ON POINT COVERS OF MULTIPLE INTERVALS AND AXIS-PARALLEL RECTANGLES

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ABSTRACT

In certain families of hypergraphs the transversal number of a hypergraph is bounded above by a function of its packing number. In this paper we study hypergraphs related to multiple intervals and axis-parallel rectangles, respectively. Essential improvements of former established upper bounds are presented here. We explore the close connection between the to problems at issue.

1 Introduction

Let $\mathcal{H}$ be a *hypergraph*: a finite family of nonempty subsets of an underlying set X. These subsets are called the *edges* of the hypergraph, the elements of X are its *vertices*. The *transversal number* $\tau(\mathcal{H})$ is the minimum k such that some set of k vertices meets

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all the edges of the hypergraph. The *packing number* $\nu(\mathcal{H})$ is defined as the maximum number of pairwise disjoint edges of $\mathcal{H}$. The inequality

$$\tau(\mathcal{H}) \geq \nu(\mathcal{H})$$

is trivial. If $\mathcal{H}$ is a finite set of intervals on a line, then $\tau(\mathcal{H}) = \nu(\mathcal{H})$ as it was first observed by Hajnal and Surányi [HaS]. We note that the result can be extended to arbitrary families of closed intervals on a line (see e.g. [Kar]). On the other hand, there are several interesting types of hypergraphs where $\tau(\mathcal{H})$ may be arbitrarily large though $\nu(\mathcal{H}) = 1$ (see e.g. [DSW] or [GyL2]).

Hypergraphs related to families of multiple intervals were investigated by Gyárfás and Lehel in [GyL1] and [GyL2], and upper bounds were established for the transversal number in terms of the packing number. To state the results we should introduce some terminology.

Let m denote an arbitrary natural number. Choose and fix m distinct parallel lines $\ell_1, \ldots, \ell_m$. For convenience, we will occasionally identify each of them with the line $\mathbb{R}$ of real numbers. An *m-interval* A is the union of m intervals one located on each of the lines $\ell_1, \ldots, \ell_m$. We use the superscript notation A^i to denote the component of A on the line ℓ_i. Here some but definitely not all of the intervals $A^1, \ldots, A^m$ may be empty. Define

$$f(k,m) = \max\{\tau(\mathcal{H}) | \mathcal{H} \text{ is a family of } m\text{-intervals with } \nu(\mathcal{H}) = k\}.$$

We obtain an other interesting type of hypergraph if we let the lines $\ell_1, \ldots, \ell_m$ coincide. A *homogeneous m-interval* A is the union of m intervals on the line $\mathbb{R}$. We denote these intervals by $A^1, \ldots, A^m$. Analogously to $f(k,m)$ we define

$$f^*(k,m) = \max\{\tau(\mathcal{H}) | \mathcal{H} \text{ is a family of homogeneous } m\text{-intervals with } \nu(\mathcal{H}) = k\}.$$

Clearly $f(k,m) \leq f^*(k,m)$.

The observation of Hajnal and Surányi now can be stated as follows.

Theorem 1.1. *[HaS]* $f(k,1) = f^*(k,1) = k$. ∎

Gyárfás and Lehel [GyL1] proved that $f(k,m)$ and $f^*(k,m)$ are finite for every $k, m \in \mathbb{N}$. To be more detailed, they proved

Theorem 1.2. *[GyL1]* $f(k,m) \leq f(k((k+1)^{m-1} - 1), m-1) + k$,

$$f^*(k,m) \leq (f^*(k,m-1))^{m-1} f(k,m) + \sum_{i=1}^{m-1} (f^*(k,m-1))^i \ .$$

∎

The recursions in Theorem 1.2 yield $f(k,m) = O(k^{m!})$ and $f^*(k,m) = O(k^{(m+1)!/2})$ for any fixed m. In Section 2 we improve these upper bounds, and also show a lower bound for the function $f(k,m)$. We have to note that a quadratic upper bound for

$f^*(k,2)$ was proved by Kostochka [Kos] earlier and later Tardos [Tar] proved a linear upper bound for the same function.

