:math.CO/0310056.

[4] P. Bacon, Equivalent formulations of the Borsuk-Ulam theorem, *Canad. J. Math.*, **18** (1966), 492–502.

[5] I. Bárány, A short proof of Kneser's conjecture *J. Combin. Theory Ser. A*, **25** (1978), no. 3, 325–326.

[6] I. Bárány, personal communication.

[7] A. Björner, Topological methods, in: *Handbook of Combinatorics* (Graham, Grötschel, Lovász eds.), 1819–1872, Elsevier, Amsterdam, 1995.

[8] A. Björner, M. de Longueville, Neighborhood complexes of stable Kneser graphs, *Combinatorica*, **23** (2003), no. 1, 23–34.

[9] J. A. Bondy, P. Hell, A note on the star chromatic number, *J. Graph Theory*, **14** (1990), 479–482.

[10] G. J. Chang, L. Huang, X. Zhu, Circular chromatic numbers of Mycielski's graphs, *Discrete Math.*, **205** (1999), 23–37.

[11] P. Csorba, Homotopy types of box complexes, manuscript, arXiv:math.CO/0406118.

[12] P. Csorba, Non-tidy Spaces and Graph Colorings, Ph.D. thesis, 2005.

[13] P. Csorba, C. Lange, I. Schurr, A. Wassmer, Box complexes, neighbourhood complexes, and chromatic number, *J. Combin. Theory Ser. A* **108** (2004), 159-168, arXiv:math.CO/0310339.

[14] P. Erdős, Z. Füredi, A. Hajnal, P. Komjáth, V. Rödl, Á. Seress, Coloring graphs with locally few colors, *Discrete Math.*, **59** (1986), 21–34.

[15] P. Erdős, A. Hajnal, On chromatic graphs, (Hungarian) *Mat. Lapok*, **18** (1967), 1–4.

[16] G. Fan, Circular chromatic number and Mycielski graphs, *Combinatorica*, **24** (2004), 127–135.

[17] K. Fan, A generalization of Tucker's combinatorial lemma with topological applications, *Annals of Mathematics*, **56** (1952), no. 2, 431–437.

[18] K. Fan, Evenly distributed subsets of $S^n$ and a combinatorial application, *Pacific J. Math.*, **98** (1982), no. 2, 323–325.

[19] Z. Füredi, Local colorings of graphs (Gráfok lokális színezései), manuscript in Hungarian, September 2002.

[20] Z. Füredi, personal communication.

[21] C. Godsil, G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics 207, Springer-Verlag, New York, 2001.

[22] J. E. Greene, A new short proof of Kneser's conjecture, *Amer. Math. Monthly*, **109** (2002), no. 10, 918–920.

[23] A. Gyárfás, T. Jensen, M. Stiebitz, On graphs with strongly independent colour-classes, *J. Graph Theory*, **46** (2004), 1–14.

[24] H. Hajiabolhassan, X. Zhu, Circular chromatic number of Kneser graphs, *J. Combin. Theory Ser. B*, **88** (2003), no. 2, 299–303.

[25] H. Hajiabolhassan, X. Zhu, Circular chromatic number and Mycielski construction, *J. Graph Theory*, **44** (2003), 106–115.

[26] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. Electronic version available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`.

[27] F. C. Holroyd, Problem 338 (BCC16.25), Erdős-Ko-Rado at the court of king Arthur, *Discrete Math.*, **197/198** (1999), 812.

[28] A. Johnson, F. C. Holroyd, S. Stahl, Multichromatic numbers, star chromatic numbers and Kneser graphs, *J. Graph Theory*, **26** (1997), no. 3, 137–145.

[29] G. Kalai, personal communication.

[30] J. Körner, personal communication.

[31] J. Körner, C. Pilotto, G. Simonyi, Local chromatic number and Sperner capacity, to appear in *J. Combin. Theory Ser. B*.

[32] I. Kříž, Equivariant cohomology and lower bounds for chromatic numbers, *Trans. Amer. Math. Soc.*, **333** (1992), no. 2, 567–577; I. Kříž, A correction to: "Equivariant cohomology and lower bounds for chromatic numbers", *Trans. Amer. Math. Soc.*, **352** (2000), no. 4, 1951–1952.

[33] P. C. B. Lam, W. Lin, G. Gu, Z. Song, Circular chromatic number and a generalization of the construction of Mycielski, *J. Combin. Theory Ser. B*, **89** (2003), no. 2, 195–205.

[34] M. Larsen, J. Propp, D. Ullman, The fractional chromatic number of Mycielski's graphs, *J. Graph Theory*, **19** (1995), no. 3, 411–416.

[35] K-W. Lih, D. D-F. Liu, Circular chromatic numbers of some reduced Kneser graphs, *J. Graph Theory*, **41** (2002), no. 1, 62–68.

[36] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Combin. Theory Ser. A*, **25** (1978), no. 3, 319–324.

[37] L. Lovász, Self-dual polytopes and the chromatic number of distance graphs on the sphere, *Acta Sci. Math. (Szeged)*, **45** (1983), 317–323.

[38] L. Lovász, personal communication.

[39] J. Matoušek, *Using the Borsuk-Ulam Theorem, Lectures on Topological Methods in Combinatorics and Geometry*, Springer-Verlag, Berlin etc., 2003.

[40] J. Matoušek, A combinatorial proof of Kneser's conjecture, *Combinatorica*, **24** (2004), 163–170.

[41] J. Matoušek, G. M. Ziegler, Topological lower bounds for the chromatic number: A hierarchy, *Jahresber. Deutsch. Math.-Verein.*, **106** (2004), no. 2, 71–90, arXiv:math.CO/0208072.

[42] F. Meunier, A topological lower bound for the circular chromatic number of Schrijver graphs, manuscript, 2004.

[43] J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.*, **3** (1955), 161–162.

[44] B. A. Reed, P. D. Seymour, Fractional colouring and Hadwiger's conjecture, *J. Combin. Theory Ser. B*, **74** (1998), 147–152.

[45] E. R. Scheinerman, D. H. Ullman, *Fractional Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, Chichester, 1997.

[46] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, *Nieuw Arch. Wisk. (3)*, **26** (1978), no. 3, 454–461.

[47] G. Simonyi, G. Tardos, Local chromatic number and distinguishing the strength of topological obstructions, submitted, arXiv:math.CO/0502452.

[48] M. Stiebitz, Beiträge zur Theorie der färbungskritischen Graphen, Habilitation, TH Ilmenau, 1985.

[49] J. Talbot, Intersecting families of separated sets, *J. London Math. Soc. (2)*, **68** (2003), no. 1, 37–51, arXiv:math.CO/0211314.

[50] C. Tardif, Fractional chromatic numbers of cones over graphs, *J. Graph Theory*, **38** (2001), 87–94.

[51] B. Toft, A survey of Hadwiger's conjecture, Surveys in graph theory (San Francisco, CA, 1995), *Congr. Numer.* **115** (1996), 249–283.

[52] A. Vince, Star chromatic number, *J. Graph Theory*, **12** (1988), no. 4, 551–559.

[53] X. Zhu, Circular chromatic number: a survey, *Discrete Math.*, **229** (2001), no. 1–3, 371–410.

[54] G. M. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, *Invent. Math.*, **147** (2002), no. 3, 671–691, arXiv:math.CO/0103146.

[55] R. T. Živaljević, $WI$-posets, graph complexes and $\mathbb{Z}_2$-equivalences, to appear in *J. Combin. Theory Ser. A*, arXiv:math.CO/0405419.