# Separating convex sets by straight lines

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**Abstract**

We answer some questions of Tverberg about separability properties of families of convex sets. In particular, we show that there is a family of infinitely many pairwise disjoint closed disks, no two of which can be separated from two others by a straight line. No such construction exists with equal disks. We also prove that every uncountable family of pairwise disjoint convex sets in the plane has two uncountable subfamilies that can be separated by a straight line.

## 1 Introduction

In 1979, Helge Tverberg [Tv79] initiated the investigation of the following problem. Given two positive integers, $k$ and $l$, what is the smallest number $n = n(k,l)$ such that for any family $\mathcal{F}$ of pairwise disjoint compact convex sets in the plane, one can find a straight line which has at least $k$ members of $\mathcal{F}$ on one of its sides and at least $l$ members on the other? Clearly, we have $n(1,1) = 2$. Improving the original bound of Tverberg, Hope and Katchalski [HK90] showed that $n(1,k) \leq 12(k-1)$ for every $k \geq 2$. (Their proof is based on an old theorem of L. Fejes Tóth [Fe53]. For some other related results, see [GG45], [Ha47], [FF73], [Fe87], [AKP89], [CRUZ92], [RT93].)

However, somewhat surprisingly, $n(2,2)$ does not exist. K. P. Villanger (see [Tv79]) constructed an infinite family $\mathcal{F}$ of pairwise disjoint segments in the plane so that there is *no* straight line that has at least two members of $\mathcal{F}$ on both of its sides. Here we describe a similar but

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somewhat simpler construction with the same property, using only *unit* segments.

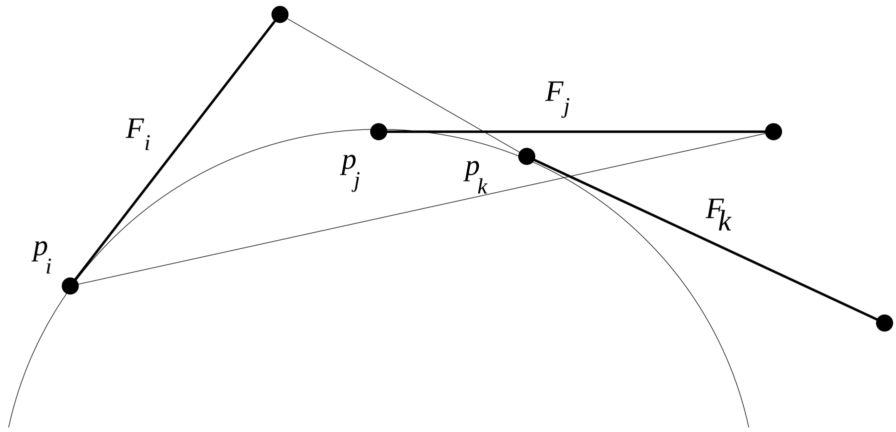


**Figure 1.**

Let $C$ be a unit circle, and let $p_1, p_2, \ldots$ be an infinite sequence of points on $C$, in clockwise order, such that $|p_i - p_{i+1}| = 10^{-3^i}$. Let $F_i$ denote the clockwise oriented unit segment starting at $p_i$ and tangent to $C$ $(i = 1, 2, \ldots)$. To see that $\mathcal{F} = \{F_1, F_2, \ldots\}$ meets the requirements, it is enough to show that, for any $1 \leq i < j < k$, every line $\ell$ separating $F_j$ from $F_k$ must intersect $F_i$. Indeed, as the segment connecting $p_k$ to the far end of $F_i$ intersects $F_j$, $F_i$ cannot lie on the same side of $\ell$ where $F_k$ is. It cannot lie on the other side of $\ell$ either, because $|p_k - p_j|$ is much smaller than $|p_j - p_i|$, so the segment connecting $p_i$ to the far end of $F_j$ must intersect $F_k$. (See Figure 1.)

**Definition.** A family of pairwise disjoint sets in the plane is said to be *separable*, if any two sets can be separated by a straight line which does not intersect any member of the family. Instead of saying that a family contains a separable subfamily of size $m$, we sometimes say that it has $m$ separable members.

Note that in some papers (e.g., in [PT00], [FF73]) families with the above property are called strongly separable or totally separable.

