Isosceles triangles determined by a planar point set

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Abstract

It is proved that, for any $\varepsilon > 0$ and $n > n_0(\varepsilon)$, every set of n points in the plane has at most $n^{\frac{11e-3}{5e-1}+\epsilon}$ triples that induce isosceles triangles. (Here e denotes the base of the natural logarithm, so the exponent is roughly 2.136.) This easily implies the best currently known lower bound, $n^{\frac{4e}{5e-1}-\epsilon}$, for the smallest number of distinct distances determined by n points in the plane, due to Solymosi–C. Tóth and Tardos.

1 Introduction

In 1946, Erdős [5] raised some notoriously difficult questions about the distribution of distances determined by finite point sets. In particular, he asked what is the smallest number of distinct distances determined by n points in the plane. Denoting this number by $g(n)$, he conjectured that $g(n) \geq cn/\sqrt{\log n}$. The best currently known lower bound follows by a combination of the results of Solymosi–C. Tóth [12] and G. Tardos [18]: for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that

$$g(n) \geq c_\varepsilon \left(n^{\frac{4e}{5e-1}-\epsilon}\right). \quad (1)$$

Here and later in this note, e stands for the base of the natural logarithm.

In a series of papers, Erdős and Purdy [6], [7] initiated the investigation of the distribution of *triangles* (more generally, simplices) in finite point sets. Pach and Sharir [10] pointed out that it readily follows from a result of Szemerédi and Trotter [16], [17] that the maximum number of triples in a set of n points in the plane that induce isosceles triangles is $O(n^{7/3})$.

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The aim of this paper is to improve this bound.

Theorem 1. *For any $\varepsilon > 0$, the number of isosceles triangles spanned by three points of an n-element point set in the plane is*

$$O_\varepsilon\left(n^{\frac{11e-3}{5e-1}+\epsilon}\right) = O(n^{2.137}).$$

The above two problems are intimately related. Indeed, if a point set P determines at most g distinct distances, then around each point $p \in P$ the remaining $n-1$ points lie on g concentric circles. If the numbers of points sitting on these circles are $n_1, n_2, \ldots, n_g$, then there are precisely $\sum_{i=1}^{g}\binom{n_i}{2} \geq g\binom{(n-1)/g}{2}$ isosceles triangles whose two equal sides meet at p. Thus, the total number of isosceles triangles is at least $\frac{n^3}{2g} + O(n^2)$. Therefore, any upper bound on the number of isosceles triangles yields a lower bound on $g(n)$. In particular, Theorem 1 immediately implies inequality (1). In this sense, our Theorem 1 can be regarded as a strengthening of (1).

Theorem 1, in turn, follows from a general upper bound for the number of incidences between a set of points and a set of circles.

Theorem 2. *Let P be a set of n distinct points and let C be a set of ℓ distinct circles in the plane. Let Q denote the set of centers of the circles in C and let $|Q| = m$.*

Then, for any $0 < \alpha < 1/e$, the number I of incidences between the points in P and the circles of C is

$$O_\alpha\left(n + \ell + n^{\frac{2}{3}}\ell^{\frac{2}{3}} + n^{\frac{4}{7}}m^{\frac{1+\alpha}{7}}\ell^{\frac{5-\alpha}{7}} + n^{\frac{12+4\alpha}{21+3\alpha}}m^{\frac{3+5\alpha}{21+3\alpha}}\ell^{\frac{15-3\alpha}{21+3\alpha}} + n^{\frac{8+2\alpha}{14+\alpha}}m^{\frac{2+2\alpha}{14+\alpha}}\ell^{\frac{10-2\alpha}{14+\alpha}}\right).$$

Figure 1 and Table 1 give the best known upper bounds on the number of incidences between n points an ℓ circles around m centers in the plane. Figure 1 defines regions according to the different settings of the parameters n, m, and ℓ, and Table 1 gives the best known bounds for each of these regions. We have $0 < \alpha < 1/e$ and $\varepsilon > 0$ in Table 1 and the constans multiplyer in the O notation depends on the choice of α or ε. As is illustrated by Figure 1, each term of the expression in Theorem 1 provides the best known bound in some nonempty region of the parameters. For all but the first term, our bound is new in the corresponding region or at least in some part of it. In two further regions, the trivial bound nm or the estimate $n^{6/11+3\epsilon}\ell^{9/11-\epsilon}$ found by Aronov and Sharir [2] are the best currently known bounds for the number of incidences.

