The k most frequent distances in the plane*

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Abstract

A new upper bound is shown for the number of incidences between n points and n families of concentric circles in the plane. As a consequence, it is shown that the number of the k most frequent distances among n points in the plane is $f_n(k) = O(n^{1.4571}k^{.6286})$ improving earlier bound of Akatsu, Tamaki, and Tokuyama.

1 Introduction

The famous theorem of Szemerédi and Trotter [16] states that the number of incidences between n points and ℓ lines in the plane is at $O(n^{2/3}\ell^{2/3} + n + \ell)$. A construction due to Erdős [9] shows that this bound is tight, and later Székely [15] gave an elegant proof to this theorem by means of geometric graphs. Ever since, considerable efforts were made to find tight bounds on the number of incidences between a set of points and other objects in d-dimensional Euclidean space. Only partial results [10, 6, 13, 5] are known so far for most other incidence problems, a tight bound of Szemerédi and Trotter is only known to generalize to incidences of points and lines in the complex plane $\mathbf{C}^2$ [18].

One interesting incidence problem is that of points and circles. It is conjectured that the number of incidences between n points and ℓ circles in the plane is at most $O(n^{2/3}\ell^{2/3}\log^c(n\ell) + n + \ell)$. This would be tight as well for some constant c in the $\sqrt{n} \times \sqrt{n}$ square grid as was shown by Erdős [7]. Recently, Aronov and Sharir [3] have improved the upper bound to the circle-point incidence number to $O_\varepsilon(n + n^{2/3}\ell^{2/3} + n^{6/11+3\varepsilon}\ell^{9/11-\varepsilon} + \ell)$ for arbitrary small positive ε. In this paper, we investigate the number of incidences between n point and n families of k concentric circles using a recent technique of [14].

Theorem 1 *Given n points and n families of concentric circles each with at most k circles in the plane, the maximal number of incidences between the points and the circles is*

$$I(n,k) = O_\varepsilon\left(n^{\frac{5}{3}} \cdot \left(\frac{k}{n^{1/3}}\right)^{\frac{5e-1}{7e+1}+\varepsilon}\right) = O(n^{1.4571}k^{.6286}),$$

where e is the base of the natural logarithm and $0 < \varepsilon \leq 1/e$ is an arbitrarily small positive number.

We expect that our bound is not best possible. Especially for small values of k there is a better bond than ours. Akatsu, Tamaki, and Tokuyama [2] showed by a straightforward

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application of Székely's [15] method that $I(n,k) = O(n^{10/7}k^{5/7})$. Theorem 1 improves this bound for $k > n^{1/3}$. It has interesting applications to the k most frequent (or favorite) distances problem and to pattern matching.

Similarly to $I(n,k)$, function $I(n,\ell,m)$ can be defined as the maximal number of incidences between n points and ℓ circles in the plane such that the circles have exactly m distinct centers. Our theorem gives a new bound for a special case of $n = m$, but function $I(n,\ell,m)$ deserves interest in itself. For $\ell = m$, that is where all circles have distinct centers, we might hope that $I(n,\ell,\ell) = O(n^{2/3}\ell^{2/3} + n + \ell)$ corresponding to the Szemerédi-Trotter bound. The number of incidences between points and circles is known to be higher than this only in highly symmetric configurations, where there are many concentric families of circles.

1.1 The k most frequent distances

For a point set P, denote by $f(P,k)$ the number of occurrences of the k most frequent distances in the point set P. Let $f_n(k) = \max_{|P|=n} f(P,k)$. For example, the maximal number of unit distances is $f_n(1)$, ($f_n(1) = O(n^{4/3})$ [15]). Akatsu, Tamaki, and Tokuyama proved [2], using Székely's method, that $f_n(k) = O(n^{10/7}k^{5/7})$. This in turn implies another result of Székely [15], i.e. the number of distinct distances determined by n points in the plane is at least $\Omega(n^{4/5})$. Theorem 1 is somewhat stronger than the corresponding bound on $f_n(k)$ implied by it (see below) in that one can choose the k most frequent distances separately for every one of the n points.

