Towards the Hanna Neumann conjecture using Dicks' method

Gábor Tardos*
Mathematical Institute
of the Hungarian Academy of Sciences
Pf. 127, Budapest, H-1364 Hungary
e-mail: tardos@cs.elte.hu

Abstract

The Hanna Neumann conjecture states that the intersection of two nontrivial subgroups of rank $k+1$ and $l+1$ of a free group has rank at most $kl+1$. In a recent paper [3] W. Dicks proved that a strengthened form of this conjecture is equivalent to his amalgamated graph conjecture. He used this equivalence to reprove all known upper bounds on the rank of the intersection. We use his method to improve these bounds. In particular we prove an upper bound of $2kl-k-l+1$ for the rank of the intersection above $(k, l \geq 2)$ improving the earlier $2kl - \min(k, l)$ bound of [1].

We prove a special case of the amalgamated graph conjecture in the hope that it may lead to a proof of the general case and thus of the strengthened Hanna Neumann conjecture.

1 Introduction

For a longer introduction to the history of the problem see [3]. Here we borrow the terminology from there to present a shorter version.

By the Nielsen-Schreier theorem [9,11] any subgroup of a free group is free, thus it is characterized up to isomorphism by its *rank*, the size of a free generating set. It is a natural question and goes back more than 40 years how the rank of the intersection relates to the rank of two subgroups of a free group. It is convenient to introduce the *reduced rank* $\bar{r}(H) = max(rank(H) - 1, 0)$. First Howson [5] proved that $H \cap K$ is finitely generated if H and K are, and gave the $\bar{r}(H \cap K) \leq 2\bar{r}(H)\bar{r}(K) + \bar{r}(H) + \bar{r}(K) + 1$ bound. Then Hanna Neumann (with the help of R. Baer [7]) improved the bound to $\bar{r}(H \cap K) \leq 2\bar{r}(H)\bar{r}(K)$ and conjectured the the stronger $\bar{r}(H \cap K) \leq \bar{r}(H)\bar{r}(K)$ bound, later to be called the Hanna Neumann conjecture. This conjectured bound is tight if true as it is easy to construct subgroups H and K for any given ranks satisfying $\bar{r}(H \cap K) = \bar{r}(H)\bar{r}(K)$.

Despite the continues interest in the conjecture since Hanna Neumann's paper (see e. g. [2,3,4,6,8,10,12]) there are only two papers improving the upper bound on the function $f(h, k) = \max\{\bar{r}(H \cap K) | \bar{r}(H) = h, \bar{r}(K) = k\}$. First Burns [1] proved $\bar{r}(H \cap K) \leq 2\bar{r}(H)\bar{r}(K) - \min(\bar{r}(H), \bar{r}(K))$ then we [13] proved the special case of Hanna Neumann conjecture when one of the subgroups H, K is of rank 2.

For subgroups H and K of a free group we define $\bar{r}(H, K) = \sum \bar{r}(g^{-1}Hg \cap K)$ where the summation extends over the representatives g of the double cosets HgK. As this sum

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includes $\bar{r}(H \cap K)$ we have $\bar{r}(H, K) \geq \bar{r}(H \cap K)$. In [8] Walter Neumann proposed the following strengthened form of the Hanna Neumann conjecture: $\bar{r}(H, K) \leq \bar{r}(H)\bar{r}(K)$. He also extended Burns' bound to $\bar{r}(H, K) \leq 2\bar{r}(H)\bar{r}(K) - \min(\bar{r}(H), \bar{r}(K))$. Our paper [13] proves the special case (when one of the ranks is two) of the strengthened conjecture, so all known upper bounds on $\bar{r}(H \cap K)$ bounds $\bar{r}(H, K)$ too.

Recently Dicks [3] proved the equivalence of the strengthened Hanna Neumann conjecture with a conjecture on bipartite graphs that he called the amalgamated graph conjecture (see below). Although equivalence with certain graph conjectures is nothing new (see e.g [10,13]) the amalgamated graph conjecture represents a strikingly new approach to the problem. Using that any bound on $\bar{r}(H, K)$ is equivalent with limiting the number of edges of a bipartite graph with a certain symmetry condition Dicks deduced both previously known bounds.

