TransvERSALS of d-intervals—comparing three approaches

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1. The problem

In this short note we compare three different methods for solving the same combinatorial problem. We start by describing the problem itself.

Consider any set system F of non-empty sets. By the *matching number* $\nu(F)$ of F we mean the maximal number of pairwise disjoint sets of F. A *transversal* of F is a set S that nontrivially intersects all sets in F. In this case S is said to *cover* F. By the *transversal number* $\tau(F)$ of F we mean the minimum cardinality of a transversal of F.

We clearly have

$$\nu(F) \leq \tau(F)$$

for any set system F, but no inequality holds in full generality in the opposite direction. Finding special classes of set systems for which equality holds or the transversal number can be bounded in terms of the matching number is a central problem in combinatorial duality theory.

The König-Hall theorem gives an example for such a class. Considering bipartite graphs as sets of edges and identifying an edge with the set of its two endpoints one gets that $\tau = \nu$ for finite bipartite graphs. For general (non-bipartite) graphs as set systems one trivially has $\tau \leq 2\nu$ which is tight as graphs consisting of triangle components show.

Another example where equality holds was noticed by Gallai (see Hajnal and Surányi [5]): For finite sets F of intervals of a line we have $\tau(F) = \nu(F)$.

The class of set systems we consider in this note is closely related to the second example mentioned. Let d be a positive integer. A *d-interval* is a non-empty union of d closed intervals on the line. We are going to compare the matching and transversal numbers of families of d-intervals.

Sometimes it is more natural to assume that the components of the d-intervals are separated, they lie on distinct lines. Let us fix d distinct parallel lines and define a *separated d-interval* to be the union of d non-empty closed intervals, one on each line. If I is a d-interval and $1 \leq i \leq d$ we say that I^i, the i'th *component* of I is the interval of I on the i'th line. We remark here that part of the literature uses the name "d-interval" for this and they use "homogeneous d-interval" for our previous definition.

Let us note here that the assumption of the (separated) d-intervals to be closed is not crucial, it can be replaced by an assumption on the finiteness of the set systems we consider. One of these assumptions is essential though, for infinite families of open intervals even Gallai's observation fails.

Let us define here the functions we are interested in. For positive integers k and d let let us define $g(d,k)$ to be the maximal transversal number of any family of d-intervals with matching number at most k. Similarly let $f(d,k)$ be the maximal transversal number of any family of separated d-intervals with matching number at most k. In what follows we survey the techniques used in upper bounding these functions. In this notation Gallai's observation can be stated as

$$f(1,k) = g(1,k) = k.$$

We discuss the three methods for bounding f and g in the order they were invented. Separated d-intervals were introduced by Gallai and he posed the problem of finding $f(d,1)$. Gyárfás and Lehel [4] established that for families of (separated) d-intervals the transversal number can be bounded in terms of d and the matching number, i.e. $f(d,k)$ and $g(d,k)$ are finite. The actual bound they got is very weak (and it is left implicit in their paper). Their strictly elementary argument is discussed in the next section. We also mention elementary lower bounds for f and g and a connection between these functions in the same section.

In the third section we discuss the topological approach. Surprisingly it is the topology of higher dimensional spaces that gives the best known upper bound for the transversal number of (separated) d-intervals. It gives tight results for both 2-intervals and separated 2-intervals. This technique was used by Tardos [12] and Kaiser [6].

In the fourth section we discuss the approach of Alon [1] based on the duality of linear programming. The bounds achieved are much better than the ones discussed in the second chapter but worse by a factor around 2 than the ones we get from the topological approach. The major advantage of this method is simplicity, Alon's paper is less than a page.

In the last section we mention related results.

2. The elementary approach

Gyárfás and Lehel [4] were first to prove that for families F of separated d-intervals $\tau(F)$ is bounded in terms of $\nu(F)$ and d. They proved

Theorem 1. [4] *For positive integers d and k we have*

$$f(d+1,k) \leq f(d,k((k+1)^d-1))+k.$$

This proves that $f(d,k))$ is always finite but the bound one gets is worse than $k^{d!}$. To illustrate the proof technique we show the similar proof of a somewhat stronger statement in Károlyi and Tardos [8]:

Theorem 2. [8] *For integers $d \geq 2$ and $k \geq 1$ we have*

$$f(d,k) \leq f(d,k-1)+f(d-1,k(d-1))+(d-1)^2k+1.$$

Proof: Let F be a family of separated d-intervals with $\nu(F)=k$. For compactness reasons we may assume F is finite. Imagine sliding a point x on the first line from left to right and stopping as soon as we have a set $S \subset F$ of $(d-1)k+1$ d-intervals satisfying that pairwise intersections of members of S are contained in the first line and the first component of each d-interval in S is to the left of x possibly containing x. We may suppose that such point x exists as otherwise clearly $\nu(F) \leq f(d-1,k(d-1))$. Let us fix x and S.

