Distinct Distances in Three and Higher Dimensions*

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Abstract

Improving an old result of Clarkson *et al.*, we show that the number of distinct distances determined by a set P of n points in three-dimensional space is $\Omega(n^{77/141-\varepsilon}) = \Omega(n^{0.546})$, for any $\varepsilon > 0$. Moreover, there always exists a point $p \in P$ from which there are at least so many distinct distances to the remaining elements of P. The same result holds for points on the three-dimensional sphere. As a consequence, we obtain analogous results in higher dimensions.

1 Introduction

"My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances" – wrote Erdős on his 80th birthday [8]. What is the minimum number of distinct interpoint distances determined by n points in $\mathbb{R}^d$? More precisely, Erdős [7] asked the following question in 1946. Given a point set P, let $g(P)$ denote the number of distinct distances between the elements of P. Let $g_d(n) = \min_P g(P)$, where the minimum is taken over all sets P of n distinct points in d-space. We want to describe the asymptotic behavior of the function $g_d(n)$. More than 50 years later, in spite of considerable efforts, we are still far from knowing the correct order of magnitude of $g_d(n)$ even in the plane ($d = 2$). This problem is more than just a "gem" in recreational mathematics. It was an important motivating

*Work on this paper has been supported by a grant from the U.S.-Israeli Binational Science Foundation. Work by Boris Aronov has also been supported by NSF Grants CCR-99-72568 and ITR CCR-00-81964. Work by János Pach and Micha Sharir has also been supported by NSF Grants CCR-97-32101 and CCR-00-98246. Work by János Pach has also been supported by a PSC-CUNY Award, and work by Micha Sharir by a grant from the Israel Science Fund (for a Center of Excellence in Geometric Computing), and by the Hermann Minkowski–MINERVA Center for Geometry at Tel Aviv University. Work by Gábor Tardos has also been supported by Hungarian Science Foundation grants OTKA T029255, OTKA T030059, and AKP 2000-78 2.1.

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trigger, along with several companion problems, such as those of repeated distances and of point-curve incidences, that was partially responsible for the discovery of important new concepts and methods (levels in arrangements, space decompositions, cuttings, epsilon-net techniques, etc.) which proved to be relevant in many areas of discrete and computational geometry, including motion planning and ray shooting.

Erdős' question is intimately related to another problem raised in the same paper: what is the maximum number of times that the *same* distance can occur among n points in d-space? Denoting this function by $f_d(n)$, we clearly have, by the pigeonhole principle, that

$$g_d(n) \geq \frac{\binom{n}{2}}{f_d(n)}. \tag{1}$$

It was proved, respectively, by Spencer et al. [16] and by Clarkson et al. [4] that $f_2(n) = O(n^{4/3})$ and $f_3(n) = O(n^{3/2}\beta(n))$, where $\beta(n) = 2^{O(\alpha^2(n))}$ and $\alpha(n)$ is the extremely slowly growing inverse Ackermann's function. Therefore, we have

$$g_2(n) = \Omega(n^{2/3}) \text{ and } g_3(n) = \Omega(n^{1/2}/\beta(n)). \tag{2}$$

Because small multiplicative factors such as $\beta(n)$ often appear in our calculations, we introduce the notation $f(n) = \widetilde{O}(g(n))$ to denote $f(n) = O(g(n)n^{\varepsilon})$, for any $\varepsilon > 0$, with the implied constant depending on ε. Similarly, $f(n) = \widetilde{\Omega}(g(n))$ means that $f(n) = \Omega(g(n)n^{-\varepsilon})$, for any $\varepsilon > 0$. With this notation, a single distance occurs at most $\widetilde{O}(n^{3/2})$ times among n points in three dimensions, and thus $g_3(n) = \widetilde{\Omega}(n^{1/2})$.

For $d \geq 4$, the above "naive" approach based on (1) cannot give any nontrivial lower bound on the number $g_d(n)$ of distinct distances determined by n points, because we have $f_d(n) \geq n^2/4$. To see that a single distance can appear $n^2/4$ times, consider two circles centered at the origin, whose planes are orthogonal to each other, and place $n/2$ points on each of them. Observe that all distances between a point on one of the circles and a point on the other are the same.

