

On roughly transitive amenable graphs and harmonic Dirichlet functions

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Abstract. We introduce the notion of rough transitivity and prove that there exist no non-constant harmonic Dirichlet functions on amenable roughly transitive graphs.

1 Introduction

Let us call a graph G *roughly transitive* if there exist quasi-isometries f_{xy} from G to G for vertices x and y of G with $f_{xy}(x) = y$ such that these quasi-isometries have bounded distortion. (See Section 2 for notations.)

Any graph quasi-isometric to a vertex-transitive graph is clearly roughly transitive. However, the converse is far from being true as we shall show in Section 3.

The study of harmonic Dirichlet functions on graphs goes back to Cheeger and Gromov [2]. They proved that there exist no non-constant harmonic Dirichlet function on the Cayley graph of an amenable group. For different proofs see also Elek [3] and Paschke [7]. Later Medolla and Soardi [5] extended this result to amenable vertex-transitive graphs. In a recent preprint Benjamini, Lyons, Peres, and Schramm [1] gave a probabilistic proof for this result. The goal of this paper is to extend the result of Medolla and Soardi to amenable roughly transitive graphs:

Theorem 1 *The only harmonic Dirichlet functions of a roughly transitive amenable graph are the constant functions.*

Note that some assumption of this kind is necessary as there exist amenable graphs with non-constant harmonic Dirichlet functions, see e.g. [10], Chapter 6.

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2 Notations

In this note we use the term *graph* for simple, connected, undirected graphs with bounded degree vertices only. We will mostly consider infinite graphs. We denote the set of vertices of a graph G by $V(G)$, we use $E(G)$ for the set of edges. By $\vec{E}(G)$ we denote the set of oriented edges: $\vec{E}(G) = \{(x, y) \mid \{x, y\} \in E(G)\}$. We denote the opposite orientation (y, x) of an oriented edge $e = (x, y)$ by \bar{e} . We consider the Hilbert space $l_2(G)$ of the l_2 functions $u : \vec{E}(G) \rightarrow \mathbb{R}$ satisfying $u(\bar{e}) = -u(e)$ with the scalar product $\langle u, u' \rangle = 1/2 \sum_{e \in \vec{E}(G)} u(e)u'(e)$. For simplicity we write $u(x, y)$ for the value $u((x, y))$. For a function $v : V(G) \rightarrow \mathbb{R}$ we define its *differential* $dv : \vec{E}(G) \rightarrow \mathbb{R}$ by $dv(x, y) = v(y) - v(x)$. We call a $v : V(G) \rightarrow \mathbb{R}$ function a Dirichlet function if $dv \in l_2(G)$ and denote the set of Dirichlet functions by $D(G)$. Let $l_2(V(G))$ denote the Hilbert space of the l_2 functions $v : V(G) \rightarrow \mathbb{R}$ (with the standard scalar product), this is clearly contained in $D(G)$.

Consider the adjoint d^* of the operator $d : l_2(V(G)) \rightarrow l_2(G)$. We call a function $u \in l_2(G)$ a *flow* if $d^*u = 0$. We call a Dirichlet function $v \in D(G)$ *harmonic* if dv is a flow. We denote the set of harmonic Dirichlet functions by $HD(G)$.

Here d^*u is given by $d^*u(x) = \sum_{\{y,x\} \in E(G)} u(y, x)$ for $u \in l_2(G)$ and $x \in V(G)$. Thus u is a flow if and only if $\sum_{\{x,y\} \in E(G)} u(x, y) = 0$ for every $x \in V$. The function $v \in D(G)$ is harmonic if and only if for every $x \in V$ the value $v(x)$ is the average of the values $v(y)$ with $\{x, y\} \in E(G)$. All constant functions $V(G) \rightarrow \mathbb{R}$ are harmonic Dirichlet functions.