In this paper we apply Dick's method to improve upon the known bounds. With hardly any modification of Dick's proof of the strengthened Hanna Neumann conjecture for the case when H or K has rank two we deduce $\bar{r}(H, K) \leq 2\bar{r}(H)\bar{r}(K) - \bar{r}(H) - \bar{r}(K) + 1$ if $\bar{r}(H), \bar{r}(K) \geq 1$ (Corollary 6). This improves the strengthened Burns bound for all pairs of ranks.

We can also prove a special case of the amalgamated graph conjecture and hope that it can serve as an intuition for a future proof of the general case. Right now we can use the method of this proof to improve the bound of Corollary 6 by one for subgroups of rank at least three (Corollary 11). This proves the Hanna Neumann conjecture in the case both subgroups have rank three.

2. Dicks' method

Definition. Following the terminology of [3] all *graphs* in this paper are simple bipartite graphs together with a 2-coloring of the vertices to *initial* and *terminal* vertices. A *subgraph* of a graph Γ is a graph consisting of a subset of the vertices of Γ together with some edges of Γ connecting such vertices. Subgraphs naturally inherit the 2-coloring. We call a subgraph *induced* (or *full*) if it consists of a subset of the vertices of Γ together with all the edges of Γ connecting such vertices. We say that two graphs are *isomorphic* if there is a graph-isomorphism between them preserving the 2-coloring, i. e. mapping initial vertices to initial vertices, terminal vertices to terminal vertices. By the *size* of a graph we mean the triple (n, m, e) of the number of initial, terminal vertices and edges.

We say that the graphs G_1, G_2 and G_3 *amend* the graph G if $G_1 \cup G_2 \cup G_3$ is a simple graph and $G_1 \cap G_2 = G_2 \cap G_3 = G_3 \cap G_1 = G$. We call the disjoint union of isomorphic copies of the graphs $G_1 \cup G_2$, $G_2 \cup G_3$ and $G_3 \cup G_1$ the *amended graph* $A(G_1, G_2, G_3)$ (the amended graph is therefore defined only up to isomorphism). We call the graph G *evenly amendable* if there exist graphs G_1, G_2 and G_3 amending it such that the connected components of $A(G_1, G_2, G_3)$ are isomorphic in pairs.

Let us recall the following definitions from the Introduction: $\bar{r}(H) = \max(rank(H) - 1, 0)$ for a free group H and $\bar{r}(H, K) = \sum \bar{r}(g^{-1}Hg \cap K)$ for subgroups H and K of a free group where the summation extends over the representatives g of the double cosets HgK.

The main result in Dicks' paper [3] is the following:

Theorem 1. *Let H and K be subgroups of a free group. Then there exists an evenly amendable graph of size $(2\bar{r}(H), 2\bar{r}(K), 2\bar{r}(H,K))$.* ∎

This result shows that the strengthened Hanna Neumann conjecture is implied by the following *amalgamated graph conjecture*: The size (n, m, e) of an evenly amendable graph satisfies $2e \le nm$. We mention that [3] (as it claims in its title) proves the equivalence of the two conjectures.

3. Induced subgraphs

In this section we prove the special case of the amalgamated graph conjecture when the original graph has to be an induced subgraph of the graphs amending it (Corollary 5).

Definition. Let i and j be non-negative integers. A graph G is called (i,j)*-trivial* if it size is (n, m, e) with $n \le i$ and $m \le j$, otherwise G is (i,j)*-nontrivial*. The induced subgraphs G_1 and G_2 of G form an (i,j)*-decomposition* of G if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is (i,j)-trivial. The graph G is called (i,j)*-decomposable* if it has proper subgraphs forming an (i,j)-decomposition, otherwise it is (i,j)*-connected*. Maximal (i,j)-nontrivial, (i,j)-connected induced subgraphs of a graph G are called the (i,j)-factors of G. The graphs G_1, G_2 and G_3 are said to (i,j)*-evenly amend* the graph G if they amend G and the (i,j)-factors of $A(G_1, G_2, G_3)$ are isomorphic in pairs. If there exist such graphs G_1, G_2 and G_3 then G is called (i,j)*-evenly amendable*.