The first bound in (2) has been subsequently improved by Chung et al. [3], Székely [17], Solymosi and Tóth [15], Tardos [19], Katz [9], and Katz and Tardos [10] culminating in the lower bound $g_2(n) = \widetilde{\Omega}(n^{(48-14e)/(55-16e)}) = \Omega(n^{0.8641})$. On the other hand, the best known upper bound, which is due to Erdős and is conjectured to be sharp, is $g_2(n) = O(n/\sqrt{\log n})$. It is attained by the set of vertices of the $n^{1/2} \times n^{1/2}$ integer lattice.

In three dimensions, however, nothing better was known than the "naive" bound in (2). The aim of this paper is to present such an improvement. Specifically, we have

Theorem 1.1. *A set P of n points in three dimensions determines at least*

$$\widetilde{\Omega}\left(n^{77/141}\right) = \Omega\left(n^{0.546}\right)$$

distinct distances. Moreover, there always exists a point $p \in P$ that determines at least so many distinct distances to the remaining points of P.

The number of distinct distances determined by the vertices of an $n^{1/3} \times n^{1/3} \times n^{1/3}$ integer lattice is $\Theta(n^{2/3})$. That is, we have $g_3(n) = O(n^{2/3})$, and it is conjectured that this bound is not far from being sharp. For more problems and results of this type, consult [12].

Let $t_p(P)$ stand for the number of distinct distances between a point p and the elements of $P \setminus \{p\}$, and put $t(P) = \max_{p \in P} t_p(P)$. Finally, let $t_3(n) = \min t(P)$, where the minimum is taken over all n-element point sets P in $\mathbb{R}^3$. Clearly, $t(P) \le g(P)$ and our result can be stated as

$$g_3(n) \ge t_3(n) = \widetilde{\Omega}\left(n^{77/141}\right) = \Omega(n^{0.546}).$$

As in most of all the earlier approaches to the planar problem [3, 4], our proof establishes an upper bound for the number $I(P, S)$ of incidences between the points in P and the set S of all spheres around the elements of P, passing through at least one other point of P. Clearly, the number of these spheres is at most $nt(P)$. Since $I(P, S)$ is $n(n-1)$, this leads to an inequality for $t(P)$, whose solution yields a lower bound for $t(P)$.

The most serious technical difficulty in the proof of our main result is that in three dimensions we may encounter 'large' configurations, each involving many points of P lying on a circle and many other points of P lying on the line orthogonal to the circle and passing through its center (the 'axis' of the circle). See Figure 1. This leads to a complete bipartite pattern of incidences between the set of points on the circle and the set of spheres centered at the points on the axis and passing through the circle. Such configurations hinder the derivation of a sharp bound for $I(P, S)$, so we start the proof by removing all points of P that lie on lines containing too many points.

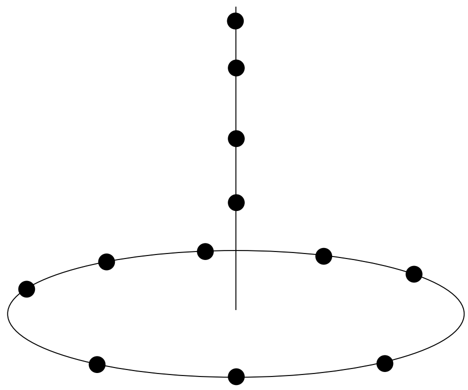

Figure 1: A complete bipartite incidence pattern.

We also extend the results of Theorem 1.1 to point sets in the three-dimensional unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$:

Theorem 1.2. *A set P of n points in $\mathbb{S}^3$ determines at least*

$$\widetilde{\Omega}\left(n^{77/141}\right) = \Omega\left(n^{0.546}\right)$$

distinct distances. Moreover, there always exists a point $p \in P$ that determines at least so many distances to the remaining points of P.

The spherical result readily generalizes to distances in higher dimensions:

Corollary 1.3. *For $d \geq 3$, any set P of n points in Euclidean d-space $\mathbb{R}^d$ or on the d-sphere $\mathbb{S}^d$ determines at least*

$$\widetilde{\Omega}\left(n^{1/(d-\frac{90}{77})}\right)$$

distinct distances. Moreover, there always exists a point $p \in P$ that determines at least so many distances to the remaining points of P.

As for $d = 3$, the number of distinct distances determined by an $n^{1/d} \times \cdots \times n^{1/d}$ portion of the integer lattice is $O(n^{2/d})$, so that we have $g_d(n) = O(n^{2/d})$.