We partition $F=F_1 \cup F_2 \cup F_3$ by putting the d-interval of $I \in F$ into F_1 if the first component I^1 is strictly to the left of x, we put $I \in F_2$ if $x \in I^1$ and finally we put $x \in F_3$ if I^1 is strictly to the right of x. We can find a transversal of F_1 consisting of $f(d-1,k(d-1))$ points outside the first line. The singleton $\{x\}$ clearly covers F_2. It remains to prove that $\tau(F_3) \leq f(d,k-1)+k(d-1)^2$. For $2 \leq i \leq d$ consider the components $\{I^i | I \in S\}$ and take a set C_i of $k(d-1)$ points on the i'th line separating these $k(d-1)+1$ disjoint intervals.

Let F_3' be the part of F_3 not covered by any of the sets C_i. Now each component of a d-interval in F_3' can intersect at most one element of S (and the first component intersects none). Thus if we had a family $S' \subset F_3'$ of k pairwise disjoint d-intervals then they intersect at most $(d-1)k$ members of S meaning that we could extend S' with an element of S to get $k+1$ pairwise disjoint d-intervals in F. The contradiction shows $\nu(F_3') < k$ and thus $\tau(F_3') \leq f(d, k-1)$. Therefore $\tau(F_3) \leq f(d, k-1) + k(d-1)^2$ as required. ■

Using the base cases $f(1,k) = k$ and $f(d,0) = 0$ the result above yields $f(d,k) = O(k^d)$ for any fixed d. This result is still far from optimal, for better bounds see the two following sections.

One can regard separated d-intervals as special cases of d intervals. This yields the trivial connection between f and g:

Lemma 3. *For every d and k we have $f(d,k) \geq g(d,k)$.*

Proof: For any family F of separated d-intervals we construct a family F' of d-intervals with $\tau(F') = \tau(F)$ and $\nu(F') = \nu(F)$. We take a function ϕ that maps the d lines of the separated d-intervals homeomorphically to d disjoint open intervals of the one line of the d-intervals. Now ϕ maps separated d-intervals to d-intervals and we can simply take $F' = \{\phi(I) | I \in F\}$ ■

Inequality in the other direction is less obvious. In [4] the finiteness of g was established by showing:

Lemma 4. [4] *For $d \geq 2$ and $k \geq 1$ we have*

$$g(d,k) \leq (g(d-1,k))^{d-1} f(d,k) + \sum_{i=1}^{d-1} (g(d-1,k))^i.$$

A stronger bound appears in [8]:

Lemma 5. [8] *For every d and k we have*

$$g(d,k) \leq f(d, 2d(d-1)k).$$

We include the sketch of the proof of this weak result because it nicely complements the trivial observation of Lemma 3.

Proof sketch: For any family F of d-intervals one can construct a family F' of separated d-intervals such that the d-intervals in F are obtained from the members of F' by perpendicular projection. Clearly $\tau(F) \leq \tau(F')$ in this case. Thus it is enough to prove that $\nu(F') \leq 2d(d-1)\nu(F)$. This is so, since for any pairwise disjoint family S of separated d-intervals the greedy algorithm finds a subset of size $\lceil |S|/(2d(d-1)) \rceil$ of whose projections are also pairwise disjoint. ■

We conclude this section with an easy lower bound on f.

Theorem 6. [8] *For positive integers d and k we have*

$$f(d,k) \geq kf(d,1) \geq dk.$$

Proof: We identify the d lines on which our separated d-intervals lie with the line of the real numbers. Let us consider the set F of separated d-intervals I defined by $I^i = [a, b_i]$ for $i = 1, \ldots, d$ for all possible values $0 < a < 1$, $a < b_i \leq 1$ for all $i = 1, \ldots d$ and $b_i = 1$ for at least one $i = 1, \ldots d$. It is easy to see that the longest component in any pair of separated d-intervals of F contains an entire component of the other separated d-interval, thus the pair is intersecting and we have $\nu(F) = 1$. One can also see that any finite transversal contains the point 1 in each of the d lines, thus $\tau(F) \geq d$ and we proved the second inequality.