The above construction also shows that there exist infinitely many pairwise disjoint straight-line segments in the plane, no three of which are separable. One may be tempted to believe that there is no such example with 'fat' sets. However, we prove that this is not the case.

**Theorem 1.** *There is a family of infinitely many pairwise disjoint disks (or squares) in the plane, which has no three separable members.*

In Section 2, we prove Theorem 1 in a somewhat stronger form (Theorem 2.3), and we also establish some simple positive results. In particular, these results imply that every infinite family of disks of roughly *equal size* has an infinite separable subfamily, and the same is true for infinite families of *axis-parallel* rectangles (Theorems 2.4 and 2.5).

The family of sets $\mathcal{F}$ depicted in Figure 1 has *countably* many members, no pair of which can be separated from another pair by a straight line. Tverberg [Tv79] asked whether there exists such a construction with *uncountably* many convex sets. We answer this question in the negative, in the following strong sense.

**Theorem 2.** *Every uncountable family of pairwise disjoint convex sets in the plane has two uncountable subfamilies that can be separated by a straight line.*

Our original proof of Theorem 2 was simplified by V. Totik [To99]. We present the simplified proof in Section 3, while the last section contains some related problems and concluding remarks.

## 2 Entangled sets

**Definition 2.1** A sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of pairwise disjoint compact convex sets in the plane is said to be *entangled*, if at least one of the following conditions is satisfied:

- for every $1 \leq i < j < k$, any straight line separating $F_i$ from $F_j$ intersects $F_k$;
- for every $1 \leq i < j < k$, any straight line separating $F_j$ from $F_k$ intersects $F_i$.

Clearly, an entangled sequence $\mathcal{F}$ cannot have three separable elements. Furthermore, there is no straight line which has at least two elements of $\mathcal{F}$ on both of its sides. The construction described in the Introduction proves the following.

**Theorem 2.2.** *There exists an infinite sequence of entangled unit segments in the plane.* $\square$

We prove Theorem 1 in the following stronger form.

**Theorem 2.3** *There exists an infinite sequence of (i) entangled disks, (ii) entangled squares in the plane.*

**Proof:** We start the construction with two disjoint, but almost touching, disks (or squares), $F_1$ and $F_2$, with the property that the counterclockwise angle between the $x$-axis and any line separating them is between $\varepsilon/4$ and $\varepsilon$, for some small positive constant $\varepsilon$. Assume that, for some $n \geq 2$, we have already found disks (squares, respectively) $F_1, \ldots, F_n$ with the property that, for every $1 \leq i < j < k \leq n$, any line separating $F_i$ and $F_j$ cuts through $F_k$. Also assume, inductively, that the angle between the $x$-axis and every line separating two members of $\{F_1, \ldots, F_n\}$ is between $\varepsilon_n = \varepsilon/2^n$ and $\varepsilon$.

Let $F$ denote the convex hull of $\cup_{i=1}^n F_i$. Take a huge disk (square, resp.) $F'_{n+1}$ touching $F$ at a point $p$ such that the angle between the $x$-axis and the tangent to $F'_{n+1}$ at $p$ is $3\varepsilon_n/4$. Clearly, every line separating two members of $\{F_1, \ldots, F_n\}$ will cut through $F'_{n+1}$, provided that the radius (sidelength, resp.) of $F'_{n+1}$ is sufficiently large.

Let $F_{n+1}$ denote the set obtained from $F'_{n+1}$ by slightly shrinking it about its center. Obviously, $F_1, \ldots, F_{n+1}$ will satisfy the induction hypothesis. That is, for every $1 \leq i < j < k \leq n+1$, any line separating $F_i$ and $F_j$ cuts through $F_k$, and the angle between the $x$-axis and any line separating two of the sets is between $\varepsilon_{n+1} = \varepsilon_n/2$ and $\varepsilon$. □

It is impossible to combine the features of Theorems 2.2 and 2.3 by constructing an infinite sequence of entangled unit disks or squares, because every large family of 'fat' sets of roughly the same size contains a large separable subfamily. We formulate this result in Euclidean spaces of arbitrary dimension. Extending the definition on page 2, we call a family of pairwise disjoint compact convex sets in $d$-space *separable*, if every pair can be separated by a hyperplane which does not intersect any member of the family.