The next three sections are devoted to the proof of Theorem 1.1. We prove Theorem 1.2 and Corollary 1.3 in Section 5. In Section 6 we make a few concluding remarks.

Very recently, Solymosi and Vu [13] have proved that $g_3(n) = \Omega(n^{0.564})$, thus improving our Theorem 1.1. They have also extended their analysis to higher dimensions, showing that $g_d(n) = \Omega\left(n^{\frac{2}{d}-\frac{2}{d(d+2)}}\right)$. Note that in their result the exponent becomes very close to the exponent of the upper bound $g_d(n) = O(n^{2/d})$, as d increases.

2 Lines with many points

We need the following two results.

Theorem A ([18, 4, 17]). *The number of incidences between n points and m pseudo-segments (i.e., Jordan arcs, any two of which have at most one point in common) is*

$$O(n^{2/3}m^{2/3} + n + m).$$

Theorem B ([1]). *The number of incidences between n points and m circles in $\mathbb{R}^d$ is*

$$\widetilde{O}(n^{6/11}m^{9/11} + n^{2/3}m^{2/3} + n + m).$$

Let P be a set of n points in $\mathbb{R}^3$. Let $t = t(P)$ and recall that our final goal is to prove a lower bound on t.

As mentioned in the introduction, we have to pay special attention to the lines containing many points of P. In this section we establish a reasonably small threshold μ_0 so that only a negligibly small number of points of P lie on lines that contain more than μ_0 points.

No line ℓ contains more than $t+1$ points of P, since the distance from an extremal point of $\ell \cap P$ to all other points of $\ell \cap P$ on the line are distinct.

For any line ℓ, let $\mu_\ell = |\ell \cap P|$ and let C_ℓ be the set of circles having ℓ as an axis and containing at least one point of P. Our goal is to bound the size $|C_\ell|$ of this set. Consider the set S_ℓ of all the spheres centered at points of $\ell \cap P$ and containing at least one point of P. We fix a halfplane π bounded by ℓ. Each sphere in S_ℓ intersects π in a semicircle, and each circle in C_ℓ intersects π at a single point. Moreover, the intersection semicircles are distinct for distinct spheres of S_ℓ, and the intersection points are distinct for distinct circles of C_ℓ. See Figure 2. Since every circle $\gamma \in C_\ell$ is contained in exactly μ_ℓ of the spheres of S_ℓ, we have $|C_\ell|\mu_\ell$ incidences between the at most $\mu_\ell t$ semicircles and the $|C_\ell|$ points within π. Clearly, these semicircles form a collection of pseudo-segments, that is, each pair of them intersects at most once. Hence, Theorem A implies that

$$|C_\ell|\mu_\ell = O\big((|C_\ell|\mu_\ell t)^{2/3} + \mu_\ell t + |C_\ell|\big). \tag{3}$$

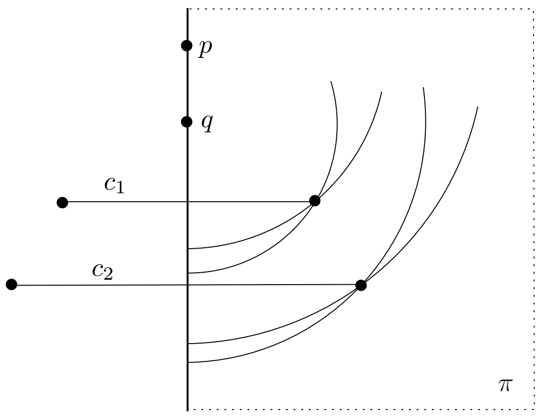

Figure 2: Circle-sphere containments along a fixed axis. The two points p, q on the axis and the two circles c_1, c_2 (shown projected orthogonally to π) induce four spheres, each centered at one of the points and containing one of the circles. Each circle-sphere containment is mapped into an incidence between a point and a semicircle.

Rewriting (3), we deduce that one of the inequalities $\mu_\ell \le a$, $|C_\ell| \le at$, or $|C_\ell| \le at^2/\mu_\ell$ must hold, for some absolute constant a. Using the fact that $\mu_\ell \le t+1$, we have $at = O(t^2/\mu_\ell)$. Hence, we have

$$|C_\ell| = O\left(\frac{t^2}{\mu_\ell}\right),$$

whenever $\mu_\ell > a$.