It is worth ponting out a simple consequence of Theorem 2, which is a generalization of the main result (Theorem 1) in [13].

Corollary 3. *Let P be a set of n distinct points and C be a set of ℓ distinct circles in the plane.*

If among the centers of the circles in C there are at most n distinct points, then for any $0 < \alpha < 1/e$ the number of incidences between the points in P and the circles in C is

$$O_\alpha\left(n^{\frac{5+3\alpha}{7+\alpha}}\,\ell^{\frac{5-\alpha}{7+\alpha}}\right).$$

Proof: Substituting $m = n$ in Theorem 2, the fifth term becomes the required bound. It dominates the other five terms, whenever $\ell < n^{(9-\alpha)/(5-\alpha)}$. For $\ell \geq n^{(9-\alpha)/(5-\alpha)}$, the trivial bound $nm = n^2$ is better than the one in Corollary 3. □

The proof of Theorem 2 is based on the same ideas as [12] and [13]. In particular, all our bounds crucially depend on the following lemma from [13], which is a slight generalization of a result of Tardos [18].

Given a real matrix A, let $S(A)$ denote the set of all reals that can be written as the sum of two distinct entries from the same row of A.

Lemma 4 [13]. *For any $0 < \alpha < 1/e$, there exists an integer $s > 1$ with the following property. For every $N \geq k \geq 0$ and for every N by s real matrix A which does not have two equal entries in the same row and in which for all but at most $k-1$ of the indices $i = 1, \ldots, N-1$, all entries of the i-th row are smaller than all entries in the next row, we have*

$$|S(A)| = \Omega_\alpha\left(\frac{N}{k^{1-\alpha}M^\alpha}\right),$$

where M is the maximum multiplicity of any entry in A.

It is not clear whether Lemma 4 holds for other values of α, larger than $1/e$. I. Ruzsa (personal communication) showed that it is certainly false for $\alpha \geq 1/2$. If Lemma 4 remains true for any $\alpha \geq 1/e$, we obtain that the number of isosceles triangles induced by triples of an n-element point set in the plane is $O_\alpha(n^{(11-3\alpha)/(5-\alpha)})$.