To prove the first inequality consider a pairwise intersecting family F of separated d-intervals with transversal number $f(d,1)$. We may suppose F covers a finite segment of the d lines and thus we may take the union of k translates of F covering pairwise disjoint segments. This construction clearly multiplies both the matching and the transversal numbers by k. ■

Notice that the the family constructed in the proof above satisfies that we get the same ordering for any $i = 1, \ldots, d$ if we order the separated d-intervals according the left endpoint of their i'th component. It was noticed in [8] that this bound is tight for such families, i.e. this property implies $\tau(F) \leq d\nu(F)$

As we are going to see in the next section (see Theorem 7) that the bound of Theorem 6 is tight for $d = 2$. It is however not tight in general. An example of 10 separated 3-intervals in [4] shows $f(3,1) \geq 4$ (equality is also established there). Theorem 6 with Lemma 3 shows $g(d,k) \geq dk$. This is not tight even for $d = 2$. An example of 6 2-intervals in [4] shows $g(2,1) \geq 3$ (equality is also established there, while $g(2,k) = 3k$ is shown in [6], see Corollary 12).

3. The topological approach

Using the topology of higher dimensional spaces in the investigation of transversals of d-intervals was initiated in [12] where the present author used simplicial complexes to establish the tight bound for separated 2-intervals:

Theorem 7. [12] $f(2,k) = 2k$.

Proof sketch: Let us take a family F of separated 2-intervals. We may suppose F covers a finite part of the two lines, say each element of F is contained in the separated 2-interval $[X_0, X_{k+1}] \cup [Y_0, Y_{k+1}]$. Now consider the $2k$-tuples of points $T = (X_1, \ldots, X_k; Y_1, \ldots, Y_k)$ with $X_0 \leq X_1 \leq \ldots \leq X_{k+1}$ and $Y_0 \leq Y_1 \leq \ldots \leq Y_{k+1}$. The space S of these candidate transversals T is the direct product of two k-simplexes. We define bad subsets $B_{i,j} \subset S$ for $0 \leq i, j \leq k$ by letting $T \in B_{i,j}$ if and only if there exists an element of F contained in $(X_i, X_{i+1}) \cup (Y_j, Y_{j+1})$.

The hard part of the proof is to show the following for any collection of open subsets $B_{i,j} \subseteq S$. If $T \notin B_{i,j}$ holds whenever $X_i = X_{i+1}$ or $Y_j = Y_{j+1}$ (boundary condition) then either the union of all the $B_{i,j}$ does not cover S or else we have a permutation σ of $\{0,\ldots,k\}$ such that the intersection of the sets $B_{i,\sigma(i)}$ is nonempty.

The proof of this claim in [12] is tedious and uses the topology of simplicial complexes. We omit it here.

To apply the above statement we have to show that our bad sets $B_{i,j}$ satisfy the conditions there. The boundary condition is trivial. The sets $B_{i,j}$ are open since the separated d-intervals in F are closed. Therefore we have one of the two possible consequences there.

By the definition of the bad sets $B_{i,j}$ any point $T = (X_1,\ldots,X_k;Y_1,\ldots,Y_k) \in S$ outside all of the bad sets represents a transversal $\{X_1,\ldots,X_k,Y_1,\ldots,Y_k\}$ of F. Thus if $\cup_{0\le i,j\le k} B_{i,j} \ne S$ we have $\tau(F) \le 2k$.

If $T = (X_1,\ldots,X_k;Y_1,\ldots,Y_k) \in \cap_{i=0}^{k} B_{i,\sigma(i)}$ for a permutation σ of $\{0,\ldots,k\}$ then the elements I_i of F contained in $(X_i,X_{i+1})\cup(Y_j,Y_{j+1})$ are pairwise disjoint for $i=0,\ldots,k$ thus $\nu(F) \ge k+1$.