We now consider the collection L_μ of all lines ℓ with $\mu_\ell \ge \mu$ for some parameter $\mu > a$. Let $\mathcal{C}_\mu$ be the union of the sets C_ℓ, for $\ell \in L_\mu$. Notice that the sets C_ℓ are disjoint for distinct lines ℓ. We count the number of incidences between the points in P and the circles in $\mathcal{C}_\mu$. Each collection C_ℓ contributes $n - \mu_\ell$ such incidences. We may assume that $n - \mu_\ell \ge n/2$, as otherwise $t = \Omega(n)$. Thus, we have at least $n|L_\mu|/2$ incidences between the n points of P and the $O(|L_\mu|t^2/\mu)$ circles of $\mathcal{C}_\mu$. Using Theorem B on circle-point incidences in three dimensions, we obtain

$$n|L_\mu|/2 = \widetilde{O}(n^{6/11}(|L_\mu|t^2/\mu)^{9/11} + n^{2/3}(|L_\mu|t^2/\mu)^{2/3} + n + |L_\mu|t^2/\mu). \tag{4}$$

Solving (4) for $|L_\mu|$, it follows that either $\mu = \widetilde{O}(t^2/n)$ or

$$|L_\mu| = \widetilde{O}\left(1 + \frac{t^4}{\mu^2 n} + \frac{t^9}{\mu^{9/2} n^{5/2}}\right). \tag{5}$$

Let us choose μ_0 such that (5) holds for all $\mu \geq \mu_0$. Let X be the set of points in P incident to at least one line in L_{μ_0}. Clearly,

$$|X| \leq \sum_{i \geq 0} 2^{i+1} \mu_0 |L_{2^i \mu_0}|.$$

Notice that $L_{2^i \mu_0} = \emptyset$ if $2^i \mu_0 > t + 1$. Hence

$$|X| = \widetilde{O}\left(t + \frac{t^4}{\mu_0 n} + \frac{t^9}{\mu_0^{7/2} n^{5/2}}\right).$$

By the definition of $\widetilde{O}$, this can be rephrased as

$$|X| \leq \left(t + \frac{t^4}{\mu_0 n} + \frac{t^9}{\mu_0^{7/2} n^{5/2}}\right) \gamma(n),$$

where we may take $\gamma(n)$ to satisfy $\log n < \gamma(n) = \widetilde{O}(1)$. We choose the value $\mu_0 = \gamma^2(n) t^{18/7}/n = \widetilde{O}(t^{18/7}/n)$. The above bound yields $|X| = o(n)$ unless $t > n^{0.7}$. In the latter case, there is nothing to prove, as our final goal is a much weaker lower bound on t. The following lemma summarizes our results so far.

Lemma 2.1. *Let P be a set of n distinct points in $\mathbb{R}^3$ and let $t = t(P)$. If $t \leq n^{0.7}$ we can set $\mu_0 = \widetilde{O}(t^{18/7}/n)$ such that the total number of points in P incident to lines containing at least μ_0 points of P is only $o(n)$.*

3 Incidences with good spheres

Let P be a set of n points in $\mathbb{R}^3$. We classify each (two-dimensional) sphere σ as being either *good* with respect to P, if no circle that lies fully in σ contains more than half of the points in $\sigma \cap P$, or *bad*, otherwise.

In this section, we bound the number of incidences between a set of good spheres and P. Let G be a finite set of good spheres.

Lemma 3.1. *The number of incidences between P and G is $O(n|G|^{3/4})$.*

Proof. Consider the following set of quintuples:

$$\mathcal{Q} = \{(p_1, p_2, p_3, p_4, \sigma) \mid \sigma \in G,\ p_1, p_2, p_3, p_4 \text{ distinct non-coplanar points of } P \cap \sigma\}.$$