Corollary 2

$$f_n(k) = O_\varepsilon\left(n^{\frac{5}{3}} \cdot \left(\frac{k}{n^{1/3}}\right)^{\frac{5e-1}{7e+1}+\varepsilon}\right) = O(n^{1.4571}k^{.6286}),$$

where e is the base of the natural logarithm and $0 < \varepsilon \leq 1/e$ is an arbitrarily small positive number.

This bound improves earlier bounds in the interval $n^{1/3} \leq k \leq n^{4/(5-1/e)} = n^{0.864}$, i.e. the same interval where the Akatsu, Tamaki, Tokuyama bound was the best. It also implies the best known lower bound [14, 17] for the number t of distinct distances of n points in the plane: putting $f_n(t) = \binom{n}{2}$ we obtain $t = \Omega_\varepsilon(n^{4e/(5e-1)-\varepsilon}) = \Omega(n^{0.864})$. Comparing with earlier bounds on $f_n(k)$, we have the following.

Corollary 3 $f_n(k) = O_\varepsilon(\min(n^{4/3}k, n^{\frac{10e+2}{7e+1}-\varepsilon}k^{\frac{5e-1}{7e+1}+3\varepsilon}, n^2))$.

Remark 4 Corollary 2 implies a new bound for the *inner product* of two planar point sets introduced by Akatsu, Tamaki, and Tokuyama [2]. The *inner product* of P and Q plays a crucial role in the complexity of the following pattern matching problem. Given two point sets P and Q, find a subset $P' \subset P$ of largest cardinality such that there is a rigid motion μ with $\mu(P') \subset Q$.

The *inner product* for P and Q is defined as $\lambda(P,Q) = \sum_{d\in D(P)} h_P(d) \cdot h_Q(d)$ where $D(P)$ denotes the set of distances in the point set P and $h_P(d)$ denotes the number of occurrences of distance d in P. $\lambda(n,m) = \max_{|P|=n,|Q|=m} \lambda(P,Q)$.

Our Corollary 2 together with the computations of [2] yield

$$\lambda(n,m) = n^{\frac{10e+2}{7e+1}-\varepsilon} m^{\frac{10e+2}{7e+1}-\varepsilon} \sum_{k=1}^{m^{\frac{4e}{5e-1}}} O_\varepsilon\left(k^{2\left(\frac{5e-1}{7e+1}+3\varepsilon\right)-2}\right) = O(n^{1.4576}m^{1.6798}),$$

improving the earlier bound $\lambda(n,m) = O(n^{10/7}m^{62/35}) = O(n^{1.429}m^{1.771})$ in the interval $n^{1/3} \leq m \leq n$.

1.2 Terminology

A topological (multi-)graph is a (multi-)graph $G(V, E)$ drawn in the plane such that the vertices of G are represented by distinct points in the plane, and its edges by simple arcs between the corresponding point pairs. Any two arcs representing distinct edges have finitely many points in common. We will make no notational distinction between the vertices (resp., edges) and the points (resp., arcs) representing them.

Unlike in the standard definition of topological graphs, we allow arcs representing edges of G to pass through other vertices. Such topological graphs were first employed by Pach and Sharir [10]. Two edges of a topological graph are said to form a *crossing*, if they have a common point which is not an endpoint of both curves. The *crossing number* of a *topological* graph or multigraph is the total number of crossing pairs of edges. The *crossing number* of an *abstract* graph or multigraph G is the minimum crossing number over all possible representations (i.e., drawings) of G as a topological graph. We remark the crossing number defined here is only one of several alternatives, see [12].