For vertices x and y of a graph G let $\delta(x, y)$ denote their distance in G . A *wobbling* is a map $f : V(G) \rightarrow V(G')$ such that $\delta(x, f(x))$ for $x \in V(G)$ is bounded. The map $f : V(G) \rightarrow V(G')$ is called a *quasi-isometry* from G to G' if there exists a positive number k —the *distortion* of f —such that for vertices x and y in $V(G)$ one has

$$\frac{1}{k}\delta(x, y) - 1 < \delta(f(x), f(y)) \leq k\delta(x, y),$$

and for every vertex $x \in V(G')$ there exists $y \in V(G)$ with $\delta(x, f(y)) < k$. A *quasi-inverse* of a quasi-isometry f from G to G' is a quasi-isometry g from G' to G such that $f \circ g$ and $g \circ f$ are wobbings.

Note that for a quasi-isometry f of distortion k one can take a quasi-inverse of distortion $2k^2$.

When speaking of subgraphs of a graph we always mean a connected full subgraph with at least one edge. Let G_0 be a subgraph of a graph G . By $\sigma(G_0)$ we denote the set of vertices of G_0 that have neighbors in G outside G_0 . We call G *amenable* if $\inf |\sigma(G_0)|/|V(G_0)| = 0$, where the infimum is taken for finite subgraphs G_0 . We call the sequence (G_i) of finite subgraphs of a graph G a *Følner sequence* if $|\sigma(G_i)|/|V(G_i)|$ tends to 0 as k tends to infinity.

3 An example

In this section we construct a roughly transitive amenable graph that is not quasi-isometric to a vertex transitive graph. Let G be a simply connected nilpotent Lie group. Then a *net* N in G is a roughly transitive amenable graph. Here the vertices of the net N form a

maximal subset of G with minimum distance 1, the edges connect vertices of distance at most 3. If G has a co-compact lattice then of course N is quasi-isometric to the lattice, that is a vertex transitive graph, so the fact that all harmonic Dirichlet function on N is constant follows from Soardi's Theorem [9] on the quasi-isometry invariance of the existence on non-constant harmonic Dirichlet function. Note however, that according to Malcev's Theorem [4] G has a co-compact lattice if and only if the Lie-algebra of G is rational i.e. has a base with rational structure constants. There are uncountable pairwise non-isomorphic non-rational Lie-algebras [8].

Proposition 2 *There exists a simply connected nilpotent Lie group G and a net N of G such that N is not quasi-isometric to any vertex transitive graph.*

Proof: Let \mathcal{G} be the graded two-step nilpotent Lie algebra $N(d)$ in [8], where d is irrational. This algebra is non-rational. Let G be the associated simply connected nilpotent Lie group and let N be a net of G . Suppose that N is quasi-isometric to a vertex transitive graph M . Then M has polynomial growth so by the famous result of Trofimov [11], M is quasi-isometric to the Cayley graph of a discrete nilpotent group Γ . Let H be the Malcev-completion of Γ . Then H is a nilpotent Lie group which contains Γ as a co-compact lattice. Moreover, the nilpotent Lie groups G and H are quasi-isometric. Now, we recall Pansu's result [6] (we thank to Martin Bridson for calling our attention to this beautiful theorem): If two nilpotent Lie groups are quasi-isometric, then their associated *graded* Lie algebras are in fact isomorphic. The Lie algebra of H is rational by Malcev's Theorem [4]. The associated graded Lie algebra of a rational Lie algebra is obviously rational as well. On the other hand the Lie algebra of G is just $N(d)$, which is itself graded. Thus G and H cannot be quasi-isometric. This proves our proposition. ■

It would be interesting to characterize roughly transitive graphs of polynomial growth up to quasi-isometry the same way as Trofimov did in the vertex transitive case.

4 The result

We borrow some notations from [1]. The *support* $\text{supp}(v)$ of a real-valued function v is the subset of the domain where v is not zero. For a graph G we define $\star(G)$ to be the closure in $l_2(G)$ of the functions dv , where $v : V(G) \rightarrow \mathbb{R}$ has finite support. Let $\diamond(G)$ be the closure in $l_2(G)$ of the flows with finite support.

