The longest segment in the complement of a packing

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1 Introduction

Let K be a compact convex body in $\mathbb{R}^n$ not contained in a hyperplane, and denote the norm whose unit ball is $\frac{1}{2}(K-K)$ by $\|\cdot\|_K$. Given a translative packing of K, we are interested in how long segments (with respect to $\|\cdot\|_K$) lie in the complement of the interiors of the translates. The main result of this note is showing the existence of a translative packing with an exponential upper bound on the length of the segments avoiding it (see below). But we start here with a lower bound.

We show that any packing of the unit Euclidean ball B^n avoids a segment of length exponential in n. It is a rather interesting question to find how long segments necessarily exist that avoid any packing of *any* convex, open body in $\mathbb{R}^n$. Our lower bound proof does not work for bodies allowing dense packings.

Let us consider any packing of B^n, and denote the area and the packing density of the unit ball by κ_n and $\delta(B^n)$, respectively. Choose a unit segment s, and denote the projection of B^n into some hyperplane orthogonal to s by B^{n-1}, and set $\lambda = \frac{\kappa_n}{3\kappa_{n-1}\cdot\delta(B^n)}$. The definition of the packing density yields that there exists a translate Z of the cylinder $\lambda\cdot s + n\cdot B^{n-1}$ which is intersected by at most

$$V(Z+B^n)\cdot\frac{\delta(B^n)}{\kappa_n} \leq \frac{\lambda+2}{3\lambda}\cdot(n+1)^{n-1} < n^{n-1}$$

balls in the packing. (The latter inequality only holds for large enough n and follows from our estimate on λ below.) Therefore the total area of the projections of these balls into the base of Z is less than the area of the base, and there exists a segment s' parallel to s in the complement of the balls such that

$$\text{length}(s') = \lambda = \frac{\kappa_n}{3\kappa_{n-1}} \cdot \frac{1}{\delta(B^n)} \geq 2^{0.599n+o(n)}. \tag{1}$$

Here we used the estimate $\delta(B^n) \leq 2^{-0.599n+o(n)}$ of Kabatjanskii & Levenstein [4].

Slight modification of the argument above yields that for any lattice packing of equal balls, there exists a line avoiding all balls (see A. Heppes [3]). On the other hand, Ch. Zong conjectured that there exists a packing where the length of the longest segment in the complement is at most c^n for some constant c, and the paper M. Henk & Ch. Zong [2] constructed a packing where the segments in the complement have bounded length.

Let $|\cdot|$ denote the n–dimensional Lebesgue measure.

Theorem 1 *Let K a compact convex body in $\mathbb{R}^n$ not contained in a hyperplane. Then there exists a periodic translative packing of K such that any segment of length $c_0 n^2 \cdot \frac{|K-K|}{|K|}$ (with respect to $\|\cdot\|_K$) intersects the interior of some translate where c_0 is an absolute constant.*

Remark: Note that the bound in the theorem is $c_0 n^2 2^n$ for centrally symmetric bodies K, while in the general case it is bounded by $c_0 n^2 4^n$, since $|K-K| \leq \binom{2n}{n} \cdot |K|$ according to the celebrated result of C.A. Rogers & G. Shepard [6], and we have $\binom{2n}{n} < 4^n$.

If the upper bound of Theorem 1 is improved to $c^{n+o(n)}$ for some $c < 2$ for the ball, then (1) yields that $\delta(B^n) \geq c^{-n+o(n)}$. Therefore such an improvement seems to be hard to prove. Actually, in order to improve on the classical lower bound $\delta(B^n) \geq 2^{-n}$, it is sufficient to construct a packing such that any segment parallel to a given direction and having length of at least $c^{n+o(n)}$, $c < 2$, intersects the interior of some of the balls.

Let us consider a consequence of Theorem 1. A *cloud* for the convex body K is defined as a packing of translates K which do not overlap K, and any half line emanating from K intersects the interior of at least one translate. It was proved in K. Böröczky & V. Soltan [1] that there always exists a finite cloud. As for the cardinality of a cloud, Ch. Zong verified the upper bound n^{n^2}, which was improved to c^{n^2} independently by I. Talata [8], I. Bárány and I. Leader (see Ch. Zong [9]). Here I. Talata [8] proved $2^{1.401n^2+o(n^2)}$ if K is a ball, $3^{n^2+o(n^2)}$ if K is centrally symmetric, and $6^{n^2+o(n^2)}$ in general. Now fix any translate of K in the packing given by Theorem 1, and consider all translates in the packing which are at most distance $c_0 n^2 \cdot \frac{|K-K|}{|K|}$ from the fixed copy. We deduce

Corollary 1 *For any centrally symmetric convex K in $\mathbb{R}^n$, there exists a cloud by $2^{n^2+o(n^2)}$ translates. For a general convex body K, a cloud can be formed using $4^{n^2+o(n^2)}$ translates.*

With respect to a lower bound, I. Talata [8] verified that a cloud of the unit ball always has at least $2^{0.599n^2+o(n^2)}$ elements. A lower bound with slightly weaker constant was independently obtained by I. Bárány (see Ch. Zong [9]).

Finally, it is customary to consider a packing $\{x_i + K\}$ such that $\|x_i - x_j\|_K \geq \varrho$ for $i \neq j$ and for a prescribed constant $\varrho \geq 2$. Our arguments show that for such a packing, there exists a segment of length $c_1(n)\varrho^n$ in the complement, and there exists a packing where the length of any segment in the complement is at most

$$c_2(n)\varrho^n \log \varrho.$$

For clouds, it easy to see that at least $c_3(n)\varrho^{n^2-n}$ translates are needed for any cloud (even if the source is only one point), and our argument yields a family consisting of at most

$$c_4(n)\varrho^{n^2-n}(\log \varrho)^n$$

translates clouding K. Here $c_1(n)$, $c_2(n)$, $c_3(n)$ and $c_4(n)$ are positive constants depending only on the dimension n.