2 Interpreting Beck's theorem

We state here the theorem of Szemerédi and Trotter, which is used all over this paper. It comes in two equivalent formulations, both stated below. The current record for the constant factor is due to Pach and Tóth [11]

Theorem 5 (Szemerédi-Trotter [16, 11]) *Given n distinct points in the plane, we call a line m-rich if it passes through at least m of them.*

(a) *Given n distinct points and ℓ distinct lines in the plane, the number of point-line incidences is*

$$O(n^{2/3}\ell^{2/3} + n + \ell).$$

(b) *Given n distinct points in the plane and an integer $m \geq 2$ the number of incidences between the points and the m-rich lines is*

$$I_m = O(n^2/m^2 + n).$$

(c) *Given n distinct points in the plane and an integer $m \geq 2$, the number of m-rich lines is*

$$L_m = O(n^2/m^3 + n/m).$$

All of these bounds are asymptotically tight.

Proof. We just give the straightforward proof of parts (b) and (c) from the standard formulation (a). Given n distinct points in the plane and $m \geq 2$ we clearly have $L_m \leq I_m/m$ and part (a) gives $I_m = O(n^{2/3}L_m^{2/3} + n + L_m) = O(n^{2/3}I_m^{2/3}/m^{2/3} + n + I_m/m)$. From which one concludes that either $I_m = O(n^2/m^2)$, or $I_m = O(n)$, or $m = O(1)$. Notice that since $m \geq 2$ we have $I_m < n^2$ and thus the case $m = O(1)$ is also covered by $I_m = O(n^2/m^2)$ proving assertion (b). Now part (c) follows from (b) and $L_m \leq I_m/m$. □

We prove here a detailed formulation of Beck's theorem [4]. The original theorem of Beck (stated as Theorem 7) will follow as a simple corollary.

Theorem 6 *Given a set P of n points in the plane and a set F of f pairs of points from P one of the following statements holds:*

1. *At least $f/4$ pairs of F are on lines incident to at least cf/n points.*
2. *At least $f/4$ pairs of F are on lines incident to at most Cn^2/f points.*

Here c and C are positive absolute constants.

The proof we present here is a simple extension of the standard proof of Beck's theorem from the Szemerédi-Trotter theorem.

Proof. For $u < v$ let $N_{u,v}$ be the number of pairs of distinct points of P determining a line going through at least u but at most v points of P. Using Theorem 5(c), the number of u-rich lines is $O(n^2/u^3 + n/u)$. Clearly each line going through at most v points of P is determined by at most v^2 pairs from P, thus we have

$$N_{u,v} = O(n^2v^2/u^3 + nv^2/u).$$

We get a better bound by partitioning the $[u, v]$ interval first:

$$N_{u,v} \le \sum_{i=0}^{\lfloor \log(v/u) \rfloor} N_{2^i u, 2^{i+1}u} = \sum_{i=0}^{\lfloor \log(v/u) \rfloor} O\left(\frac{4n^2}{2^i u} + 4n2^i u\right) =$$

$$O\left(\frac{n^2}{u} \sum_{i=0}^{\lfloor \log(v/u) \rfloor} 2^{-i} + nu \sum_{i=0}^{\lfloor \log(v/u) \rfloor} 2^i\right) = O(n^2/u + nv).$$

So we have

$$N_{u,v} \le C_0(n^2/u + nv)$$

for an appropriate constant $C_0 > 0$.

We let now $C = 4C_0$ and $c = 1/C$ and set $u = Cn^2/f$ and $v = cf/n$. If $u \ge v$ the statement of the Theorem is trivial. Otherwise we have

$$N_{u,v} \le C_0(f/C + cf) = f/2.$$

Thus at most $f/2$ of the pairs in P^2 determine a line going through at least u but at most v points of P. Therefore at least $f/2$ of the f pairs in F are not in this category. Thus either $f/4$ of the pairs in F determine a line going through less than u points of P or at least $f/4$ of the pairs in F determine a line going through more than v points of P, as required. □

Specifically, taking F to be all the pairs of distinct points from P, we obtain the following.