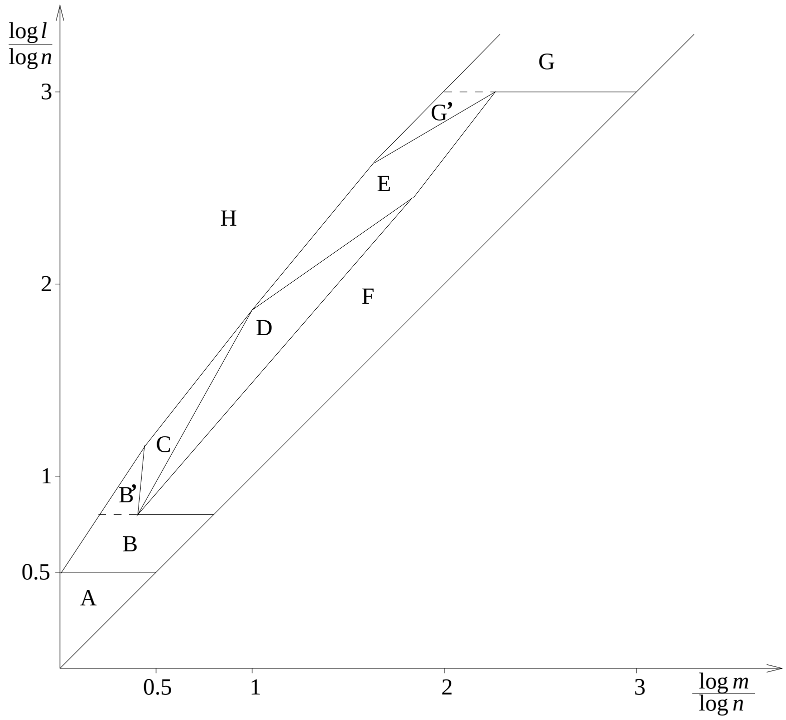

Figure 1.

region	**best known bound**	**source**
A	$O(n)$	[4, 10]
B	$O\left(n^{\frac{2}{3}}\ell^{\frac{2}{3}}\right)$	[2]
B'	$O\left(n^{\frac{2}{3}}\ell^{\frac{2}{3}}\right)$	Theorem 1
C	$O\left(n^{\frac{4}{7}}m^{\frac{1+\alpha}{7}}\ell^{\frac{5-\alpha}{7}}\right)$	Theorem 1
D	$O\left(n^{\frac{12+4\alpha}{21+3\alpha}}m^{\frac{3+5\alpha}{21+3\alpha}}\ell^{\frac{15-3\alpha}{21+3\alpha}}\right)$	Theorem 1
E	$O\left(n^{\frac{8+2\alpha}{14+\alpha}}m^{\frac{2+2\alpha}{14+\alpha}}\ell^{\frac{10-2\alpha}{14+\alpha}}\right)$	Theorem 1
F	$O\left(n^{\frac{6}{11}+3\varepsilon}\ell^{\frac{9}{11}-\varepsilon}\right)$	[2]
G	$O(\ell)$	[4, 10]
G'	$O(\ell)$	Theorem 1
H	nm	trivial

Table 1.

2 An important special case

The aim of this section is to establish the following important special case of Theorem 2, where C consists of the same number, k, of concentric circles around each element of Q.

Proposition 2.1. *Let P be a set of n distinct points, let Q be a set of m distinct points in the plane, and let C be a family of mk circles, consisting of k concentric circles around each point in Q.*

Then, for any $0 < \alpha < 1/e$, the number of incidences between the points in P and the circles in C is

$$O_\alpha\left(n + mk + n^{\frac{2}{3}}m^{\frac{2}{3}}k^{\frac{2}{3}} + n^{\frac{4}{7}}m^{\frac{6}{7}}k^{\frac{5-\alpha}{7}} + n^{\frac{12+4\alpha}{21+3\alpha}}m^{\frac{18+2\alpha}{21+3\alpha}}k^{\frac{15-3\alpha}{21+3\alpha}} + n^{\frac{8+2\alpha}{14+\alpha}}m^{\frac{12}{14+\alpha}}k^{\frac{10-2\alpha}{14+\alpha}}\right).$$

Let I be the set of all pairs (p, q) such that $p \in P$, $q \in Q$, and P is incident to one of the circles around q. We have to give an upper bound on $|I|$.

First, we outline the proof of Proposition 2.1.

We use three parameters, $a, b, s \geq 2$, to partition I as follows. The value of s will solely depend on the choice of $0 < \alpha < 1/e$, so it will be regarded as a constant. The values of a and b will depend on n, m, and k.

For any $(p, q) \in I$, we consider the number of points in P on the line l_{pq} connecting p and q, which are incident to a circle in C around q. We use the Szemerédi–Trotter theorem (Lemma 2.3 below) to bound the number of pairs, for which this is greater than our parameter a. By losing just a few more pairs from I, we partition the remaining pairs into s-tuples and bound their number. The elements of an s-tuple will correspond to s distinct points of P, incident to the same circle in C. If we can choose two of these points so that their perpendicular bisector contains less than b elements of Q, we connect them along the circle C. In this way, we obtain a so-called *topological graph*, a graph Γ drawn by (possibly crossing) continuous arcs. Then we apply Székely's lemma on crossing numbers (Lemma 2.2) to bound the number of edges of Γ and thus the number of s-tuples satisfying this condition. To bound the number of remaining s-tuples, we use Lemma 4 and again the Szemerédi–Trotter theorem.