**Theorem 2.4.** *Let $R > r > 0$ be fixed, and let $\mathcal{F}$ be a family of $n$ pairwise disjoint compact convex sets in $d$-space, each containing a ball of radius $r$ and contained in another ball of radius $R$. Then $\mathcal{F}$ has a separable subfamily with at least $cn$ members, where $c = c(r, R, d) > 0$ is a constant.*

**Proof:** Choose a number $s$ randomly and uniformly in $[0, 4dR]$, and cut the space into cubes along the hyperplanes $x_i = 4dRk + s$, for every integer $k$ $(i = 1, \ldots, d)$. The expected number of members of $\mathcal{F}$ intersected by these hyperplanes is at most $n/2$.

Let $v_d$ denote the volume of the $d$-dimensional unit ball. There are at most $(4dR)^d/(v_d r^d)$ members of $\mathcal{F}$ contained in the same cube, so we can find a separable subfamily of size at least $\left(v_d r^d/(2(4dR)^d)\right) n$. □

One cannot strengthen Theorem 2.3(ii) by exhibiting an infinite sequence of entangled *axis-parallel* squares, because of the following observation.

**Theorem 2.5.** *Any family $\mathcal{F}$ of $n$ pairwise disjoint axis-parallel boxes in $R^d$ has at least $n/(c\log n)^d$ separable members, where $c > 0$ is an absolute constant.*

**Proof:** Let the projection of the box $B \in \mathcal{F}$ to the $i$-th coordinate be $[B_{i,1}, B_{i_2}]$. For the separation we use only axis-parallel hyperplanes. This allows us to assume without loss of generality that the sets $\{B_{i,b}|B \in \mathcal{F}, b \in \{0,1\}\}$ consists of (at most $2n$) consecutive integers for every $i = 1, \ldots, d$, as changing these values but leaving their order does not alter the problem. Our assumption implies that all sides of the boxes in $\mathcal{F}$ are between 1 and $2n$. There are positive numbers $l_1, \ldots, l_d$ such that $\mathcal{F}$ has at least $n/\lceil \log_{3/2}(2n)\rceil^d$ members whose sidelength in the $i$-th coordinate belong to the interval $[l_i, 3l_i/2]$, for every $i = 1, \ldots, d$. Let $\mathcal{F}'$ denote the subfamily consisting of these members. As in the proof of the previous statement, for every $i = 1, \ldots, d$, pick a number $s_i$ randomly and independently in $[0, 2l_i]$. The expected number of members of $\mathcal{F}'$, disjoint from all axis-parallel hyperplanes $x_i = 2jl_i + s_i$ (where $i = 1, \ldots, d$, and $j$ is an integer), is at least $|\mathcal{F}'|/4^d$. As no two members of $\mathcal{F}'$ fit into the same cell determined by these hyperplanes, we obtain that $\mathcal{F}$ has at least $|\mathcal{F}'|/4^d \geq n/\left(4\lceil \log_{3/2}(2n)\rceil\right)^d$ separable members. □

## 3 Proof of Theorem 2

Our original proof of Theorem 2 was greatly simplified by V. Totik [To99].

Let $\mathcal{F}$ be an uncountable family of pairwise disjoint convex sets in the plane. Since there are no more than countably many disjoint sets of positive measure, we may assume that every member of $\mathcal{F}$ has zero measure. That is, $\mathcal{F}$ consists of points, segments, half-lines, and lines. There are uncountably many members that fall into one of these four categories, so we can ignore all other members of $\mathcal{F}$. If all members of $\mathcal{F}$ are points, then the proof is straightforward. If $\mathcal{F}$ consists of straight lines, then the situation is even simpler, because two disjoint lines must be parallel. So we can assume that all elements of $\mathcal{F}$ are segments or all of them are half-lines.

If all sets in $\mathcal{F}$ share an endpoint, we are done. Thus, we may assume without loss of generality that every point is an endpoint of only countably many members of $\mathcal{F}$. Similarly, we may assume that there are no more than countably many pairwise parallel half-lines in $\mathcal{F}$.