For families F of separated 2-intervals with $\nu(F) \le k$ we thus must have $\tau(F) \le 2k$ and therefore $f(2,k) \le 2k$ as claimed. The inequality in the opposite direction comes from Theorem 6. ■

Notice the striking similarity of the core topological statement of the proof to Sperner's Lemma. As it was already pointed out in [12] the same proof can bound $f(d,k)$ for arbitrary d if the right analog of the topological statement on the alternatives is proved. This was implicitly done by Kaiser [6]. He also simplified the presentation of the argument and through the use of the Borsuk-Ulam theorem greatly simplified the proof.

We present here Kaiser's upper bound on $g(d,n)$, i.e. the transversal number of d-intervals.

Theorem 8. [6] $g(d,k) \le (d^2-d+1)k$

Proof: Take a family F of d-intervals. We may suppose again that all the d-intervals of F are contained in the unite interval $[0,1]$.

Let us fix $n \ge 1$ and define the n-simplex $S = \{(y_1,\ldots,y_n)|0 \le y_1 \le \ldots \le y_n \le 1\}$. We think of $(y_1,\ldots,y_n) \in S$ to represent the set $\{y_1,\ldots,y_n\}$ and thus S is the space of transversal-candidates for F of size at most n. We set $V = \{0,1,\ldots,n\}$. A point $y = (y_1,\ldots,y_n) \in S$ determines the open intervals $L_i(y) = (y_i,y_{i+1})$ for $i \in V$ where we take $y_0 = 0$ and $y_{d+1} = 1$. (Note that some of these intervals may be empty.) For a set $e \subset V$ we define

$$w_y(e) = \sup_I dist(y,I)$$

with I ranging over the d-intervals of F contained in $\cup_{i\in e} L_i(y)$ but intersecting every $L_i(y)$ with $i \in e$. Here $dist$ denotes minimum distance between sets on the line.

For $i \in V$ and $y \in S$ we define

$$w_y(i) = \sum_{e\ni i} w_y(e).$$

Observation 9.

(a) $w_y(e) \ge 0$, *furthermore* $w_y(e) = 0$ *for every set* e *of size* $|e| > d$ *or for* $e = \emptyset$.

(b) *$w_y(e) = 0$ if there exist an index $i \in e$ with $L_i(y) = \emptyset$ and thus $w_y(i) = 0$ if $L_i(y) = \emptyset$*
(c) *if $w_y(e) = 0$ for every set $e \subset V$ then y represents a transversal of F.*
(d) *as a function of $y \in S$ all the functions $w_y(e)$ $(e \subset V)$ and thus the functions $w_y(i)$ $(i \in V)$ are continuous.*

Proof: (a) follows from the fact that F is a collection of nonempty d-intervals. Parts (b) and (d) are trivial. For (c) notice that any element of F disjoint from the set represented by y has a positive contribution to the supremum in the definition of $w_y(E)$ for a (unique) set E. Here we also use that F consists of closed sets. ■

The following observation is the heart of this proof. This is the point, where topology is used. Note that we did not suppose anything about the family F or the parameter n for the observation.

Lemma 10. *There exist a point $y \in S$ for which $w_y(i)$ is independent of $i \in V$.*

Proof: To be able to use the Borsuk-Ulam theorem we have to shift attention from the simplex S to the sphere $S^n = \{(z_0, \ldots, z_n | \sum z_i^2 = 1\} \subset R^{n+1}$. We do it through the map $g : S \to S^n$ defined by

$$g(z_0, \ldots, z_n) = (t_1, \ldots, t_n),$$

$$t_i = \sum_{j=0}^{i-1} z_i^2, \quad \text{for } i = 1, \ldots, n.$$

Note that $g(z)$ does not depend on the sign of the coordinates of z.

We combine g with the weight functions $w_y(i)$ to define the functions $h_i : S^n \to R$ for $i = 1, \ldots, n$ by

$$h_i(z) = \mathrm{sign}(z_i) w_{g(z)}(i) - \mathrm{sign}(z_0) w_{g(z)}(0)$$

where $z = (z_0, \ldots, z_n)$.

Notice that by Observation 9 (d) all functions in the above definition are continuous except for the sign function. But at the jump of the sign function $\mathrm{sign}(z_i)$ $(i \in V)$ we have $L_i(g(z)) = \emptyset$ and thus by Observation 9 (b) we have $w_{g(z)}(i) = 0$. Therefore the functions h_i are continuous everywhere.