Next, we work out the details. Let

$$I' = \{(p, q) \in I \ : \ |\{p' \in l_{pq} \cap P \mid (p', q) \in I\}| \leq a\}.$$

For any $q \in Q$, let $P_q = \{p \in P \mid (p, q) \in I'\}$, and identify a set D_q of pairwise disjoint circular arcs on the circles in C around q so that each arc contains precisely s elements of P_q and together they cover all but at most $k(s-1)$ points of P_q.

We can assume without loss of generality that none of these arcs intersects a fixed half-line l_q emanating from q.

Call a line l *rich* if $|l \cap Q| \geq b$. We say that an arc in D_q is *good*, if it contains two points $p, p' \in P_q$ such that the perpendicular bisector of pp' is not rich. Denote by G the set of good arcs in $\cup_{q \in Q} D_q$, and let $B = \cup_{q \in Q} D_q \setminus G$ be the set of all *bad arcs*. Construct a topological graph Γ on the vertex set P, by connecting a single pair of points for each good arc $\beta \in G$. If $\beta \in D_q$, choose these two points $p, p' \in \beta \cap P_q$ so that their perpendicular bisector is not rich and connect them along β. The graph Γ is not necessarily *simple*, i.e., it may contain parallel edges connecting the same pair of points. However, it is not hard to bound the multiplicity of these edges, as follows. All edges between two vertices p and p' are drawn along separate circles in C, whose centers lie on the perpendicular bisector of pp'. If this line is not rich, there are fewer than b such edges. If this line is rich, then by our construction p and p' are not connected at all. Thus, the maximum edge-multiplicity, $m(\Gamma)$, of Γ satisfies

$$m(\Gamma) < b.$$

Let $c(\Gamma)$ denote the *crossing number* of Γ. Since each crossing between two edges of Γ occurs at an intersection point of two circles in C, we clearly have

$$c(\Gamma) \leq 2\binom{|C|}{2} < m^2k^2.$$

On the other hand, the following useful generalization of a well known theorem of Ajtai et al. [1] and Leighton [8], due to L. Székely [15], provides a lower bound for crossing numbers.

Lemma 2.2. [15]. *Let Γ be a topological multigraph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, in which every pair of vertices is connected by at most $m(\Gamma)$ edges.*

If $|E(\Gamma)| \geq 5m(\Gamma)|V(\Gamma)|$, then the crossing number of Γ satisfies

$$c(\Gamma) = \Omega\left(\frac{|E(\Gamma)|^3}{m(\Gamma)|V(\Gamma)|^2}\right).$$

Plugging the last two inequalities into Lemma 2.2, we conclude that the number of good arcs satisfies

$$\begin{aligned} |G| = |E(\Gamma)| &= O\left(|V(\Gamma)|m(\Gamma) + c^{\frac{1}{3}}(\Gamma)m^{\frac{1}{3}}(\Gamma)|V(\Gamma)|^{\frac{2}{3}}\right) \\ &= O\left(nb + n^{\frac{2}{3}}m^{\frac{2}{3}}k^{\frac{2}{3}}b^{\frac{1}{3}}\right). \end{aligned} \quad (2)$$

Now we focus on the set B of bad arcs and estimate their number. Fix $0 < \alpha < 1/e$ and s so that they satisfy the conditions in Lemma 4. Construct an N_q by s

real matrix A_q, where N_q is the number of bad arcs in D_q and each row corresponds to a bad arc. Let the row of A_q assigned to a bad arc $\beta \in B \cap D_q$ consist of the entries $c_1, \ldots, c_s$, where $\beta \cap P_q = \{p_1, \ldots, p_s\}$ and c_i is the angle of the smallest counter-clockwise rotation that takes the reference half-line l_q to the half-line qp_i.