2 The proof of Theorem 1

Let K be a compact convex body in $\mathbb{R}^n$ not contained in a hyperplane. All distances and lengths below are measured with respect to $\|\cdot\|_K$.

Our proof is probabilistic: we select random translates of K for the packing and show that with high probability their collection satisfies the requirement of Theorem 1. More precisely, we consider a large enough compact factor T^n of $\mathbb{R}^n$ and throw uniform random translates of K into T^n one by one. By keeping those that are disjoint from all earlier translates we obtain our periodic packing. Note that this method is not greedy, as our rule excludes a translate from the packing if it intersects some earlier translates even though all those translates may have been excluded themselves. This suboptimal rule is necessary to obtain *independence* between the configurations of regions far from each other.

The detailed argument is as follows. We set $c_0 = 10\,000$ and assume $n > 2$ for simplicity. According to the Minkowski-Hlawka theorem, there exists a lattice Λ such that $\Lambda + 2c_0 n^2 \cdot 4^n (K-K)$ is a packing and

$$\det \Lambda \leq 2^n \cdot \left|2c_0 n^2 \cdot 4^n (K-K)\right|. \tag{2}$$

The condition on Λ yields that if $\|x-y\|_K < 2c_0 n^2 4^n$ then the distance of images of x and y in the torus $T^n = \mathbb{R}^n/\Lambda$ is still $\|x-y\|_K$.

We throw points $x_1, x_2, \ldots$ into T^n independently with uniform distribution with respect to the Lebesgue measure. We color an x_i red if $\|x_j - x_i\|_K > 2$ holds for any $j < i$, or in other words, if $x_i + K$ is disjoint from any $x_j + K$ for $j < i$. For a measurable $A \subset T^n$, denote the probability that A contains no red point by $P(A)$.

Lemma 1 *Let $A, B \subset T^n$ be measurable such that the diameter of B is less than 2, and there exist translates $y_i + B \subset A$, $i = 1, \ldots, N$ with $\|y_i - y_j\|_K \geq 6$ for $i \neq j$. Then*

$$P(A) \leq \left(1 - \frac{|B|}{|K-K|}\right)^N.$$

Proof: First we calculate the probability that B contains a red point. The probability that x_i lands in B and it is colored red is

$$P_i = |B| \cdot (1 - |K-K|)^{i-1}.$$

Since the diameter of B is less than 2, only at most one $x_i \in B$ is colored red, and we deduce that

$$1 - P(B) = \sum_{i \geq 1} P_i = \frac{|B|}{|K-K|}. \tag{3}$$

Now the sets $y_i + B - (K-K)$ are disjoint, and hence the events that $y_i + B$ contains no red point, $i = 1, \ldots, N$, are independent. Each of these events have equal probability as calculated in (3), hence the lemma follows. Q.E.D.

According to C.A. Rogers [5], there exists a covering $\{z + \frac{1}{n}K | z \in Z\}$ of T^n whose density is at most $n \ln n + n \ln \ln n + 4n$. Therefore we deduce by (2) that

$$|Z| \leq (n \ln n + n \ln \ln n + 4n) \cdot n^n \cdot \frac{\det \Lambda}{|K|} \leq 2^{10n^2}.$$

Let S be the family of segments in T^n whose length is between $c_0 n^2 \cdot \frac{|K-K|}{|K|} - 1$ and $c_0 n^2 \cdot \frac{|K-K|}{|K|} + 1$, and the endpoints are chosen from Z. Clearly, $\#S \leq (\#Z)^2 \leq 2^{20n^2}$.

Now Lemma 1 can be applied to $A = s_k - (1 - \frac{2}{n})K$ with $B = -(1 - \frac{2}{n})K$ and $N = \lfloor \frac{c_0}{6} \cdot n^2 \frac{|K-K|}{|K|} \rfloor$. We deduce that the probability P_0 that there exists an $s \in S$ such that $s - (1 - \frac{2}{n})K$ contains no red point is

$$P_0 \leq \#S \cdot \left(1 - \frac{|B|}{|K-K|}\right)^N \tag{4}$$

$$\leq 2^{20n^2} \left(1 - \left(1 - \frac{2}{n}\right)^n \frac{|K|}{|K-K|}\right)^N < 1. \tag{5}$$

Therefore there exists a sequence $x_1, x_2, \ldots$ such that for any $s \in S$, the set $s - (1 - \frac{2}{n})K$ contains a red point. Denote the family of red points by $r_1, \ldots, r_m$.

Now $\Lambda + \{r_1 + K, \ldots, r_m + K\}$ is a periodic translative packing in $\mathbb{R}^n$. Let us consider a segment $s_0 = aa'$ with length $c_0 n^2 \frac{|K-K|}{|K|}$. Embedding a and a' into T^n, there exist points $z, z' \in Z$ with $a \in z + \frac{1}{n}K$, $a \in z' + \frac{1}{n}K$. We now have $s = zz' \in S$ and $s \subset s_0 - \frac{1}{n}K$. We have seen that that there exists some $r_i \in s - (1 - \frac{2}{n})K \subset s_0 - (1 - \frac{1}{n})K$, and hence s intersects the interior of $r_i + K$. In turn, we conclude Theorem 1.

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