We start with simple observations.

Lemma 2. *Let i and j be non-negative integers.*

a. *The $(0,0)$-factors are the connected components, thus $(0,0)$-evenly amendable is evenly amendable.*
b. *If G_1 and G_2 form an (i,j)-decomposition of a graph G and $H \subset G$ then $H \cap G_1$ and $H \cap G_2$ form an (i,j)-decomposition of H.*
c. *If G_1 and G_2 form an (i,j)-decomposition of a graph G then the (i,j)-factors of G are the disjoint union of the (i,j)-factors of G_1 and the (i,j) factors of G_2.*
d. *If $i' \ge i$, $j' \ge j$ then the (i',j')-factors of G are the disjoint union of the (i',j')-factors of the (i,j)-factors of G.*
e. *If $i' \ge i$, $j' \ge j$ then graphs (i,j)-evenly amending a graph also (i',j')-evenly amend it.*

PROOF: Points a and b are trivial.

For c let $H \subset G$ be (i,j)-connected. By b it has to be contained in either of G_1 or G_2. By the definition if contained in both then H is (i,j)-trivial.

As (i,j)-factors can be obtained by repeated (i,j)-decompositions (which are also (i',j')-decompositions) d follows from c.

Point e trivially follows from d. ∎

The following lemma is central in the proof our main results.

Lemma 3. *Let i and j be non-negative integers. Suppose G_1, G_2 and G_3 (i,j)-evenly amend the graph G, G' and G'' form an (i,j)-decomposition of $G_1 \cup G_2$ and G has i' initial* *and j' terminal vertices outside G'. If $G \cap G''$ is an induced subgraph of $G_1 \cup G_2 \cup G_3$ then $G' \cap G_1$, $G' \cap G_2$ and G_3 $(i+i', j+j')$-evenly amend $G' \cap G$.*

PROOF: Let $G'_i = G' \cap G_i$ and $G''_i = G'' \cap G_i$ for $i = 1, 2$. It is easy to see that our assumption that $G \cap G''$ is an induced subgraph of $G_1 \cup G_2 \cup G_3$ implies that for $i = 1, 2$ G''_i is an induced subgraph of G'' and both G''_i and $G'_i \cup G_3$ are induced subgraphs of $G_i \cup G_3$. Our goal is to show that these graphs form $(i+i', j+j')$-decompositions of the corresponding larger graphs.

The graph $G''_1 \cap G''_2$ has at most i initial and j terminal vertices in G' and i' initial and j' terminal vertices outside G', thus it is $(i+i', j+j')$-trivial. Since $G''_1 \cup G''_2 = G''$ the graphs G''_1 and G''_2 form an $(i+i', j+j')$-decomposition of G''. Similarly for $i = 1, 2$ the graph $G''_i \cap (G'_i \cup G_3)$ has at most i initial and j terminal vertices in G' and at most i' initial and j' terminal vertices outside G' thus it is $(i+i', j+j')$-trivial. Since $G''_i \cup (G'_i \cup G_3) = G_i \cup G_3$ the graphs G''_i and $G'_i \cup G_3$ form an $(i+i', j+j')$-decomposition of $G_i \cup G_3$.