We say that two members of $\mathcal{F}$ are *close* to each other, if their closures have a point in common, or they are parallel half-lines. Consider three distinct elements of $\mathcal{F}$. We claim that $\mathcal{F}$ has only a countable number of members that are close to all three of them. To see this, notice that every member of $\mathcal{F}$ close to $F \in \mathcal{F}$

- either contains an endpoint of $F$,
- or shares an endpoint with $F$,
- or is a half-line parallel to $F$,
- or has an endpoint in $F$.

The members of $\mathcal{F}$ satisfying any of the first three conditions form a countable set. Obviously, no member of $\mathcal{F}$ satisfies the last condition for three distinct $F$'s, as every member of $\mathcal{F}$ has at most two endpoints.

This implies that for all but at most two members of $\mathcal{F}$ there are uncountably many members in $\mathcal{F}$ *not* close to them. Notice that if two members of $\mathcal{F}$ are not close to each other, then there is a straight line separating them, which passes through at least two points of rational coordinates. Let us call such a line *rational*.

Since there are only countably many rational lines, every member $F \in \mathcal{F}$, with at most two exceptions, can be separated from uncountably many other members by a single rational line $\ell_F$. We conclude that there is an uncountable subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $\ell_F$ is the same for every $F \in \mathcal{F}'$. Obviously, this line has uncountably many members on both of its sides.

## 4 Remarks

**4.1.** As was mentioned in the Introduction, Tverberg [Tv79] discovered that, for every large family $\mathcal{F}$ of pairwise disjoint compact convex sets in the plane (even for an entangled sequence of sets), there is a straight line separating one member of $\mathcal{F}$ from many other members. It was also pointed out in [Tv79] that no such theorem holds in 3-space. To see this, take a *finite* family $\mathcal{F}$ of pairwise disjoint straight lines, no three of which are parallel to the same plane. Then any plane separating two members of $\mathcal{F}$ must cross every other member, and this property is preserved when we intersect all members of $\mathcal{F}$ with a sufficiently large ball to obtain a family of *compact* sets.

However, if we start with an *infinite* family of lines, the above property may be violated when we replace the lines by their intersection with the ball. Nevertheless, it is not hard to establish the following

**Proposition 4.2.** *There exists an infinite family $\mathcal{F}$ of pairwise disjoint unit segments in 3-space such that there are no two members that can be separated from a third by a plane.*

**Proof:** We fix a unit segment $pq$ and define the segments $F_1, F_2, \ldots,$ recursively. Assume that we have already defined the first $n$ pairwise disjoint segments, $F_1, F_2, ..., F_n$, such that

- $F_i$ and $pq$ have an interior point in common $(1 \leq i \leq n)$,
- the directions of $F_1, \ldots, F_n$ and $pq$ are in general position,
- no two members of $\{F_1, \ldots, F_n\}$ can be separated from a third by a plane.

Let $r$ be a point contained in the open segment $pq$ that does not belong to any $F_i$ $(1 \leq i \leq n)$. Let $F_{n+1}$ be a unit segment passing through $r$, whose direction is in general position with respect to the directions of $F_1, \ldots, F_n$ and $pq$. If the endpoints of $F_{n+1}$ are close enough to $p$ and $q$, then $F_1, \ldots, F_{n+1}$ satisfy the above conditions for $n+1$. $\square$

**4.3** Notice that it was a crucial feature of the above construction that all segments $F_i$ cross a fixed unit segment. Indeed, every family $\mathcal{F}$ satisfying the condition in Proposition 4.2 must be bounded, which implies that its members have an accumulation point $pq$ with respect to the Hausdorff distance. If the closure of a member $F_i \in \mathcal{F}$ is disjoint from the closed segment $pq$, then $F_i$ can be separated by a plane from infinitely many members of $\mathcal{F}$.

Although one can find a *continuum* of pairwise skew lines in general position in 3-space, no two of which can be separated by a plane from a third, Proposition 4.2 guarantees the existence of only a *countably* infinite family of unit segments with the same property. Is it true that, for any uncountable family $\mathcal{F}$ of pairwise disjoint bounded convex sets in 3-space, there is a plane which has uncountably many members of $\mathcal{F}$ on both of its sides?

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