If the rows corresponding to the bad arcs on a circle follow each other in the natural order, the matrix A_q meets the requirements of Lemma 4. By the definition of I' and P_q, we have that the maximum multiplicity of any entry in A_q is $M_q \leq a$. All values in $S(A_q)$ are twice the angles of rich lines going through q, thus Lemma 4 implies that q is incident to $\Omega_\alpha(N_q/(k^{1-\alpha}a^\alpha))$ rich lines. Hence, the total number of incidences between the points in Q and the rich lines is $\Omega_\alpha(|B|/(k^{1-\alpha}a^\alpha)$.

On the other hand, the Szemerédi-Trotter theorem gives an upper bound on the same quantity.

Lemma 2.3 [16],[17]. *(i) The number of lines passing through at least $b \geq 2$ elements of a set of m points in the plane is $O(m/b + m^2/b^3)$.*

(ii) The number of incidences between m points in the plane and all lines passing through at least $b \geq 2$ of them is $O(m + m^2/b^2)$.

(iii) The number of incidences between m points and ℓ lines in the plane is $O(m^{2/3}\ell^{2/3} + m + \ell)$.

Comparing Lemma 2.3 (ii) with the above lower bound for the same quantity, we obtain

$$|B| = O_\alpha\left(mk^{1-\alpha}a^\alpha + m^2k^{1-\alpha}a^\alpha/b^2\right). \tag{3}$$

As each arc in D_q covers a constant number s of the points in P_q, and at most $(s-1)k$ points are not covered, in view of the inequalities (2) and (3), we get

$$|I'| = \sum_{q \in Q} |P_q| \leq s|G| + s|B| + (s-1)mk$$

$$= O_\alpha\left(nb + mk + m^2k^{1-\alpha}a^\alpha/b^2 + k^{\frac{2}{3}}m^{\frac{2}{3}}n^{\frac{2}{3}}b^{\frac{1}{3}}\right). \tag{4}$$

The term $mk^{1-\alpha}a^\alpha$ in the upper bound on $|B|$ is dominated by mk, if we choose our parameter a so that it satisfies $2 \leq a \leq k$. (Such a choice is impossible if $k = 1$, but in that case the bound in Proposition 2.1 is significantly worse than the previously known bounds, cf. [4], [10], [2].)

It remains to bound the number of pairs $(p,q) \in I \backslash I'$. Now we use the Szemerédi-Trotter theorem separately for P and Q. By Lemma 2.3 (i), for any $t \geq 2$, the number of straight lines passi1ng through more than t points of P is $O(n/t + n^2/t^3)$. By Lemma 2.3 (iii), the number of incidences between these lines and the m points of Q is

$$O(m + n/t + n^2/t^3 + n^{2/3}m^{2/3}/t^{2/3} + n^{4/3}m^{2/3}/t^2).$$

Let I_t denote the number of pairs $(p,q) \in I$ such that $t < |\{p' \in l_{pq} \cap P \ : \ (p',q) \in I\}| \leq 2t$. Clearly, each incidence counted above is responsible for at most $2t$ pairs in I_t, whence

$$|I_t| = O(mt + n + n^2/t^2 + n^{2/3}m^{2/3}t^{1/3} + n^{4/3}m^{2/3}/t).$$

Using the fact that $I \setminus I' = \cup_{i=0}^{\lfloor \log(k/a) \rfloor} I_{2^i a}$, we obtain

$$|I \setminus I'| = O\left(mk + n\log k + n^2/a^2 + n^{2/3}m^{2/3}k^{1/3} + n^{4/3}m^{2/3}/a\right).$$