Clearly G'_1, G'_2 and G_3 amend $G \cap G'$. By Lemma 2.e the $(i+i', j+j')$-factors of $A(G_1, G_2, G_3)$ are isomorphic in pairs. By the above observations and Lemma 2.c these factors are isomorphic to the $(i+i', j+j')$-factors of $A(G'_1, G'_2, G_3)$ plus twice the $(i+i', j+j')$-factors of G''_1 and G''_2. Therefore the $(i+i', j+j')$-factors of $A(G'_1, G'_2, G_3)$ have to be also isomorphic in pairs. ∎

Notice the unfortunate condition in Lemma 3 requiring that $G \cap G''$ is an induced subgraph. We need this because in our definition of (i, j)-decomposition both components have to be induced subgraphs. We require that in turn because otherwise all $(1, 1)$-connected graphs would be $(1, 1)$-trivial.

In case G is an induced subgraph then this condition is automatically satisfied. Corollary 5 proves the amalgamated graph conjecture in this case. In the general case however we can only prove much weaker results.

Theorem 4. *Let i and j be non-negative integers. Suppose G_1, G_2 and G_3 (i, j)-evenly amend the (i, j)-nontrivial graph G and G is an induced subgraph of $G_1 \cup G_2 \cup G_3$. Then the size (n, m, e) of G satisfies $e \le nm - (n-i)(m-j)/2$.*

PROOF: The proof is by induction on the number of vertices of $A(G_1, G_2, G_3)$. All the graphs $G_1 \cup G_2$, $G_2 \cup G_3$ and $G_3 \cup G_1$ are (i, j)-nontrivial, thus if all are (i, j)-connected, then $A(G_1, G_2, G_3)$ has three (i, j)-factors, a contradiction. Therefore by symmetry we may assume that $G_1 \cup G_2$ has an (i, j)-decomposition to proper subgraphs G' and G''. Let i' and j' be the number of initial and terminal vertices of G outside G', let i'' and j'' be the number of initial and terminal vertices of G are outside G'', finally let i_0 and j_0 the number of initial and terminal vertices of $G \cap G' \cap G''$. We have $i_0 \le i$, $j_0 \le j$, $i_0 + i' + i'' = n$ and $j_0 + j' + j'' = m$. By symmetry we may assume $j' \le j''$.

The graphs $G' \cap G_1$, $G' \cap G_2$ and G_3 $(i+i', j+j')$-evenly amend $G \cap G'$ by Lemma 3. Here $G \cap G'$ is an induced subgraph of $G' \cup G_3$ and $A(G' \cap G_1, G' \cap G_2, G_3)$ is a proper subgraph of $A(G_1, G_2, G_3)$ since G' is a proper subgraph of $G_1 \cup G_2$. Thus we may use the inductive hypothesis for $G \cap G'$ unless it is $(i+i', j+j')$-trivial.

Suppose first that $G \cap G'$ is $(i+i', j+j')$-nontrivial. By the inductive hypothesis $G \cap G'$ has at most $(i_0+i'')(j_0+j'') - (i_0+i''-i-i')(j_0+j''-j-j')/2$ edges. G clearly has at most $i'j' + i'j_0 + i_0j'$ edges outside G'. Thus the total number of edges of

G is $e \le i'j' + i_0 j' + i' j_0 + (i_0 + i'')(j_0 + j'') - (i_0 + i'' - i - i')(j_0 + j'' - j - j')/2 = nm - (n-i)(m-j)/2 - i'(j - j_0) - j'(i - i_0) \le nm - (n-i)(m-j)$ as claimed.

If $n \le i$ then $m > j$ since G is not (i,j)-trivial and thus $e \le nm \le nm-(n-i)(m-j)/2$ trivially holds.

It remained to show the inequality in the theorem when $n > i$ and $G \cap G'$ is $(i+i', j+j')$-trivial and thus $i'' + i_0 \le i + i'$. No edge of G connects a vertex outside G' to a vertex outside G''. Thus in this case we have $e \le nm - i'j'' - i''j' = nm - (n-i)(m-j_0)/2 - (i + i' - i_0 - i'')(j'' - j')/2 - (i - i_0)j' \le nm - (n-i)(m-j)/2$ as claimed. ∎

Corollary 5. *Suppose that the graph G of size (n, m, e) is an induced subgraph of three graphs evenly amending it. Then $2e \le nm$.*

PROOF: Case $i = j = 0$ of Theorem 4. ∎

4. The general case

In this section we prove upper bounds on the number of edges of an evenly amendable graph without the induced subgraph condition of Corollary 5. Our main tool is still Lemma 3 so we need to make sure its induced subgraph condition is still satisfied. In case a graph has no initial vertices it is an induced subgraph of all graphs containing it. This gives us Theorem 7 and Corollary 8.