Notice that we always have $dl_2(V(G)) \subseteq \star(G)$. The following lemma is well known. We prove it to be self contained.

Lemma 3 *Let G be any graph. We have the following orthogonal decomposition:*

$$l_2(G) = \star(G) + \diamond(G) + dHD(G).$$

The flows constitute the subspace $\diamond(G) + dHD(G)$, and $dD(G) = \star(G) + dHD(G)$.

Proof: First note that if $v \in D(G)$ and u is a flow with finite support then we have $\langle dv, u \rangle = \langle dv_0, u \rangle = \langle v_0, d^*u \rangle = 0$, where v_0 is v restricted to the finite support consisting

of the endpoints of the oriented edges in the support of u . Thus $dD(G)$ is orthogonal to $\diamond(G)$. To see that they are orthogonal complements consider any function u orthogonal to $\diamond(G)$. Fix a vertex $x_0 \in V(G)$ and define $v : V(G) \rightarrow \mathbb{R}$ by $v(x_n) = \sum_{i=1}^n u(x_{i-1}, x_i)$ for any path (x_0, \dots, x_n) ($n \geq 0$) in G . The orthogonality proves that v is well defined. We have $dv = u$ thus $d \in D(G)$ as required.

Next we claim that the orthogonal complement of $\star(G)$ consists of all the flows. Indeed a function $u \in l_2(G)$ belongs to this complement if and only if $\langle dv, u \rangle = \langle v, d^*u \rangle = 0$ for every $v : V(G) \rightarrow \mathbb{R}$ with a finite support. This is satisfied if and only if $d^*u = 0$.

To finish the proof of the lemma notice that the orthogonal complement of $\star(G)$ contains $\diamond(G)$ and the intersection of the orthogonal complements of $\star(G)$ and $\diamond(G)$ are flows in $dD(G)$, and by definition, this is $dHD(G)$. ■

Let G_0 be a finite subgraph of G and L a closed subspace of $l_2(G)$. We define

$$\dim_{G_0}(L) = \frac{\sum_{e \in \vec{E}(G_0)} \langle P_L e, e \rangle}{|\vec{E}(G_0)|},$$

where P_L is the orthogonal projection to L and the oriented edge e is identified with the element of $l_2(G)$ mapping e to 1, \bar{e} to -1 and all other oriented edges to 0.

Claim 4 *Let G_0 be a finite subgraph of a graph G . We have $\dim_{G_0}(l_2(G)) = 1$. For orthogonal closed subspaces L and L' of $l_2(G)$ we have $\dim_{G_0}(L + L') = \dim_{G_0}(L) + \dim_{G_0}(L')$. For closed subspaces $L \subseteq L'$ one has $0 \leq \dim_{G_0}(L) \leq \dim_{G_0}(L')$. If the support of all $u \in L$ is contained in $\vec{E}(G_0)$ then $\dim_{G_0}(L) = \dim(L)/|E(G_0)|$.*

Proof: The first statement follows from noting that $P_{l_2(G)}$ is the identity. The second statement follows from the equality $P_{L+L'} = P_L + P_{L'}$ for orthogonal subspaces L and L' . The non negativity is trivial and implies the monotonicity. For the last statement note that we can work in $l_2(G_0)$. Half of the vectors $e \in \vec{E}(G_0)$ (taking one of the pairs e and \bar{e}) form an orthonormal bases of $l_2(G_0)$, and $|E(G_0)|\dim_{G_0}(L)$ is the trace of P_L restricted to $l_2(G_0)$ computed in this bases. ■

Lemma 5 *If G_0 is a finite subgraph of the graph G then $\dim_{G_0}(\star(G) + \diamond(G)) \geq 1 - |\sigma(G_0)|/|E(G_0)|$.*

Proof: First note that $dHD(G_0) = 0$ since every flow on the finite graph G_0 has finite support. So by Lemma 3 we have $l_2(G_0) = \star(G_0) + \diamond(G_0)$.