It is not hard to get rid of the logarithmic factor in the last formula. To see this, notice that the $n + n^2/t^2$ terms in the bounds on $|I_t|$ actually bound a value proportional to the number of incidences between P and some lines going through at least t points of P. By Lemma 2.3 (ii), the total number of such incidences for any $t \geq a$ is $O(n + n^2/a^2)$. (Alternatively, one can get rid of the extra logarithmic factor by using the result of [2], which provides better bounds for I in all cases where $n \log n$ would be the leading term.) Thus, we have

$$|I \setminus I'| = O\left(n + mk + n^2/a^2 + n^{2/3}m^{2/3}k^{1/3} + n^{4/3}m^{2/3}/a\right). \tag{5}$$

Putting (4) and (5) together, we get

$$|I| = O_\alpha\left(nb + mk + n^{2/3}m^{2/3}k^{2/3}b^{1/3} + \frac{n^2}{a^2} + \frac{n^{4/3}m^{2/3}}{a} + \frac{m^2k^{1-\alpha}a^\alpha}{b^2}\right). \tag{6}$$

Notice that the above bound holds all $k \geq a \geq 2$ and $b \geq 2$. To minimize this expression, set

$$a = \min\left(k, \max\left(2, n^{\frac{10}{14+\alpha}}m^{\frac{-6}{14+\alpha}}k^{\frac{-5+\alpha}{14+\alpha}}, n^{\frac{16}{21+3\alpha}}m^{\frac{-4}{21+3\alpha}}k^{\frac{-15+3\alpha}{21+3\alpha}}\right)\right),$$

$$b = \max\left(2, n^{\frac{-2}{7}}m^{\frac{4}{7}}k^{\frac{1-3\alpha}{7}}a^{\frac{3\alpha}{7}}\right).$$

In case $a = k$, we have $I = I'$ and Proposition 2.1 follows from (4). In all other cases, the result is true by (6).

3 Proof of Theorem 2

Partition Q into the sets

$$Q_0 = \{q \in Q \ : \ |\{c \in C : \text{the center of } c \text{ is } q\}| \leq \ell/m\},$$

$$Q_i = \{q \in Q \ : \ 2^{i-1}\ell/m < |\{c \in C : \text{the center of } c \text{ is } q\}| \leq 2^i\ell/m\},$$

for $i \geq 1$. We also partition C into the sets

$$C_i = \{c \in C \ : \ \text{the center of } c \text{ is in } Q_i\},$$

for $i \geq 0$. Let C'_i denote the sets obtained from C_i by adding dummy circles to bring the number of circles around each $q \in Q_i$ up to $k_i = \lfloor 2^i \ell / m \rfloor$. Clearly, we have $m_i := |Q_i| \leq m/2^{i-1}$, and the values $\ell_i := |C'_i|$ add up to $O(\ell)$.

Applying Proposition 2.1 to the system (P, Q_0, C'_0), we get that the number of incidences between the points in P and the circles in C'_0 does not exceed the bound in Theorem 2. For the systems (P, Q_i, C'_i), we obtain similar bounds, but their last three terms are multiplied by some constant negative power of 2^i. Notice that we can assume $Q_i = \emptyset$ for $i > \log n$, for a concentric family of circles has at most n elements incident to at least one point in P. Hence, adding up the upper bounds that follow from Proposition 2.1, we readily obtain a weaker version of the bound in Theorem 2, in which the first three terms are multiplied by $\log n$.

In the rest of this proof, we get rid of these unwanted logarithmic factors. In the case of the first term, n, of the expression, this can be achieved by noticing that for all settings of the parameters, when $n \log n$ would be the leading term, the upper bound

$$O(n + \ell + n^{2/3}\ell^{2/3} + n^{6/11+3\epsilon}\ell^{9/11-\epsilon})$$

established by Aronov and Sharir [2] is better and gives $O(n + n^{2/3}\ell^{2/3})$ incidences.

It is even easier to argue for the second term, as not only each $m_i k_i = \ell_i$ is bounded by $O(\ell)$, but we also have $\sum_i \ell_i = O(\ell)$.