Theorem 7 (Beck [4]) *Given n points in the plane, at least one of the following two statements holds:*

1. *There is a line incident to $\Omega(n)$ points.*
2. *There are at least $\Omega(n^2)$ lines incident to at least two points.*

3 Proof of Theorem 1

Consider a set P of n points, a set Q of n center points (points of P and Q may coincide), and at most k concentric circles around each point of Q for a fixed natural number k. Let F denote the pairs of points (p, q) where $p \in P$, $q \in Q$, and p is incident to a circle around q. Put $f = |F|$ be the number of point-circle incidences and apply Theorem 6 to $Q \cup P$ and F.

In the first case, at least $f/4$ pairs are on lines incident to $\Omega(f/n)$ points. According to the Szemerédi-Trotter Theorem 5 (b), the number of incidences between such lines and points is $O(n^4/f^2 + n)$. On each line, one point of Q may occur in at most $2k$ pairs. So we have $f/4 \le 2kO(n^4/f^2 + n) = O(n^4k/f^2 + nk)$, which clearly implies the required bound for $1 \le k \le n$.

The second case of Theorem 6 is considered in the rest of the proof. Let F' be the set of at least $f/4$ pairs of F on lines incident to $O(n^2/f)$ points. For a point $q \in Q$ let $P_q = \{p \in P | \ pq \in F'\}$. Clearly $\sum_{q \in Q} |P_q| = |F'| \ge f/4$. Consider the set $\mathcal{C}_q$ of all circles centered at $q \in Q$ that contain at least one point of P_q.

Let s be a large integer to be chosen later. The value of s will depend on ε alone. We treat s and all other parameters depending on ε alone as constants. After deleting at most $(s-1)$ points from each circle in $\mathcal{C}_q$, partition the remaining points into pairwise disjoint consecutive s-tuples $(x_1, x_2, \ldots, x_s) \in P^s$. We can also make sure that the circular arcs corresponding to an s-tuple never intersects the ray parallel to the positive x axis and starting at q.

The number of such s-tuples over all circles is $t = \Omega(f/s) = \Omega_\varepsilon(f)$, because we deleted at most $(s-1)kn < f/8$ points (or otherwise $f = O_\varepsilon(kn)$ and we are done).

A line ℓ is called *rich* if ℓ is incident to at least m points in Q, where m is a number to be specified later. An s-tuple $(x_1, x_2, \ldots, x_s)$ is said to be *good* if the bisector of at least one of the segments $x_i x_j$, $1 \le i < j \le s$ is not rich; otherwise it is called *bad*.

Denote by g the number of good s-tuples.

Define a topological multigraph G on the vertex set $V = P$, as follows. If an s-tuple $(x_1, x_2, \ldots, x_s)$ is good, add to the graph one edge between a pair of points from $\{x_1, x_2, \ldots, x_s\}$ whose bisector is not rich. We generate exactly one edge for each good s-tuple. Draw each such edge along the circular arc determined by the s-tuple.

The number of vertices of G is $|V| = n$; the number of edges of G is $|E| = g$. The graph G may have multiple edges when two points, u and v, happen to belong to more than one good s-tuples, associated with different points of Q (as centers of the corresponding circles). However, the multiplicity of each edge is at most m, because all of these points of Q must lie on the bisector of u and v, which, by assumption, is not rich.

The following lemma of [15] is a straightforward extension of a result of Ajtai, Chvátal, Newborn, and Szemerédi [1] and of Leighton [8], to topological *multi*graphs. As we pointed out in the introduction, we use a slightly non-standard definition of topological multigraphs, which allows edges to pass through vertices, but Székely's proof applies *verbatim* to this case as well.

Lemma 8 (Székely [15]) *Let $G(V, E)$ be a topological multigraph, in which every pair of vertices is connected by at most m edges. If $|E| \ge 5|V|m$, then the crossing number of G is*

$$cr(G) \ge \frac{\beta |E|^3}{m|V|^2},$$

for an absolute constant $\beta > 0$.