We identify functions $u \in l_2(G_0)$ and $v \in D(G_0)$ with their extension in $l_2(G)$ (respectively $D(G)$) that is zero outside the original domain. We have to distinguish two differential operators: $d_{G_0} : D(G_0) \rightarrow l_2(G_0)$ is not the restriction of $d : D(G) \rightarrow l_2(G)$ as $d(D(G_0)) \not\subseteq l_2(G_0)$ unless $G = G_0$. But using the formula for d^* one sees that $d_{G_0}^*$ is the restriction of d^* . Thus $\diamond(G_0) \subseteq \diamond(G)$. Let $D_1 = \{v \in D(G) \mid \text{supp}(v) \subseteq \sigma(G_0)\}$ and $D_2 = \{v \in D(G) \mid \text{supp}(v) \subseteq V(G_0) \setminus \sigma(G_0)\}$. Clearly, $D(G_0) = D_1 + D_2$ and $\star(G_0) = d_{G_0}D_1 + d_{G_0}D_2$. Notice that d and d_{G_0} are identical in D_2 , thus $d_{G_0}D_2 = dD_2 \subseteq \star(G)$.

Let $L = dD_2 + \diamond(G_0)$. We have $d_{G_0}D_1 + L = \star(G_0) + \diamond(G_0) = l_2(G_0)$, thus $\dim(L) \geq \dim(l_2(G_0)) - \dim(d_{G_0}D_1) \geq |E(G_0)| - \dim(D_1) \geq |E(G_0)| - |\sigma(G_0)|$.

We have $L \subseteq (\star(G) + \diamond(G)) \cap l_2(G_0)$, thus by Claim 4 $\dim_{G_0}(\star(G) + \diamond(G)) \geq \dim_{G_0}(L) = \dim(L)/|E(G_0)| \geq 1 - |\sigma(G_0)|/|E(G_0)|$ as claimed. ■

Corollary 6 For a Følner sequence (G_i) of finite subgraphs of a graph G we have $\dim_{G_i}(dHD(G))$ tends to zero as i tends to infinity.

Proof: By Lemma 3, Claim 4, and Lemma 5 we have $0 \leq \dim_{G_i}(dHD(G)) \leq |\sigma(G_i)|/|E(G_i)|$. Here $|\sigma(G_i)|/|E(G_i)|$ tends to zero as (G_i) is a Følner sequence. Note that for traditional reasons we used $|\sigma(G_i)|/|V(G_i)|$ in the definition of Følner sequences, but since G_i is connected and has bounded degree $|V(G_i)|$ and $|E(G_i)|$ are proportional. ■

Corollary 6 indicates that the harmonic Dirichlet functions on an amenable graph form a “small” subspace. It does not, however, imply that $dHD(G) = 0$, this is false for some amenable graphs. Our goal is to prove that if non-constant harmonic functions exist on a roughly transitive graph then they form a “large” subspace contradicting Corollary 6. This is immediate for transitive graphs: with any harmonic Dirichlet function all its translates are harmonic. The case of roughly transitive graphs require more care. We study next how quasi-isometries act on Dirichlet functions and on $dHD(G)$.

Let G_1 and G_2 be graphs. For a map $f : V(G_1) \rightarrow V(G_2)$ we define the function $f^* : v \mapsto f \circ v$ on the functions $v : V(G_2) \rightarrow \mathbb{R}$. Let A be a subset of the vertices of a graph G . For $k > 0$ we define the k -neighborhood $C_k(A)$ of A to be $\{x \in V(G) \mid \exists y \in A : \delta(x, y) \leq k\}$. We define $\chi_A : \vec{E}(G) \rightarrow \mathbb{R}$ to be the characteristic function of the oriented edges $(x, y) \in \vec{E}(G)$ with $x \in A$ and $y \in A$.