We have to work most for the third term, $n^{2/3} m_i^{2/3} k_i^{2/3} = O(n^{2/3}\ell^{2/3})$. In this case, we have to look into the proof of Proposition 2.1. This can be the dominant term for some i only if we choose the parameter b to be 2, and in this case the term appears in our bound, because the number of edges of a certain topological graph Γ_i is at most $O(n + n^{2/3} m_i^{2/3} k_i^{2/3})$. Notice, however, that the union Γ of all topological graphs Γ_i, for which the parameter b was set to be equal to 2, is still a topological graph on n vertices, it still does not have any parallel edges, and its crossing number is at most ℓ^2 (there are at most two crossing pairs for each pair of circles in C). Thus, by Lemma 2.2, Γ has $O(n + n^{2/3}\ell^{2/3})$ edges. Using this bound, instead of bounding the number of edges in each of the graphs Γ_i separately, we can replace the $O(n^{2/3}\ell^{2/3} \log n)$ term with $O(n^{2/3}\ell^{2/3})$.

4 Proof of Theorem 1

The common endpoint of two equal sides of an isosceles triangle is called its *apex.* (An equilateral triangle has three apices.) Consider an n-element point set P in the plane, and let T be the set of ordered triples pqr that induce an isosceles triangle in P, with apex q. Thus, $|T|$ is equal to the number of isosceles triangles induced by P, counted with multiplicities (equilateral triangles are counted six times, all other isosceles triangles twice).

For any $pqr \in T$, let $c(pqr)$ denote the circle centered at q, which passes through p and r. We classify the elements of T according to the order of magnitude of $|c(pqr) \cap P|$, and bound the sizes of the classes separately. Setting a threshold $t := n^{(1-\alpha)/(5-\alpha)}$, let

$$T' = \{pqr \in T \ : \ |c(pqr) \cap P| \le t\},$$

$$T_i = \{pqr \in T \ : \ 2^i t \le |c(pqr) \cap P| \le 2^{i+1} t\},$$

for $i = 0, 1, \ldots, \lfloor \log n \rfloor$.

For any points $p, q \in P$ there are at most $t - 1$ choices for r such that $pqr \in T'$. Thus, we have

$$|T'| < n^2 t = n^{\frac{11-3\alpha}{5-\alpha}}.$$

Let $C_i = \{c(pqr) \ : \ pqr \in T_i\}$, for $0 \le i \le \log n$. Letting $\ell_i := |C_i|$, we have at least $2^i t \ell_i$ incidences between the n points in P and the ℓ_i circles in C_i. Moreover, the center of each circle in C_i is among the n points of P, so we can apply Corollary 3, which yields

$$2^i t \ell_i = O_\alpha\left(n^{\frac{5+3\alpha}{7+\alpha}} \ell_i^{\frac{5-\alpha}{7+\alpha}}\right),$$

for an arbitrary $0 < \alpha < 1/e$. Rearranging the terms, we get for every i that

$$\ell_i = O_\alpha\left(\frac{n^{\frac{5+3\alpha}{2+2\alpha}}}{(2^i t)^{\frac{7+\alpha}{2+2\alpha}}}\right).$$

Using the fact that $|T_i| < (2^{i+1}t)^2 \ell_i$, we obtain

$$|T_i| = O_\alpha\left(\frac{n^{\frac{5+3\alpha}{2+2\alpha}}}{(2^i t)^{\frac{3-3\alpha}{2+2\alpha}}}\right) = O_\alpha\left(\frac{n^{\frac{11-3\alpha}{5-\alpha}}}{2^{i\frac{3-3\alpha}{2+2\alpha}}}\right).$$

Adding up these bounds, it follows that

$$|T| = |T'| + \sum_{i=0}^{\lfloor \log n \rfloor} |T_i| = O_\alpha\left(n^{\frac{11-3\alpha}{5-\alpha}}\right),$$

as required.

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