In [GyL2] Gyárfás and Lehel raised the following problem. Given a family $\mathcal{H}$ of rectangles with parallel sides in the plane, is there any constant c such that $\tau(\mathcal{H}) < c\nu(\mathcal{H})$? Though the question is still open, the following approximative result was proved recently by Károlyi [Kar]. Later Kostochka [FFK] gave a simpler proof.

Theorem 1.3. *[Kar] If $\mathcal{H}$ is an arbitrary family of axis-parallel rectangles in the plane $\mathbb{R}^2$, then*

$$\tau(\mathcal{H}) \le \nu(\mathcal{H})(\lfloor \log \nu(\mathcal{H}) \rfloor + 2) \ .$$

■

A discrete version of this problem is the following. Let us call a finite hypergraph $\mathcal{H}$ *rectangular*, if its set of vertices X can be placed in the plane so that every edge of $\mathcal{H}$ is of the form $X \cap R$ where R is a suitable rectangle with sides parallel to the coordinate axes. Rectangular hypergraphs were first investigated by Ding, Seymour and Winkler [DSW] in a more general context. As a consequence of their general theorem for arbitrary hypergraphs, they proved the following result.

Theorem 1.4. *[DSW] For any rectangular hypergraph $\mathcal{H}$,*

$$\tau(\mathcal{H}) \le (\nu(\mathcal{H}) + 63)^{127} \ .$$

■

Later Pach and Törőcsik [PaT] observed a connection with 4-interval hypergraphs, and proved that $\tau(\mathcal{H}) \le f(1,4)(\nu(\mathcal{H}))^8$ holds for rectangular hypergraphs $\mathcal{H}$. It is worth for noting that the constant $f(1,4)$ here comes from the observation that for rectangular hypergraphs $\mathcal{H}$ with $\nu(\mathcal{H}) = 1$ one has $\tau(\mathcal{H}) \le f(1,4)$.

In Section 3 we explore the close connection with 2-interval hypergraphs which yields the following improvement of Theorem 1.4.

Theorem 1.5. *For any rectangular hypergraph $\mathcal{H}$,*

$$\tau(\mathcal{H}) = 4\nu(\mathcal{H})(\lfloor \log \nu(\mathcal{H} \rfloor + 1)^2 \ .$$

This bound of τ is almost linear in ν. Conjecture 3.5 would even shave off one of the two $\log \nu(\mathcal{H})$ factors.

2 Multiple intervals

The following theorem (using Theorem 1.2 to see that $f^*(m,m)$ is finite) yields $f^*(k,m) = O(k^m)$ for any fixed m.

Theorem 2.1. *For $k \geq m$ we have $f^*(k,m) \leq km + \binom{k}{m} f^*(m,m)$.*

Proof. Let $\mathcal{H}$ be a system of homogeneous m-intervals with $\nu(\mathcal{H}) = k$. Let, in particular, $H_1, H_2, \dots, H_k$ be pairwise disjoint elements of $\mathcal{H}$. The km left endpoints of the intervals H_i^j $(1 \leq i \leq k,\ 1 \leq j \leq m)$ will cover the homogeneous m-intervals $H_1, H_2, \dots, H_k$; and also those members H of $\mathcal{H}$ for which some H^j $(1 \leq j \leq m)$ has a common point with at least two different H_i^l $(1 \leq i \leq k,\ 1 \leq l \leq m)$. Therefore it remains to cover those members H of $\mathcal{H}$, for which every H^j $(1 \leq j \leq m)$ meets at most one H_i $(1 \leq i \leq k)$. As these homogeneous m-intervals meet at most m of $H_1, \dots, H_k$ they are contained in one of the sets

$$\mathcal{H}_S = \{H \in \mathcal{H} | H \cap U = \emptyset \text{ for any } U \in S\}$$

where $S \subset \{H_1, \dots, H_k\}$ and $|S| = k - m$. By definition all elements of S are disjoint from any element of $\mathcal{H}_S$ and therefore

$$\nu(\mathcal{H}_S) \leq k - |S| = m \ .$$

Thus the homogeneous m-intervals in $\mathcal{H}_S$ can be covered by at most $f^*(m,m)$ points. The statement of the theorem follows by taking the km left endpoints and then covering each $\mathcal{H}_S$ separately. ∎

The same proof yields the analogous theorem for m-intervals:

Theorem 2.2. *For $k \geq m$ we have $f(k,m) \leq km + \binom{k}{m} f(m,m)$.* ∎

The best upper bound we can show to $f(k,m)$ comes from the recursion in the following Lemma. Although it yields stronger bounds, the statement and the proof is similar to Theorem 1.2.