Lemma 7 For a quasi-isometry f from a graph G_1 to a graph G_2 we have $f^*(D(G_2)) \subseteq D(G_1)$. Furthermore there is a constant c depending on the distortion k of f and the maximum degree of G_1 such that $|df^*v| \leq c|dv|$ and $|df^*v \cdot \chi_A| \leq c|dv \cdot \chi_B|$ for any $v \in D(G_2)$ and $A \subseteq V(G_1)$ if $B = C_k(f(A))$.

Proof: Let f be as in the lemma. We fix a path $f(x) = x_0^e, x_1^e, \dots, x_{k_e}^e = f(y)$ in G_2 of length $0 \leq k_e \leq k$ for each oriented edge $e = (x, y) \in \vec{E}(G_1)$. For any $v \in D(G_2)$ we have $|df^*v|^2 = 1/2 \sum_{(x,y) \in \vec{E}(G_1)} (v(f(y)) - v(f(x)))^2 \leq k/2 \sum_{e \in \vec{E}(G_1)} \sum_{i=1}^{k_e} (v(x_i^e) - v(x_{i-1}^e))^2$. The summands in this last expression all appear in the summation $|dv|^2 = 1/2 \sum_{(x,y) \in \vec{E}(G_2)} (v(y) - v(x))^2$ thus for the first statement we only have to limit the multiplicity of a summand $(v(y) - v(x))^2$ in the first sum for any $(x, y) \in \vec{E}(G_2)$. This is the number of oriented edges in $\vec{E}(G_1)$ such that the corresponding (oriented) path in G_2 contains (x, y) . As the endpoints of these edges form a subset in $V(G_1)$ of maximum distance at most $2k^2$ this multiplicity can be bounded in terms of k and the maximum degree of G_1 .

For the last statement notice that if an edge is spanned by a subset $A \subseteq \vec{E}(G_2)$ then the corresponding path is within the set $B = C_k(f(A))$. ■

Lemma 8 For a wobbling f of a graph G we have $f^*v - v \in l_2(V(G))$ for every $v \in D(G)$.

Proof: For a vertex $x \in V(G)$ take a path $x = x_0, x_1, \dots, x_k = f(x)$ in G . We have $((f^*v - v)(x))^2 \leq k \sum_{i=1}^k (v(x_i) - v(x_{i-1}))^2$. Thus to bound the l_2 norm of $f^*v - v$ in terms of $|dv|$ it is enough to note that we can choose the paths with bounded length and every edge appears in a bounded number of paths. ■

For a function $f : A \rightarrow B$ and $S \subseteq B$ we use $f^{-1}(S)$ to denote $\{x \in A \mid f(x) \in S\}$.

Lemma 9 For a graph G we have $dHD(G) \cong D(G)/d^{-1}(\star(G))$. For a quasi-isometry f from G_1 to G_2 f^* induces an isomorphism between $D(G_2)/d^{-1}(\star(G_2))$ and $D(G_1)/d^{-1}(\star(G_1))$ thus between $dHD(G_2)$ and $dHD(G_1)$. For quasi-inverses f and g the functions f^* and g^* induce inverse isomorphisms.

Proof: The $dD(G) = \star(G) + dHD(G)$ claim of Lemma 3 proves the first statement.

Let f be as in the lemma and $v \in d^{-1}(\star(G_2))$. Then there exist functions $v_i \in D(G_2)$ with finite support such that dv_i tend to dv in norm. By Lemma 7 df^*v_i tend to df^*v , and as the functions f^*v_i also have finite support $df^*v \in \star(G_1)$. Thus f^* maps $D(G_2)/d^{-1}(\star(G_2))$ linearly to $D(G_1)/d^{-1}(\star(G_1))$.