We mention here that this argument uses (i,j)-decompositions, (i,j)-factors etc. only in the special case $i = 0$. These concepts were already defined in [3] (under the names j-decomposition, j-atomic factor). Our proof is also essentially identical to the proof there. It is surprising that we are able to prove much better bounds.

Corollary 8 improves the best previously known bound and is almost the best bound we can prove. Corollary 11 of the next section improves the bound of Corollary 8 by just one.

Definition. Let j be a non-negative integer. For simplicity we use *j-decomposition*, *j-decomposable*, *j-connected* and *j-factor* for $(0,j)$-decomposition, $(0,j)$-decomposable, $(0,j)$-connected and $(0,j)$-factor. We call a graph *trivial* if it contains no initial vertices otherwise it is *nontrivial*. Clearly all $(0,j)$-trivial graphs are trivial and all j-factors ($j \ge 1$) are nontrivial.

Lemma 6. *Let j and k be non-negative integers. If the nontrivial j-evenly amendable graph G consists of a subgraph G_0 plus k isolated terminal vertices then G_0 is $(j+k)$-decomposable.*

PROOF: Let G_1, G_2 and G_3 be graphs j-evenly amending G. We prove the lemma by induction on the number of vertices of $A(G_1, G_2, G_3)$.

Consider the graphs $G_1 \cup G_2$, $G_2 \cup G_3$ and $G_3 \cup G_1$. All are nontrivial, therefore in case all are j-indecomposable $A(G_1, G_2, G_3)$ has three j-factors, a contradiction. We may assume therefore by symmetry that $G_1 \cup G_2$ has a j-decomposition to proper subgraphs G' and G''. Now $G' \cap G_0$ and $G'' \cap G_0$ form an j-decomposition of G_0 proving the claim unless one of these graphs coincides with G_0. We may therefore suppose by symmetry that $G_0 \subset G'$.

Here $G \cap G''$ is trivial as it consists of some of the terminal vertices from $G' \cap G''$ and all $j' \leq k$ terminal vertices of G outside G'. Thus every graph containing $G \cap G''$ contains it as an induced subgraph, so Lemma 3 is applicable and we have that $G_1 \cap G'$, $G_2 \cap G'$ and G_3 $(j + j')$-evenly amend $G \cap G'$.

The graph $G \cap G'$ is nontrivial and it consists of G_0 and $k - j'$ isolated vertices. As G' is a proper subgraph of $G_1 \cup G_2$ the amended graph $A(G_1 \cap G', G_2 \cap G', G_3)$ is a proper subgraph of $A(G_1, G_2, G_3)$. Therefore we can apply the inductive hypothesis for $G \cap G'$ and we get that G_0 is $(j + j') + (k - j') = (j + k)$-decomposable as claimed. ∎

We have proved this lemma to limit the number of the edges in an evenly amendable graph.

Theorem 7. *If the graph G of size (n, m, e) is evenly amendable and $n, m \geq 2$ then $e \leq mn - m - n + 2$.*

PROOF: If G is the disjoint union of two graphs, each containing both initial and terminal vertices then the bound follows.

Otherwise G must consist of a connected subgraph G_0 and some number $k \geq 0$ of isolated vertices of the same color. By symmetry we may suppose they are terminal vertices. Lemma 6 tells us that G_0 is k-decomposable. Thus we have $k \neq 0$ and the number of terminal vertices connected to all initial vertices is at most k. The rest of the terminal vertices have degree at most $n - 1$ while k of them are isolated. Thus we have $e \leq kn + (m - 2k)(n - 1) = mn - m - k(n - 2) \leq mn - m - n + 2$. ∎

Using Theorem 1 and 7 we can immediately deduce a new bound on the rank of the intersection of subgroups of a free group.