Theorem 2.3. *For positive integers k and m we have $f(k, m+1) \leq m^2k + 1 + f(k-1, m+1) + f(mk, m)$.*

Proof. Let $\mathcal{H}$ be a family of $(m+1)$-intervals with $\nu(\mathcal{H}) = k$. We must cover $\mathcal{H}$ with at most $m^2k + 1 + f(k-1, m+1) + f(mk, m)$ points. For compactness reasons it is enough to prove for finite families $\mathcal{H}$.

Let us identify the first line ℓ_1 with the real line $\mathbb{R}$. Take a subset $I \subset \ell_1$ and consider the set

$$\mathcal{H}_I = \{H \in \mathcal{H} | H^1 \subset I\}.$$

Let $\mathcal{H}'_I$ consist of the m-intervals obtained from the $(m+1)$-intervals in $\mathcal{H}_I$ by removing their first component. Let us take

$$x_0 = \sup\{x | \nu(\mathcal{H}'_{(-\infty,x)}) \leq mk.$$

As $\nu(\mathcal{H}'_{(-\infty,x_0)}) \leq mk$ we can cover $\mathcal{H}'_{(-\infty,x_0)}$ and thus $\mathcal{H}_{(-\infty,x_0)}$ by $f(mk,m)$ points. In case x_0 is infinity this finishes the proof. Suppose therefore that x_0 is finite.

By the finiteness of $\mathcal{H}$ we have $\nu(\mathcal{H}'_{(-\infty,x_0]}) > mk$. We can take therefore an $mk+1$ element set $S \subset \mathcal{H}_{(\infty,x_0]}$ of $(m+1)$-intervals that are pairwise disjoint except for their first components. For $i = 2, \ldots, m+1$ we can take mk points on ℓ_i that separates the $mk+1$ disjoint ith components of S. Let T be the set of all these m^2k points.

Let us take $\mathcal{H}^*$ to be the set of $(m+1)$-intervals of $\mathcal{H}_{(x_0,\infty)}$ not covered by T. We claim that $\nu(\mathcal{H}^*) < k$. We prove this by contradiction. Suppose $S^* \subset \mathcal{H}^*$ consists of k pairwise disjoint $(m+1)$-intervals. The any element H of S_* the first component H^1 is disjoint from the elements of S and any other component H^i $(i = 2, \ldots m+1)$ can only intersect at most one of the elements in H since T does not cover H. Thus at least one of the elements in S is disjoint from each elements of S^*. This would mean $k+1$ pairwise disjoint elements of $\mathcal{H}$, a contradiction.

By the observation above $\mathcal{H}^*$ can be covered by $f(k-1, m)$ points. Thus $\mathcal{H}_{(x_0,\infty)}$ can be covered by $f(k-1, m+1) + |T| = f(k-1, m+1) + m^2k$ points. We finish the proof by recalling that $\mathcal{H}_{(-\infty,x_0)}$ can be covered by $f(mk, m)$ points and observing that the rest of $\mathcal{H}$ is covered by the single pont x_0. ∎

Using $f(k, 1) = k$ and $f(0, m) = 0$ as the base case for the recursion in Lemma 2.3 one obtains $f(k, m) = O(k^m)$ for any fixed m again. We remark that this proof proves the existence of a covering set of this size which has only k points on the first line. (The number k is optimal here.)

We get better upper bounds for $f(k, m)$ if we use the following theorem from [Tar] as the base case when applying Lemma 2.3.

Theorem 2.4. *[Tar] $f(k, 2) = 2k$.* ∎

Corollary 2.5. *For any fixed $m \geq 2$ we have $f(k, m) = O(k^{m-1})$.*

Proof. Lemma 2.3 and Theorem 2.4 yield the proof. ∎

Let us remark that using the $f^*(k, m) \leq f(2m(m-1)k, m)$ bound in [Tar] Corollary 2.5 implies the same bound for homogeneous m-intervals: $f^*(k, m) = O(k^{m-1})$ for any fixed $m \geq 2$.