To see that this map is an isomorphism take a quasi-inverse g of the quasi-isometry f . For any $v \in D(G_2)$ we have by Lemma 8 that $g^*f^*v - v \in l_2(V(G_2)) \subseteq d^{-1}(\star(G_2))$. For $v \in D(G_1)$ we similarly have $f^*g^*v - v \in d^{-1}(\star(G_1))$ thus the maps between $D(G_2)/d^{-1}(\star(G_2))$ and $D(G_1)/d^{-1}(\star(G_1))$ induced by f^* and g^* are inverses of each other. ■

Lemma 10 Let the graph G and the positive numbers k and Δ be given. If $dHD(G) \neq 0$ then there exist a finite set $A \subseteq V(G)$ and a number $\epsilon > 0$ with the following property. For any quasi-isometry f of distortion at most k to G from a graph G' of maximum degree at most Δ one has a function $w \in HD(G')$ with $|dw| = 1$ and $|dw \cdot \chi_B| > \epsilon$ for $B = f^{-1}(A)$.

Proof: Choose a non-constant function $v \in HD(G)$. We fix $\epsilon > 0$ later and choose $A = C_{k+1}(A_0)$ with a finite set $A_0 \subseteq V(G)$ such that $|dv \cdot \chi_{V(G) \setminus A_0}| < \epsilon$.

By “constant” we mean a quantity depending on G , v , k and Δ but not on G' , f or ϵ .

We can take a quasi-inverse g of f of distortion bounded by a constant. By Lemma 9 we have the decomposition $f^*v = v' + z$ with some $v' \in HD(G')$ and $z \in d^{-1}(\star(G'))$. Similarly $g^*v' = v + t$ with some $t \in d^{-1}(\star(G))$. By Lemma 7 we have $0 < |dv| \leq |dg^*v'| \leq c|dv'|$ with some constant c . Thus $|dv'| \geq c_0 = |dv|/c > 0$. We use Lemma 7 for f^* to get $|dz| \leq |df^*v| \leq c'|dv| = c_1$ with some constants c' and c_1 . Now consider $B = f^{-1}(A) \subseteq V(G')$ and $C = (V(G') \setminus B) \cup \sigma(B)$. Notice that $\chi_C \geq 1 - \chi_B$ and $D = C_k(f(C))$ is disjoint from A_0 .

Consider the orthogonal decompositions $dv' = u_1 + u_2$ with $u_1 = dv' \cdot \chi_B$ and $u_2 = dv' \cdot (1 - \chi_B)$, and $dz = s_1 + s_2$ with $s_1 = dz \cdot \chi_B$ and $s_2 = dz \cdot (1 - \chi_B)$. By Lemma 7 we have $|u_2 + s_2| = |df^*v \cdot (1 - \chi_B)| \leq |df^*v \cdot \chi_C| \leq c_2|dv \cdot \chi_D| \leq c_2|dv \cdot \chi_{V(G) \setminus A_0}| < c_2\epsilon$ with a constant c_2 . We can write $0 = \langle dv', dz \rangle = \langle u_1, s_1 \rangle + \langle u_2, s_2 \rangle$. Here $|\langle u_1, s_1 \rangle| \leq |u_1| \cdot |s_1| \leq |u_1| \cdot |dz| \leq c_1|u_1|$ and $\langle u_2, s_2 \rangle = \langle u_2, u_2 + s_2 \rangle - |u_2|^2 \leq |u_2| \cdot |u_2 + s_2| - |u_2|^2 < -|u_2|(|u_2| - c_2\epsilon)$. We have $|u_2| \geq |dv'| - |u_1| \geq c_0 - |u_1|$ hence $c_1|u_1| \geq \langle u_1, s_1 \rangle = -\langle u_2, s_2 \rangle \geq |u_2|(|u_2| - c_2\epsilon) \geq (c_0 - |u_1|)(c_0 - c_2\epsilon - |u_1|)$ if $c_0 - c_2\epsilon - |u_1| > 0$.