Corollary 8. *For subgroups H and K of a free group with $\bar{r}(H) \geq 1$ and $\bar{r}(K) \geq 1$ we have $\bar{r}(H, K) \leq 2\bar{r}(H)\bar{r}(K) - \bar{r}(H) - \bar{r}(K) + 1$.* ∎

3. One step further

The bound in Corollary 8 is tight if one of the ranks is two. The plus one term from the bound can be removed otherwise. We present this rather small improvement to show the limits of this proof technique. After the improved bound (Corollary 11) we indicate why it is hard to go beyond that with this method (Lemma 12).

Lemma 9. *Let i and j be non-negative integers. The complete bipartite graph of size (n, m, nm) with $n > i$ and $m > j$ is not (i, j)-evenly amendable.*

PROOF: Any (bipartite) graph containing a complete bipartite graph as a subgraph contains it as induced subgraph. Thus we can apply Theorem 4 and since $nm > nm - (n - i)(m - j)/2$ we get that the complete graph is not (i, j)-evenly amendable. ∎

We remark here that a single application of Lemma 3 is also sufficient to prove Lemma 9. For the converse see Theorem 13.

Theorem 10. *If the graph G of size (n, m, e) is evenly amendable and $n, m > 2$ then $e \leq 2nm - n - m$.*

PROOF: As the number of edges of an evenly amendable graph is trivially even we only have to rule out equality in Theorem 7.

If G is the disjoint union of two graphs each containing both initial and terminal vertices then equality in Theorem 7 holds only if G is the disjoint union of a complete bipartite graph on $n-1$ initial and $m-1$ terminal vertices and an edge.

Otherwise by the proof of Theorem 7 we may suppose that G consists of a connected component G_0 and some number $j \geq 1$ of isolated terminal vertices. Here G_0 is j-decomposable and equality in Theorem 7 implies $j = 1$ and that G_0 has one terminal vertex of degree n and $m-2$ terminal vertices of degree $n-1$. The initial vertex not connected to these $m-2$ terminal vertices must coincide otherwise G_0 would not be 1-decomposable. Thus in this case G consists of a complete bipartite graph on $n-1$ initial and $m-1$ terminal vertices, plus an initial vertex of degree one connected to one of the $m-1$ terminal vertices of the complete graph plus an isolated terminal vertex.

In both extremal cases G contains a complete subgraph K on $n-1$ initial and $m-1$ terminal vertices. We need to show that neither extremal graph is evenly amendable.

Let therefore G be one of the two extremal graphs and we deduce contradiction from the assumption that the graphs G_1, G_2 and G_3 evenly amend it. Without loss of generality we may suppose that all components of the graphs G_i $(i = 1, 2, 3)$ intersect G as components disjoint from G can be removed. One of $G_1 \cup G_2$, $G_2 \cup G_3$ and $G_3 \cup G_1$ has to be disconnected as otherwise $A(G_1, G_2, G_3)$ has three components, a contradiction. (These observations are valid for all evenly amendable graphs and appear in [3] to prove that such graphs are disconnected.) We may suppose by symmetry that $G_1 \cup G_2$ is disconnected. As G has two components $G_1 \cup G_2$ must also have two components G' and G'' both intersecting G. By symmetry we may assume $K \subset G'$.

Here $G'' \cap G$ is either a vertex or two vertices connected by an edge. In both cases all graphs containing it contains it as an induced subgraph thus Lemma 3 is applicable. In the second case we get that K is $(1,1)$-evenly amendable, contradicting Lemma 9. Thus we only have the first case and there we get that the graph K_0 consisting of K and a new initial vertex connected to one of the terminal vertices of K is 1-evenly amendable.