We clearly have $|\mathcal{Q}| \leq n^4$, because any quadruple $\{p_1, p_2, p_3, p_4\}$ of non-coplanar points determine at most one sphere of G that contains them all. To obtain a lower bound on $|\mathcal{Q}|$, we fix a sphere $\sigma \in G$ and put $n_\sigma = |P \cap \sigma|$. Notice that, as σ is good, we have $n_\sigma \geq 6$. We construct the quintuples in $\mathcal{Q}$ involving σ by selecting the points p_1, p_2, p_3, and p_4 one by one. Clearly, p_1 can be chosen in n_σ different ways, while we have $n_\sigma - 1 \geq \frac{5}{6} n_\sigma$ choices for p_2 and $n_\sigma - 2 \geq \frac{2}{3}n_\sigma$ choices for p_3. After p_1, p_2, and p_3 have been selected, p_4 must be chosen in $\sigma \cap P$ but off the circle determined by the first three points. Since σ is good, we have at least $n_\sigma/2$ choices for p_4. We thus obtain the following lower bound on the size of $\mathcal{Q}$:

$$|\mathcal{Q}| \geq \sum_{\sigma \in G} \frac{5}{18} n_\sigma^4 \geq \frac{5}{18} \frac{(\sum_{\sigma \in G} n_\sigma)^4}{|G|^3}.$$

Comparing this with the n^4 upper bound on the same quantity, we get the asserted upper bound on the number $\sum_{\sigma \in G} n_\sigma$ of incidences between points in P and spheres in G. □

4 Decomposition

In this section, we complete the proof of Theorem 1.1. First we use Lemma 2.1 to eliminate all points of P that lie on lines containing many points, and then we combine Lemma 3.1 with a standard random sampling argument, to obtain the desired lower bound on $t(P)$.

We are given a set P of n points in $\mathbb{R}^3$. Let $t = t(P)$. If $t > n^{0.7}$ we are done. Otherwise we choose $\mu_0 = \widetilde{O}(t^{18/7}/n)$ so that Lemma 2.1 holds. Using the lemma, we may assume, without loss of generality, that no line contains μ_0 or more points from P. Indeed, since $t \leq n^{0.7}$, we simply remove from P those $o(n)$ points that lie on lines containing at least μ_0 points, and apply the argument to the remaining set.

Consider the set S of spheres centered at the points of P, each containing at least one element of P. Clearly, we have $|S| \leq nt$.

We fix a parameter r, whose value will be determined below, and construct a $(1/r)$-*cutting* of S. Using an extension of the method of Chazelle and Friedman [2], reviewed in [11], which is based on the vertical decomposition technique presented in [4], we obtain a cutting consisting of $\widetilde{O}(r^3)$ connected, relatively open cells of dimension 0, 1, 2, or 3, so that each cell is *crossed* by (i.e., intersected by, but not fully contained in) at most nt/r spheres in S.

We first bound the number of incidences involving points that lie in some cell and spheres that fully contain that cell. Consider a fixed cell τ. If τ is 0-dimensional, i.e., a single point, it contributes at most n incidences, for a total of $\widetilde{O}(nr^3)$. If τ is two-dimensional, it can be fully contained in at most one sphere of S, and these full containments produce at most n incidences in total, because each point of P lies in at most one such cell. If τ is three-dimensional, no sphere can fully contain it. If τ is one-dimensional, the only nontrivial case is when τ is a circular arc contained in several spheres. However, the maximum number of such spheres is at most μ_0, because the centers of all these spheres lie on the axis of the circle containing τ. Hence, the number of incidences produced by the full containments by one-dimensional cells is at most $O(n\mu_0)$ again, each point lies in at most one one-dimensional cell. In summary, full containments generate $\widetilde{O}(n\mu_0 + nr^3)$ incidences.

It remains to bound the number of incidences between points in a cell and spheres crossing it. For any cell τ, put $P_\tau \equiv P \cap \tau$, let S_τ denote the set of the at most nt/r spheres of S that cross τ, and let G_τ (resp., B_τ) denote the subset of good (resp., bad) spheres of S_τ, with respect to P_τ. By Lemma 3.1, the number of incidences between P_τ and G_τ is $O(|P_\tau|\cdot|G_\tau|^{3/4})$. Summing this bound over all cells τ, we obtain a contribution of

$$\sum_\tau O(|P_\tau|\cdot|G_\tau|^{3/4}) = O\left(n\left(\frac{nt}{r}\right)^{3/4}\right) = O\left(\frac{n^{7/4}t^{3/4}}{r^{3/4}}\right) \tag{6}$$

incidences.

We next estimate the contribution of bad spheres to the number of incidences. Fix a cell τ. For each bad sphere $\sigma \in B_\tau$, we can just consider the more than $|\sigma \cap P_\tau|/2$ points that lie on a common circle γ along σ. We choose one such circle $\gamma = \gamma(\sigma, \tau)$, and we lose at most half the incidences between P_τ and σ in doing so.