Notice also the antipodality: $h_i(-z) = -h_i(z)$ follows from $g(z) = g(-z)$. Now we can use the Borsuk-Ulam theorem. It states that n continuous, antipodal, real functions on S^n always have a common zero. (See e.g. [11].) In particular there exists a $z \in S^n$ for which $h_i(z) = 0$ for $i = 1, \ldots, n$. This clearly implies that for $y = g(z)$ we have $w_y(i) = w_y(0)$ for $i = 1, \ldots, n$ as claimed. ■

For the rest of the proof we fix $y \in S$ to be the point claimed in Observation 9 and let x be the common value of $w_y(i)$ for $i \in V$. The connection between the matching and transversal numbers of F will follow from the following alternative. If $x = 0$ then the transversal number is small, if $x \neq 0$ then the matching number is large.

Suppose first that $x = 0$. Then clearly $w_y(e) = 0$ for all $e \subset V$. By Observation 9 (c) this implies $\tau(F) \leq n$.

Let us now suppose that $x > 0$. We define $M = (V, E)$ to be the hypergraph on the vertex set V with the edge set $E = \{e \subset V | w_y(e) > 0\}$.

Let us first observe that $\nu(M) \le \nu(F)$. Indeed, we can choose a d-interval $I_e \in F$ for every edge $e \in E$ such that $I_e \subset \cup_{i \in e} L_y(i)$. The d-intervals chosen for disjoint edges are clearly disjoint, thus the inequality holds as claimed. Thus it is enough for us to lower bound $\nu(M)$.

A *partial matching* of a hypergraph N is an assignment of non-negative weights to the edges such that the sum of weights on edges containing any particular vertex is at most 1. The *partial matching number* $\nu^*(N)$ is the supremum of the total weights assigned by any partial matching of M. We clearly have $\nu^*(N) \ge \nu(N)$. We will later need inequalities in the other direction.

The function $w_y(e)/x$ clearly forms a partial matching of M. Therefore $\nu^*(M) \ge \sum_{e\in E} w_y(e)/x$. Here by Observation 9 (a) all edges in E have size at most d and thus $d \sum_{e\in E} w_y(e) \ge \sum_{e\in E} |e| w_y(e) = \sum_{i\in V} \sum_{e \ni i} w_y(e) = (n+1)x$. Thus

$$\nu^*(M) \ge \frac{n+1}{d}.$$

The last task to finish the proof is to connect $\nu^*(M)$ and $\nu(M)$. The following observation is trivial. For every hypergraph M with edges having at most d elements we have:

$$\nu^*(M) \le d\nu(M).$$

For the proof notice that if $M' \subset M$ is any maximal matching then every edge of M contains one of the points in $H = \cup M'$ and thus the weight of any edge in a fractional matching is counted at the vertex weights of vertices in H thus $\nu^*(M) \le |H| \le d|M'|$. Notice that this already implies

$$g(d,k) \le d^2 k.$$

Slight improvement of the trivial argument above is given by Füredi in [3]:

Lemma 11. [3] *For every hypergraph M with edges having at most d elements we have: $\nu^*(M) \le (d - 1 + 1/d)\nu(M)$.*

Now we have that either $x = 0$ and then $\tau(F) \le n$ or $x \ne 0$ and then $\nu(F) \ge \nu(M) \ge \nu^*(M)/(d - 1 + 1/d) \ge (n + 1)/(d^2 - d + 1)$. Thus with the choice $n = (d^2 - d + 1)k$ for any family F with $\nu(F) \le k$ we must not have $x \ne 0$ thus we must have $\tau(F) \le n$. This finishes the proof of Theorem 8. ■

We further remark here that in case $d > 2$ and there is no projective plane of order $d - 1$ the statement of Lemma 11 can be made stronger to $\nu^*(M) \le (d - 1)\nu(M)$, see [3]. This leads to the tighter bound $g(d,k) \le (d^2 - d)k$ for such d. Let us also mention that the bound of Theorem 8 is tight for the $d = 2$ special case:

Corollary 12. [6] $g(2,k) = 3k$

Proof: $g(2,k) \le 3k$ is a special case of Theorem 8. The example in [4] already mentioned at the end of Section 2 shows $g(2,1) \ge 3$ and $g(2,k) \ge kg(2,1)$ is trivial (similarly to the first inequality of Theorem 6). ■

We remark that the special case $d = 2$ of Lemma 11 was first proved by Lovász in [9].