Notice that we can choose a small enough ϵ depending on c_0 , c_1 , and c_2 such that $|u_1| \leq c_1\epsilon$ contradicts our last inequality. For this ϵ we have $|u_1| > c_1\epsilon$. We take $w = v'/|dv'| \in HD(G')$ and notice that $|dw| = 1$ and $|dw \cdot \chi_B| = |u_1|/|dv'| > \epsilon$ as $|dv'| \leq |df^*v| \leq c_1$. The choice of ϵ (and thus of A) depends only on G , v , k and Δ . ■

Now we finish the proof of our Theorem. The proof is by contradiction. Let G be a roughly transitive amenable graph with $dHD(G) \neq 0$. Let k be the bound on the distortion of the quasi-isometries f_{xy} from G to G mapping $x \in V(G)$ to $y \in V(G)$. Choose the finite set $A \subseteq V(G)$ and the number $\epsilon > 0$ for G , k and the maximum degree Δ of G as claimed

in Lemma 10. We fix a vertex $y \in A$ and use the statement of Lemma 10 to obtain a function $w_x \in HD(G)$ for each $x \in V(G)$ such that $|dw_x| = 1$ and $|dw_x \cdot \chi_{B_x}| > \epsilon$ for $B_x = f_{xy}^{-1}(A)$. Let $a = \max_{y' \in A} \delta(y, y')$. By the bound on the distortion of f_{xy} we have $\delta(x, x') < k(a + 1) = b$ for any $x \in V(G)$ and $x' \in B_x$.

Let (G_i) be a Følner sequence of finite subgraphs of G . Let $S_i = V(G_i) \setminus C_b(\sigma(G_i))$. Note that as b is constant and the degree of the vertices is limited, $|C_b(\sigma(G_i))|$ is proportional to $|\sigma(G_i)|$ thus $|S_i|/|V(G_i)|$ tends to 1 as i tends to infinity. For $x \in S_i$ we have $B_x \subseteq C_b(\{x\}) \subseteq V(G_i)$. Consider the projection P in $l_2(G)$ to $dHD(G)$ and let P_x the projection to the one dimensional subspace of $dHD(G)$ generated by dw_x . Recall that we identify an oriented edge $e \in \vec{E}(G)$ with the function in $l_2(G)$ mapping e to 1, \bar{e} to -1 and everything else to zero. We have $\langle Pe, e \rangle \geq \langle P_x e, e \rangle = \langle dw_x, e \rangle^2 = (dw_x(e))^2$ for every oriented edge e and vertex x in G . We can write $\epsilon^2 < |dw_x \cdot \chi_{B_x}|^2 = 1/2 \sum_{e \in \vec{E}(G)} \chi_{B_x}(e) (dw_x(e))^2 \geq 1/2 \sum_{e \in \vec{E}(G)} \chi_{B_x}(e) \langle Pe, e \rangle$ for any $x \in V(G)$. By summation we get $2\epsilon^2|S_i| \leq \sum_{e \in \vec{E}(G)} \langle Pe, e \rangle \sum_{x \in S_i} \chi_{B_x}(e)$ for any index i . Notice that $\sum_{x \in S_i} \chi_{B_x}(e)$ is zero for oriented edges e outside $\vec{E}(G_i)$, while it is bounded by a constant C for any $e \in \vec{E}(G)$, one can take C to be the maximum size of the b -neighborhood of a single vertex. Thus we have $2\epsilon^2|S_i|/C \leq \sum_{e \in \vec{E}(G_i)} \langle Pe, e \rangle = |\vec{E}(G_i)| \dim_{G_i}(dHD(G))$. Consequently, $\dim_{G_i}(dHD(G)) \geq \epsilon^2/C \cdot |S_i|/|E(G_i)|$. By Corollary 6 the left hand side of this last inequality tends to zero as i tends to infinity, but as $|S_i|/|V(G_i)|$ tends to 1 and $|V(G_i)|$ is proportional to $E(G_i)$ the right hand side does not. The contradiction proves the theorem. \blacksquare

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