In other words, we have constructed a set C_τ of circles, where each circle $\gamma \in C_\tau$ appears with some *multiplicity* $\mu_{\gamma,\tau}$, which is the number of bad spheres $\sigma \in B_\tau$ that satisfy $\gamma(\sigma, \tau) = \gamma$. We wish to bound the number of incidences between the points of P_τ and the circles of C_τ, where each such incidence is to be counted with the multiplicity of the corresponding circle. Namely, we wish to bound the sum

$$\sum_\tau \sum_{\gamma\in C_\tau} \mu_{\gamma,\tau}|\gamma \cap P_\tau| \;.$$

Fix a parameter $\mu > 0$, consider the subset $C_\tau^{(\mu)}$ of circles in C_τ with multiplicity between μ and 2μ. We have $\mu|C_\tau^{(\mu)}| \le |S_\tau| \le nt/r$, so that $N_\tau \equiv |C_\tau^{(\mu)}| \le nt/(r\mu)$. By Theorem B the number of incidences between N_τ distinct circles and $n_\tau \equiv |P_\tau|$ points in 3-space is

$$\widetilde{O}\left(n_\tau^{6/11}N_\tau^{9/11} + n_\tau^{2/3}N_\tau^{2/3} + n_\tau + N_\tau\right).$$

We multiply this bound by 2μ, the bound on the multiplicity of any circle in $C_\tau^{(\mu)}$, and sum it over all cells τ, to obtain that the total number of incidences between the points of P in a cell τ and the bad spheres whose representing circles are in the subset $C_\tau^{(\mu)}$, summed over all cells τ, is

$$\begin{aligned}
&\widetilde{O}\left(\mu\left(\sum_\tau n_\tau^{6/11}\right)\left(\frac{nt}{r\mu}\right)^{9/11} + \mu\left(\sum_\tau n_\tau^{2/3}\right)\left(\frac{nt}{r\mu}\right)^{2/3} + \mu\sum_\tau\left(n_\tau + \frac{nt}{r\mu}\right)\right) \\
&\qquad = \widetilde{O}\left(\mu n^{6/11}(r^3)^{5/11}\left(\frac{nt}{r\mu}\right)^{9/11} + \mu n^{2/3}(r^3)^{1/3}\left(\frac{nt}{r\mu}\right)^{2/3} + n\mu + ntr^2\right) \\
&\qquad\qquad = \widetilde{O}\left(n^{15/11}t^{9/11}r^{6/11}\mu^{2/11} + n^{4/3}t^{2/3}r^{1/3}\mu^{1/3} + n\mu + ntr^2\right).
\end{aligned} \tag{7}$$

If a circle γ appears in C_τ with multiplicity μ, then there are μ spheres whose centers all lie on the axis of γ. By our initial pruning process, we have $\mu \le \mu_0$. We can therefore bound the number of all "bad" incidences by summing (7) over an appropriate geometric progression of μ ending at $\mu_0 = \widetilde{O}(t^{18/7}/n)$, and then combine the sum with the bound (6) on "good" incidences and with the bounds for incidences between points in a cell and spheres containing the entire cell, to obtain (the fifth term has an additional logarithmic factor, which is subsumed by the factor implied by the $\widetilde{O}$-notation)

$$n(n-1) \;=\; I(P,S) \;=\; \widetilde{O}\left(\frac{n^{7/4}t^{3/4}}{r^{3/4}} + n^{13/11}t^{9/7}r^{6/11} + nt^{32/21}r^{1/3} + t^{18/7} + ntr^2 + nr^3\right).$$

We choose $r = n^{25/57}/t^{55/133} > 1$ to equalize the first two terms on the right-hand side, obtaining

$$n^2 = \widetilde{O}\left(n^{27/19}t^{141/133} + n^{196/171}t^{79/57} + t^{18/7} + n^{107/57}t^{23/133} + n^{132/57}/t^{165/133}\right).$$

Solving this inequality for t yields that one of the following five relations must hold:

$$t = \widetilde{\Omega}\left(n^{77/141}\right),\; t = \widetilde{\Omega}\left(n^{146/237}\right),\; t = \widetilde{\Omega}\left(n^{7/9}\right),\; t = \widetilde{\Omega}\left(n^{49/69}\right),\; \text{or } t = \widetilde{O}\left(n^{14/55}\right).$$

Here the last relation is impossible, as we already know that $t = \widetilde{\Omega}(n^{1/2})$. From the first four relations the weakest one—the first—is the bound claimed in Theorem 1.1. $\square$

5 Distinct distances in $\mathbb{S}^d$ and $\mathbb{R}^d$

We start with proving Theorem 1.2. We only point out the few places where the proof of Theorem 1.1 has to be altered to apply to the spherical case.