Apply Lemma 8 to the graph G defined above, with

$$m = c_\varepsilon f^3/(n^4 k^2),$$

where $c_\varepsilon > 0$ is a small constant only depending on ε. We distinguish two cases. If the condition in the lemma is not satisfied, then $g = |E| < 5|V|m = 5c_\varepsilon f^3/(n^3k^2)$. Using the Akatsu, Tamaki, Tokuyama bound [2] on f and, by choosing c_ε sufficiently small, we have $g \le t/2$. Otherwise, according to the statement,

$$cr(G) \ge \frac{\beta g^3}{(c_\varepsilon f^3/(n^4k^2)) \cdot n^2} = \frac{\beta g^3}{c_\varepsilon f^3} \cdot n^2 k^2.$$

As the edges of G are constructed along at most nk circles (at most k concentric circles around each point of Q), and two circles have at most two common points, each responsible for at most a single crossing, so we clearly have

$$cr(G) \le 2 \cdot \binom{nk}{2} \le n^2 k^2.$$

Comparing the last two inequalities, we obtain, just as in the previous case, that $g \le t/2$, provided that c_ε is chosen sufficiently small.

Therefore, we can conclude that the number of bad s-tuples is $t - g \ge t/2 = \Omega_\varepsilon(f)$.

Let us recall the main result of the paper [17]. For a real N by s matrix $A = (a_{ij})$ we define

$$S(A) = \{a_{ij} + a_{ij'} \mid 1 \leq i \leq N, 1 \leq j < j' \leq s\}.$$

Let e stand for the base of the natural logarithm.

Lemma 9 (Tardos [17]) *For every $\varepsilon > 0$, there exists an integer $s > 1$ such that for a positive integer N and a real N by s matrix $A = (a_{ij})$, consisting of all distinct entries, we have*

$$|S(A)| = \Omega_\varepsilon(N^{1/e-\varepsilon}).$$

This lemma is used there to prove the following generalization:

Lemma 10 (Tardos [17]) *For every $\varepsilon > 0$, there exists an integer $s > 1$ such that the following holds.*

Let $N \geq k$ be positive integers and let $A = (a_{ij})$ be an N by s real matrix consisting of Ns pairwise distinct entries. If $\max_j a_{ij} < \min_j a_{i+1,j}$ holds for all but at most $k-1$ of the indices $i = 1, \ldots, N-1$ we have

$$|S(A)| = \Omega_\varepsilon\left(\frac{N}{k^{1-1/e+\varepsilon}}\right).$$

Notice that Lemma 9 is the $k = N$ special case of Lemma 10. Here we need a further generalization of Lemma 10 for the case where not all the entries of A are distinct. The proof of Lemma 10 readily generalizas to this case.

Lemma 11 *For every $\varepsilon > 0$, there exist an integer $s > 1$ such that the following holds.*

Let $N \geq k$ be positive integers and let $A = (a_{ij})$ be an N by s real matrix not having two equal entries in the same row. If $\max_j a_{ij} < \min_j a_{i+1,j}$ holds for all but at most $k-1$ of the indices $i = 1, \ldots, N-1$ we have

$$|S(A)| = \Omega_\varepsilon\left(\frac{N}{k^{1-1/e+\varepsilon} M^{1/e-\varepsilon}}\right),$$

where M is the maximum multiplicity with which a number appears as an entry of A.

Notice here that $M \leq k$ by definition. In case we do not bound M at all we only have the trivial bound $|S(A)| \geq N/k$.

Proof. The dependence of s on ε is the same as in Lemma 9. For a matrix A we find $z \geq N/(3k)$ pairwise disjoint real intervals $I_1, \ldots, I_z$ each containing all entries of at least k rows of A. This can be done from left to right on the real line using a greedy strategy. Let A_i be the submatrix of A consisting of the k rows fully contained in the interval I_i for $i = 1, \ldots, z$. Now choose an N_0 by s submatrix A'_i of A_i with all entries distinct. Greedy strategy can ensure $N_0 \geq k/(Ms)$ since selecting a row of A_i to be contained in A'_i rules out at most $s(M-1)$ other rows. Now apply Lemma 9 to get $|S(A'_i)| = \Omega_\varepsilon(N_0^{1/e-\varepsilon})$. Clearly $S(A'_i) \subset 2I_i$, thus sets $S(A'_i)$ are pairwise disjoint. All of them are contained in $S(A)$, so we have