The upper bounds above for f and f^* are probably far from tight for higher values of m. There are however special types of families of multiple intervals for which the dependence between τ and ν can be computed exactly.

For an interval I denote the endpoints of I by $l(I)$ and $r(I)$. Choosing this notation so that $l(I) \leq r(I)$, we call $l(I)$ and $r(I)$ the *left and right endpoints* of I, respectively. The endpoints of I are not defined if $I = \emptyset$. We say that the family $\mathcal{H}$ of m-intervals is *left-ordered*, if $l(A^i) < l(B^i)$ implies $l(A^j) \leq l(B^j)$ for every pair $A, B \in \mathcal{H}$ and superscripts $1 \leq i, j \leq m$, whenever $l(A^i)$, $l(B^i)$, $l(A^j)$ and $l(B^j)$ are defined. Introduce

$$g(k, m) = \max\{\tau(\mathcal{H}) | \mathcal{H} \text{ is a left-ordered family of } m\text{-intervals with } \tau(\mathcal{H}) = k\}$$

Theorem 2.6. $g(k, m) = km$.

Proof. The upper bound $g(k, m) \leq km$ is an easy consequence of Theorem 1.1. Indeed, let $\mathcal{H}$ be a family of m-intervals with $\nu(\mathcal{H}) = k$, and suppose that $\mathcal{H}$ is left-ordered. We may assume that $l(A^j) \neq l(B^j)$ for every $A, B \in \mathcal{H}$ $(A \neq B)$ and $1 \leq j \leq m$. Therefore we may choose homeomorphisms

$$g_i : \ell^i \longrightarrow \mathbb{R} \ (1 = 1, 2, \ldots, m)$$

with the following property: for every $H \in \mathcal{H}$ $g_i(l(H^i)) = g_j(l(H^j))$ for every $1 \leq i, j \leq m$ if H^i and H^j are nonempty. Then $g(H) = \cup_{j=1}^m g_j(H^j)$ is an interval for every $H \in \mathcal{H}$. If the intervals $g(H_1), \ldots, g(H_l)$ are pairwise disjoint for some $H_1, \ldots, H_l \in \mathcal{H}$, then the m-intervals $H_1, \ldots, H_l$ are also pairwise disjoint, thus $l \leq k$. Therefore, by Theorem 1.1, the family of intervals $\{g(H) \mid H \in \mathcal{H}\}$ can be covered by at most k points, $x_1, \ldots, x_k$. Obviously, the inverse images $g_j^{-1}(x_i)$ ($1 \leq i \leq k$, $1 \leq j \leq m$) cover all the elements of $\mathcal{H}$, proving the assertion.

To show that the upper bound is tight we construct a left-ordered family $\mathcal{H} = \mathcal{H}_{k,m}$ of m-intervals with $\nu(\mathcal{H}) = k$ and $\tau(\mathcal{H}) \geq km$. In fact it is enough to construct $\mathcal{H}_m = \mathcal{H}_{1,m}$ with the desired properties, then $\mathcal{H}_{k,m}$ is obtained as the union of k pairwise disjoint translates of $\mathcal{H}_m$.

In the construction we identify the lines ℓ_i with the real line $\mathbb{R}$ for $i = 1, \ldots, m$. Let us take $\mathcal{H}_m$ as the set of the m-intervals H_i for $1 \leq i \leq m^2$ where we define

$$H_i^j = [-i, 0] \text{ if } (j-1)m < i \leq jm \quad \text{and} \quad H_i^j = [-i, -i + 1/2] \text{ otherwise}$$

for all $1 \leq i \leq m^2$ and $1 \leq j \leq m$.

Now it is easy to check that

1) $\mathcal{H}_m$ is left-ordered;
2) for $1 \leq i \leq i' \leq m^2$ the m-intervals H_i and H_i' both contain the point $-i$ on the line ℓ_j where $(j-1)m < i' \leq jm$ and therefore $\nu(\mathcal{H}_m) = 1$;
3) the intervals H_i^j ($i \notin \{(j-1)m + 1, \ldots, jm\}$) are pairwise disjoint for each $1 \leq i \leq m$, and therefore each point of the line ℓ_i ($1 \leq i \leq m$) covers at most $m + 1$ distinct elements of $\mathcal{H}_m$.