We derive contradiction from the assumption that K_0 is 1-evenly amendable the usual way. Let K_1, K_2 and K_3 be the smallest (in total number of vertices) set of graphs 1-evenly amending K_0. If all the graphs $K_1 \cup K_2$, $K_2 \cup K_3$ and $K_3 \cup K_1$ are 1-connected then $A(K_1, K_2, K_3)$ has three 1-factors, a contradiction. Thus we may suppose by symmetry that $K_1 \cup K_2$ has a 1-decomposition to proper subgraphs K' and K''. As the subgraph K is 1-connected it must be contained in one of them, say $K \subset K'$. The graphs $K_0' = K' \cap K_0$ and $K_0'' = K'' \cap K_0$ form a 1-decomposition of K_0 thus we have two possibilities. Either $K_0' = K_0$ and then K_0'' is empty or it consists of a single terminal vertex or else $K_0' = K$ and then K_0'' consists of the edge of K_0 outside K and the vertices it connects. As K_0'' is a complete bipartite graph in both cases and thus always an induced subgraph we can apply Lemma 3. In the first case we get a smaller triple of graphs 1-evenly amending K' and this contradicts the minimality of K_1, K_2 and K_3. In the second case we get that K is $(1,1)$-evenly amendable, contradicting Lemma 9.

The contradictions prove the theorem. ∎

Corollary 11. *For subgroups H and K of a free group with $\bar{r}(H) \geq 2$ and $\bar{r}(K) \geq 2$ we have $\bar{r}(H,K) \leq 2\bar{r}(H)\bar{r}(K) - \bar{r}(H) - \bar{r}(K)$.*

PROOF: Theorems 1 and 10 give the proof. ∎

To show the limits of this method we close the paper by showing the converse of Lemma 9.

Lemma 12. *If a graph G is not a complete bipartite graph then it is* $(1,1)$*-evenly amendable.*

PROOF: Let x be an initial and y a terminal vertex of G not connected in G. Let e be the number of edges in G and let us number these edges. For $k = 1, \ldots, e$ let x_k be the initial and y_k the terminal vertex of the kth edge and let H_k be the subgraph of G consisting of all G's vertices and the first $k-1$ edges. Let H'_k be isomorphic to H_k with the vertices of H'_k corresponding to x_k and y_k of H_k coinciding with x and y. Let all other vertices of H'_k be outside G and outside all other H'_l $(l \neq k)$.

The graphs $G_1 = G$, G_2 consisting of G plus an edge E connecting x and y and $G_3 = \cup_{k=1}^{e} H'_k \cup G$ amend G. We have $G_1 \cup G_2 = G_2$. The graph $G_1 \cup G_3 = G_3$ can be decomposed through repeated $(1,1)$-decompositions to G and the graphs H'_k $(k = 1, \ldots, e)$. The graph $G_2 \cup G_3$ can be similarly decomposed to G_2 and the graphs H''_k consisting of H'_k plus the edge E $(k = 1, \ldots, e)$. Here H''_k is isomorphic to H_{k+1} for $k = 1, \ldots, e-1$ and H''_e is isomorphic to G. Thus repeated $(1,1)$-decompositions break up $A(G_1, G_2, G_3)$ to subgraphs that are isomorphic in pairs plus the subgraph H'_1 containing no edges and having therefore no $(1,1)$-factors. Thus G_1, G_2 and G_3 $(1,1)$-evenly amend G proving the lemma.

Theorem 13. *Let i and j be positive integers. A graph G of size (n,m,e) is (i,j)-evenly amendable if and only if $n \leq i$ or $m \leq j$ or $e < nm$.*

PROOF: The only if part is proved by Lemma 9. If $e < nm$ then G is $(1,1)$-evenly amendable by Lemma 12 and therefore it is (i,j)-evenly amendable. Finally if $n \leq i$ or $m \leq j$ then G has no (i,j) factors thus $G_1 = G_2 = G_3 = G$ (i,j)-evenly amends it. ∎

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