$$|S(A)| = \Omega_\varepsilon(zN_0^{1/e-\varepsilon}) = \Omega_\varepsilon\left(\frac{N}{k}\left(\frac{k}{M}\right)^{1/e-\varepsilon}\right),$$

as claimed. □

The present authors expect that Lemma 9 and thus lemma 11 are not tight. However, Ruzsa gave an construction showing that Lemma 9 would be false with the stronger statement $|S(A)| = \Omega(N^{1/2})$.

We set the constant s depending on the desired constant $\varepsilon > 0$ as provided in Lemma 11 and let $\alpha = 1/e - \varepsilon$.

We apply Lemma 11 to the system N_q of bad s-tuples along the circles $\mathcal{C}_q$ centered at a point $q \in Q$. Consider the mapping that maps each point $u \neq q$ to the *orientation* of the ray $\vec{qu}$, i.e., to the counterclockwise angle in $[0, 2\pi)$ between the positive x-axis and $\vec{au}$. Note that when forming the s-tuples we made sure that the circular arc corresponding to an s-tuple of N_q does not intersect the ray mapped to 0. We construct an $|N_q|$ by s matrix $A_q = (a_{ij}^{(q)})$. The images of the points of the s-tuples in N_q form the rows of A_q. Notice that if the rows corresponding to s-tuples on the same circle form consequitive blocks with their natural order then the condition $\max_j a_{ij}^{(q)} < \min_j a_{i+1,j}^{(q)}$ holds for all but at most $k-1$ of the indices $i = 1, \ldots, |N_q| - 1$. By the construction of P_q, any number in $[0, 2\pi)$ appears as an entry in A_q at most $M = O(n^2/f)$ times. By construction, each orientation in the set $S(A_q)$ is twice the orientation of a rich line, which is the bisector of some pair of points on the same circle around $q \in Q$, and thus passes through q. Hence q is incident to at least $|S(A_q)|/2$ rich lines as two orientations can correspond to the same line. Now Lemma 11 gives that

$$|S(A_q)| = \Omega_\varepsilon\left(\frac{|N_q|}{k^{1-\alpha}M^\alpha}\right).$$

Therefore, I, the number of incidences between rich lines and points of Q, satisfies

$$I = \sum_{q \in Q} \Omega_\varepsilon\left(\frac{|N_q|}{k^{1-\alpha}M^\alpha}\right) = \Omega_\varepsilon\left(\frac{f}{k^{1-\alpha}M^\alpha}\right) = \Omega_\varepsilon\left(\frac{f^{1+\alpha}}{k^{1-\alpha}n^{2\alpha}}\right), \tag{1}$$

as we have $\sum_{q \in Q} |N_q| = t - g = \Omega_\varepsilon(f)$.

The same number can be estimated from above, using the Szemerédi-Trotter theorem. By Theorem 5(b),

$$I = O\left(\frac{n^2}{m^2} + n\right) = O_\varepsilon\left(\frac{n^{10}k^4}{f^6} + n\right). \tag{2}$$

If, instead of (1) we use the simpler consequence $I = \Omega(f/k)$ and contrast it with (2), we obtain the Akatsu, Tamaki, Tokuyama bound of $f = O(n^{10/7}k^{5/7})$. But for $k > n^{1/3}$ contrasting Equalities (1) and (2) gives more, we obtain that

$$\Omega_\varepsilon\left(\frac{f^{1+\alpha}}{k^{1-\alpha}n^{2\alpha}}\right) = O_\varepsilon\left(\frac{n^{10}k^4}{f^6} + n\right),$$

yielding the statement of Theorem 1 by simple rearrangement. □

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