First, the *axis* of a circle (the locus of points from which every point of the circle is equidistant) is a great circle in $\mathbb{S}^3$. Thus, we need to modify the analysis so that it handles axes that are great circles, rather than lines.

Second, it is sufficient to prove our lower bound for point sets contained in an open hemisphere of $\mathbb{S}^3$. This assumption has two advantages. In $\mathbb{R}^3$, the axis of a circle cannot contain more than $t+1$ points. This remains true on a hemisphere (within which the axis is a great semi-circle), but a full great circle could contain as many as $2t+1$ distinct points. A more important consequence of this assumption is that when we consider spheres centered at points of our set, no sphere arises more than once. In the full $\mathbb{S}^3$, spheres around diagonally opposite points could coincide.

The only subsequent place in the proof where the analysis of the spherical and the Euclidean cases differ is in the proof of the bound (3) in Section 2. We recall the setting: P is set of n points, now in a hemisphere of $\mathbb{S}^3$. We let $t = t(P)$. We pick a great circle ℓ that contains μ_ℓ points of P. We define C_ℓ to be the set of circles, each having ℓ as an axis and containing at least one point of P. Our goal is to bound $|C_\ell|$ using the bound (3), restated here:

$$|C_\ell|\mu_\ell = O\big((|C_\ell|\mu_\ell t)^{2/3} + \mu_\ell t + |C_\ell|\big). \tag{3}$$

We leave $\mathbb{S}^3$ and consider the Euclidean 4-space $\mathbb{R}^4$ containing it. Let O be the (two-dimensional) plane in $\mathbb{R}^4$ containing the great circle ℓ. We project the spheres in S_ℓ and the circles in C_ℓ orthogonally onto O. Any sphere in S_ℓ is in fact the intersection of two spheres in $\mathbb{R}^4$, one of which is the unit sphere $\mathbb{S}^3$ containing ℓ, and the other is a sphere centered at a point of ℓ. The projection of this intersection is a chord of ℓ. Note also that distinct spheres project to distinct chords. The circles in C_ℓ are contained in planes orthogonal to O, so each of them projects to a single point of O (inside the circle ℓ). Here again, distinct circles in C_ℓ project to distinct points. Each circle in C_ℓ lies in μ_ℓ spheres in S_ℓ, creating at least $|C_\ell|\mu_\ell$ incidences between the points and the chords in the projection. Thus, by Theorem A, the bound (3) also holds in this case.

The rest of the proof of Theorem 1.1 applies essentially *verbatim* in this case. Notice that for the Euclidean case one has to use Theorem B on point-circle incidences in three dimensions, whereas to derive the same bound (4) in the spherical case we use the result in four dimensions. The portion of the proof in Section 3 proceeds without any change. For the final part in Section 4, one has to use $(1/r)$-cuttings within $\mathbb{S}^3$, whose existence and properties can be established following the approach mentioned in Section 4.

Finally, we prove Corollary 1.3. The proof proceeds by induction on d. The base case $d = 3$ is covered by Theorems 1.1 and 1.2. For $d > 3$, fix an arbitrary point $p \in P$. There are $t_p(P)$ $(d-1)$-dimensional spheres centered at p that collectively contain the $n-1$ points of $P \setminus \{p\}$. Hence, there is a sphere σ passing through at least $(n-1)/t_p(P)$ elements of P. If $t_p(P)$ is smaller than the asserted bound, then σ contains more than $n^{1-1/(d-\frac{90}{77})}$ points of P. By the induction hypothesis, we may apply Corollary 1.3 to $\sigma \cap P$, and conclude that there exists a point $q \in \sigma \cap P$ that determines at least

$$\widetilde{\Omega}\left(\left(n^{1-1/(d-\frac{90}{77})}\right)^{1/(d-1-\frac{90}{77})}\right) = \widetilde{\Omega}\left(n^{1/(d-\frac{90}{77})}\right)$$

distinct distances to the other points of $\sigma \cap P$, completing the proof of Corollary 1.3.