Since $|\mathcal{H}_m| = m^2 > (m+1)(m-1)$, 3) implies that $m-1$ points are not enough to cover all the elements of $\mathcal{H}_m$. Therefore we have $\nu(\mathcal{H}_m) = 1$ and $\tau(\mathcal{H}_m) \geq m$, as claimed. ∎

Let us remark that the trivial lower bound $f(k, m) \geq g(k, m) = km$ is not tight as [GyL1] proves $f(1, 3) = 4$. Proving better lower bounds seems to be hard, even for $f(1, m)$ we are unable to improve the trivial lower bound by more than a constant.

3 Axis-parallel rectangles

To prove covering theorems for rectangular hypergraphs (Theorem 1.5) we need covering results about the following special type of rectangular hypergraphs.

The hypergraph $\mathcal{H}$ is called *pointed rectangular* if its finite vertex set X can be placed into the plane so that for every $H \in \mathcal{H}$ there exist an axis-parallel rectangle R_H such that $H = X \cap R_H$ and $\cap_{H \in \mathcal{H}} R_H \neq \emptyset$.

Bounding the transversal number of rectangular hypergraphs shows close connection to bounding the transversal number of families of multiple intervals. A straightforward generalization of Lemma 1 in [PaT] yields that $\tau(\mathcal{H}) \leq f(\nu(\mathcal{H}), 4)$. The following statement is an improvement upon this result.

Lemma 3.1. *For any pointed rectangular hypergraph* $\mathcal{H}$

$$\tau(\mathcal{H}) \leq 2f(\nu(\mathcal{H}), 2) \ .$$

Proof. Let X be the vertex set of $\mathcal{H}$ placed in the plain according to the definition. For an edge $H \in \mathcal{H}$ let R_H be the axis parallel rectangle with $H = X \cap R_H$, such that these rectangles R_H contain a common point. We may assume that the common point is the origin $(0,0)$.

Let us define $X_1 = \{(x,y) \in X | x \geq 0\}$ and $X_2 = \{(x,y) \in X | x \leq 0\}$. Let p_1 be the projection to the x axis and p_2 the projection to a different line parallel to the x axis. For an edge $H \in \mathcal{H}$ we define the following 2-interval I_H. The first component I_H^1 is the convex hull of $p_1(H \cap X_1)$ while I_H^2 is the convex hull of $p_2(H \cap X_2)$. Let $\mathcal{H}' = \{I_H | H \in \mathcal{H}\}$ be the family of 2-intervals so obtained.

Let H and H' be two intersecting edges of $\mathcal{H}$. If their common vertex is $x \in X_i$ ($i = 1$ or 2) then $p_i(x)$ is a common point of I_H and $I_{H'}$. Thus $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$.

Let P be a point covering some of the 2-intervals in $\mathcal{H}'$. We claim that the corresponding edges of $\mathcal{H}$ can be covered by two points. By symmetry we may suppose P is on the first line, the x axis, thus $P = (x_0, 0)$. Consider the set $\{(x,y) \in X_1 | x \leq x_0 \text{ and } y \geq 0\}$ and let $P_1 = (x_1, y_1)$ be an element of the set with minimal y-coordinate. Similarly let $P_2 = (x_2, y_2)$ an element of the set $\{(x,y) \in X_2 | x \leq x_0 \text{ and } y \leq 0\}$ with maximal y-coordinate. If the corresponding sets are empty then P_1 or P_2 or both are undefined. Take an edge $H \in \mathcal{H}$ such that $P \in I_H$. By definition there are points $P_3 = (x_3, y_3)$ and $P_4 = (x_4, y_4)$ in $H \cap X_1$ such that $x_3 \leq x_0 \leq x_4$. We claim that if $y_3 \geq 0$ then P_1 covers H and if $y_3 \leq 0$ then P_2 covers H. By symmetry it is enough to prove the first assertion. Since $P_3 \in \{(x,y) \in X_1 | x \leq x_0 \text{ and } y \geq 0\}$ P_1 is defined and $0 \leq y_1 \leq y_3$. Therefore any axis-parallel rectangle containing P_3, P_4, and the origin also contains P_1. Thus $P_1 \in H$ as claimed.