Using Lemma 3 one sees that the bound of Theorem 8 also applies for f in place of g. Using the proof technique directly the paper [6] establishes the slightly better bound:

Theorem 13. [6] $f(d,k) \leq (d^2-d)k$.

The proof is similar but one has to use a recent extension of the Borsuk-Ulam theorem by Ramos [10]. This theorem is about the common zeros of functions defined on product of spheres. Notice that Theorem 7 is a special case of this Theorem 13.

4. Using the duality theorem

In this section we survey the short note of Noga Alon [1]. This paper uses the method of Alon and Kleitman [2], it is based on the duality theorem for linear programming and on Turán's theorem in extremal graph theory. It is worth to include the short proof in its entirety.

Theorem 14. [1] $g(d,k) \leq 2d^2k-1$.

Proof: Let F be a family of d-intervals with $\nu(F) \leq k$. For compactness reasons it is again enough to consider finite families, let $n=|F|$. As there are no $k+1$ pairwise disjoint elements of F Turán's theorem yields that there are more than $n^2/(2k)$ unordered pairs of (not necessarily) distinct intersecting members of F. For any such pair P one finds two ordered triplets (p,I,I') with p being an endpoint in a member of the d-interval I, $p \in I'$ and $\{I,I'\}=P$. There are at most $2dn$ endpoints p thus one of them is contained in $n/(2dk)$ d-intervals of F.

Let us assign non negative rational weights a_I/b to the d-intervals $I \in F$. Applying the above argument to a multiset F' obtained from F via replicating each d-interval I a_I times, one gets a point p that lies on a collection of d-intervals representing more than $1/(2dk)$ fraction of the total weight. We apply the min-max theorem of linear programming to deduce the existence of a multiset M of m points such that each d-interval in F contains more than $m/(2dk)$ of them. Now we select the points in M whose rank in the linear ordering is divisible by $\lfloor m/(2d^2k)\rfloor+1$. Any interval disjoint from these points contains at most $m/(2d^2k)$ points of M thus non of the d-intervals in F can be disjoint of the less than $2d^2k$ points selected. ■

The bound obtained here is worse than Kaiser's bound discussed in the preceding section by a factor approaching 2 for large d.

5. Applications and open problems

It is not surprising that one can use the connection of the matching and transversal numbers of (separated) d-intervals to obtain similar connection for other families of (mostly geometrical) sets. We give one example here. Let S be a closed set in the Euclidean plane. We define an S-*box* to be the intersection of an axis-parallel rectangle with S. In [8] Theorem 7 is used to prove

Theorem 15. [8] *For any family F of S-boxes*

$$\tau(F) \leq 4\nu(F)\lfloor \log \nu(F)+1\rfloor^2.$$

It is perhaps more noteworthy when a topological method similar to that of Section 3 is proved to be applicable in a different setting. An example is the paper of Kaiser and

Rabinovich [7] about a Helly-type problem. They consider *convex* (n,d)*-bodies*, ordered n-tuples of convex sets in R^d. A family F of convex (n,d)-bodies is called *weakly intersecting* if there is an n-tuple $(p_1,\dots,p_n)$ of points in R^d such that for every convex (n,d)-body $(K_1,\dots,K_n) \in F$ one has $p_i \in K_i$ for at least one $i = 1,\dots,n$. The family F is called strongly intersecting if there exist a point $p \in R^d$ and an index $1 \le i \le n$ with $p \in K_i$ for each convex (n,d)-body $(K_1,\dots,K_n) \in F$.

Theorem 16. [7] *Suppose every subfamily F' of the family F of convex (n,d)-bodies is strongly intersecting if $|F'| \le \lceil \log_2(n+2) \rceil^d$. Then F is weakly intersecting.*

Finding out if there are more combinatorial problems where a similar line of reasoning is fruitful and what these problems are can be an interesting research project.

Writing a survey paper about a simple combinatorial problem of finding the values of the function $f(d,k)$ and $g(d,k)$ we must admit that none of these values is known exactly for $d \ge 3$ and $k \ge 1$ except for the single case $f(3,1) = 4$ settled in [4]. It seems that the most intriguing question is to find the order of magnitude for the maximum transversal number of an *intersecting* family of d-intervals. Note that the upper bound for this number $g(d,1)$ is quadratic (Theorem 8) while the lower bound is linear (Theorem 6).

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