6 Concluding remarks

Clearly, the main open problem is to close the gap between the upper and lower bounds.

An earlier draft of this paper used results of Elekes [5, 6] on the number of distinct distances determined by two sets of points, each consisting of μ points on a line. If the lines are neither parallel nor orthogonal, then there are $\Omega(\mu^{5/4})$ distinct distances between the two sets. This result was applied to points lying on axes of the circles that contain points of P. In the present analysis, however, the bottleneck is the case where the axes of circles contain approximately $\mu \approx (n^{77/141})^{18/7}/n = n^{19/47}$ points of P and $\mu^{5/4} < n^{77/141}$, thus Elekes' result cannot be used. Elekes' bound, however, is conjectured (e.g., by Elekes himself) not to be tight, and a major improvement of it would lead to a stronger lower bound on $g_3(n)$ (but not on the number $t_3(n)$ of distinct distances from a single point).

Any improvement of the bound in Theorem B [1] for the number of incidences between points and circles in three and four dimensions would also lead to an improvement of our main result.

For so-called *dense* point sets in $\mathbb{R}^3$, for which the ratio between the maximum and minimum distances between their elements is $O(n^{1/3})$, one can improve the result on the number of distinct distances by replacing Lemma 2.1 by the trivial $O(n^{1/3})$ bound on the number of points on any line. Another recent paper by Solymosi and Vu [14] pushes the bound further for this special case.

Acknowledgments

We would like to thank Pankaj K. Agarwal for helpful discussions concerning this problem.

References

[1] B. Aronov, V. Koltun and M. Sharir, Incidences between points and circles in three and higher dimensions, *Discrete Comput. Geom.*, submitted. (An earlier version appeared in *Proc. 18th ACM Symp. on Computational Geometry* (2002), 116–122.)

[2] B. Chazelle and J. Friedman, A deterministic view of random sampling and its use in geometry, *Combinatorica* 10 (1990), 229–249.

[3] F. Chung, E. Szemerédi, and W. T. Trotter, The number of distinct distances determined by a set of points in the Euclidean plane, *Discrete Comput. Geom.* 7 (1992), 1–11.

[4] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* 5 (1990), 99–160.

[5] G. Elekes, A note on the number of distinct distances, *Periodica Math. Hung.* 38(3) (1999), 173–177.

[6] G. Elekes, Sums versus products in number theory, algebra and Erdős geometry—a survey, *Paul Erdős and His Mathematics II*, Bolyai Math. Soc. Stud. 11, Budapest (2002), 241–290.

[7] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* 53 (1946), 248–250.

[8] P. Erdős, On some of my favourite theorems, in: *Combinatorics, Paul Erdős is Eighty, Vol. 2* (D. Miklós et al., eds.), *Bolyai Society Mathematical Studies* **2**, Budapest, 1996, 97-132.

[9] N. Katz, An improvement of a lemma of Tardos, submitted to *Combinatorica.*

[10] N. Katz and G. Tardos, A new entropy inequality for the Erdős distance problem, in *Towards a theory of geometric graphs* (J. Pach, ed.), Contemporary Mathematics, Amer. Math. Soc., Providence, RI, to appear.

[11] J. Matoušek, *Lectures on Discrete Geometry*, GTM Series, Springer Verlag, New York, 2002.

[12] J. Pach and P.K. Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995.

[13] J. Solymosi and V. Vu, Near optimal bounds for the number of distinct distances in high dimensions, manuscript, 2003.

[14] J. Solymosi and V. Vu, Distinct distances in high-dimensional homogeneous sets, in *Towards a theory of geometric graphs* (J. Pach, ed.), Contemporary Mathematics, Amer. Math. Soc., Providence, RI, to appear. Also in *Proc. 19th ACM Symp. on Computational Geometry* (2003), 104–105.

[15] J. Solymosi and Cs. Tóth, Distinct distances in the plane, *Discrete Comput. Geom.* 25 (2001), 629–634.

[16] J. Spencer, E. Szemerédi, and W. T. Trotter, Unit distances in the Euclidean plane, in: *Graph Theory and Combinatorics* (B. Bollobás, ed.), Academic Press, New York, 1984, 293–303.

[17] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* 6 (1997), 353–358.

[18] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* 3 (1983), 381–392.

[19] G. Tardos, On distinct sums and distinct distances, *Advances in Mathematics*, to appear.