The last paragraph implies $\tau(\mathcal{H}) \leq 2\tau(\mathcal{H}')$. As $\mathcal{H}'$ is a system of 2-intervals with $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$ we have $\tau(\mathcal{H}') \leq f(\nu(\mathcal{H}), 2)$. The statement of the lemma follows. ∎

Let us remark that an upper bound on the transversal number of rectangular hypergraph follows from Theorem 1.3, Theorem 2.4, and Lemma 3.1. Let $\mathcal{H}$ be a rectangular hypergraph. As the packing number of the system of rectangles in the definition is at most $\nu(\mathcal{H})$. Thus by Theorem 1.3 $\nu(\mathcal{H})(\lfloor \log \nu(\mathcal{H}) \rfloor + 2)$ points of the plain is enough to cover all rectangles. For any covering point P the edges of $\mathcal{H}$ for which the corresponding rectangle is covered by P is a pointed rectangular hypergraph, with packing number at most $\nu(\mathcal{H})$. Thus it can be covered by $2f(\nu(\mathcal{H}), 2) = 4\nu(\mathcal{H})$ points by Lemma 3.1 and Theorem 2.4. Thus $\mathcal{H}$ can be covered by the covering all these subsystems, proving

$$\tau(\mathcal{H}) \leq 4(\nu(\mathcal{H})^2)(\lfloor \log \nu(\mathcal{H}) \rfloor + 2).$$

To prove the much stronger Theorem 1.5 we replace Theorem 1.3 with Corollary 3.4.

We call a hypergraph $\mathcal{H}$ *disintegrated* if there is a partition $\mathcal{H} = \cup_{i \in I} \mathcal{H}_i$ such that $\cap_{H \in \mathcal{H}_i} H \neq \emptyset$ for each $i \in I$ and the elements of $\mathcal{H}_i$ are disjoint from the elements of $\mathcal{H}_{i'}$ for each $i \neq i' \in I$. We call a line *horizontal* if it is parallel to the x axis, and we call it *vertical* if it is parallel to the y axis.

Lemma 3.2 Let $\mathcal{H}'$ be a disintegrated family of axis-parallel rectangles and X a finite set in the plane. *The rectangular hypergraph $\mathcal{H} = \{R \cap X | R \in \mathcal{H}'\}$ satisfies $\tau(\mathcal{H}) \leq 4\nu(\mathcal{H})$.*

Proof. Let $\mathcal{H}' = \cup_{i\in I}\mathcal{H}'_i$ be the partition in the definition of disintegrated for $\mathcal{H}'$. Let $\mathcal{H}_i = \{R \cap X | R \in \mathcal{H}'_i\}$, these sets form a partition of $\mathcal{H}$ into pointed rectangular hypergraphs. Thus by Lemma 3.1 and Theorem 2.4 we have $\tau(\mathcal{H}_i) \leq 2f(\nu(\mathcal{H}_i), 2) = 4\nu(\mathcal{H}_i)$. By the disjointness property in the definition of disintegrated we have $\nu(\mathcal{H}) = \sum_{i\in I} \nu(\mathcal{H}_i)$ and $\tau(\mathcal{H}) = \sum_{i\in I} \tau(\mathcal{H}_i)$. The lemma follows. ∎

Lemma 3.3. *Let $\mathcal{H}$ be a system of axis-parallel rectangles in the plane and let i and j be positive integers. Suppose that there are $2^i - 1$ horizontal lines such that each rectangle in $\mathcal{H}$ intersects one of these lines. Suppose that there are $2^j - 1$ vertical lines with the same property. Then $\mathcal{H}$ can be partitioned into ij disintegrated hypergraphs.*

Proof. The proof is by induction on i and j. If $i = j = 1$ then all rectangles in $\mathcal{H}$ contain the intersection of the only horizontal and the only vertical line. Thus $\mathcal{H}$ is disintegrated as claimed.

Suppose one of i and j is not one. By symmetry we may suppose $i > 1$. Let us call the central of the $2^i - 1$ horizontal lines ℓ. We partition $\mathcal{H}$ with respect to ℓ: let $\mathcal{H}_0$ consist of the rectangles in $\mathcal{H}$ intersecting ℓ, $\mathcal{H}_1$ consist of the rectangles on the one side of ℓ, while $\mathcal{H}_2$ consist of the rectangles in $\mathcal{H}$ on the other side of ℓ. By induction $\mathcal{H}_0$ can be partitioned into j disintegrated hypergraphs, while $\mathcal{H}_1$ and $\mathcal{H}_2$ can be partitioned into $(i-1)j$ disintegrated hypergraphs each. As the elements of $\mathcal{H}_1$ are separated from the elements of $\mathcal{H}_2$ by ℓ we can take the union of a disintegrated subset of $\mathcal{H}_1$ and a disintegrated subset of $\mathcal{H}_2$ and still get a disintegrated hypergraph. Thus matching the parts of $\mathcal{H}_1$ to the parts of $\mathcal{H}_2$ we get a partition of $\mathcal{H}$ into $(i-1)j+j = ij$ disintegrated hypergraphs as claimed. ∎

Corollary 3.4. *Let $\mathcal{H}$ be a family of axis-parallel rectangles in the planes with finite packing number. Then $\mathcal{H}$ can be partitioned into $(\lfloor \log \nu(\mathcal{H}) \rfloor + 1)^2$ disintegrated hypergraphs.*

Proof. Let $\mathcal{H}'$ be the set of projections to the x axis of the rectangles in $\mathcal{H}$. Using Theorem 1.1 one finds $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$ vertical lines such that each rectangles in $\mathcal{H}$ intersects one of them. Similarly one can find at most $\nu(\mathcal{H})$ horizontal lines with the same property. Applying Lemma 3.2 yields the result. ∎

We are ready now to prove Theorem 1.5.

Proof of Theorem 1.5. Let x be the set of vertices of $\mathcal{H}$ and let $\mathcal{H}'$ be a collection of axis-parallel rectangles such that $\mathcal{H} = \{R \cap X | R \in \mathcal{H}'\}$. Apply Corollary 3.4 to partition $\mathcal{H}'$ into disintegrated hypergraphs. As $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$ the number of the parts $\mathcal{H}'_i$ $(i \in I)$ is at most $(\lfloor \log \nu(\mathcal{H}) \rfloor + 1)^2$. For $i \in I$ let $\mathcal{H}_i = \{R \cap X | R \in \mathcal{H}'_i\}$, these sets form a partition of $\mathcal{H}$. Lemma 3.2 yields $\tau(\mathcal{H}_i) \leq 4\nu(\mathcal{H}'_i)$. Using the trivial $\nu(\mathcal{H}_i) \leq \nu(\mathcal{H})$ and $\tau(\mathcal{H}) \leq \sum_{i\in I} \tau(\mathcal{H}_i)$ bounds this last observation proves the theorem. ∎

One way to improve on the bound in Theorem 1.5 would be to improve Corollary 3.4. Unfortunately it is optimal except for a constant factor. Let $\mathcal{H}$ be the family of all rectangles with integer coordinates inside a k by k square. It is easy to show that $\nu(\mathcal{H}) = \Theta(n^2)$ and the minimum number of disintegrated hypergraphs $\mathcal{H}$ can be partitioned to is $\Theta(\log^2 n)$. One can gain though by restricting attention to Sperner systems, i. e. hypergraphs with no edge containing another edge. As the packing and transversal numbers of a finite hypergraph does not change by removing all non-minimal edges we can assume without loss of generality that the hypergraph is a Sperner system. In the case of rectangular hypergraphs the corresponding family of rectangles is also a Sperner system then. So the next conjecture would be enough to prove

$$\tau(\mathcal{H}) = O(\nu(\mathcal{H}) \log \nu(\mathcal{H}))$$

for any rectangular hypergraph. It is worth noting that this bound matches the best known bound for families of rectangles.

Conjecture 3.5. *Let $\mathcal{H}$ be a Sperner system of axis-parallel rectangles. Then it can be partitioned into $O(\log \nu(\mathcal{H}))$ disintegrated hypergraphs.*

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