Transversals of 2-intervals, a topological approach

Gábor Tardos*
Mathematical Institute
of the Hungarian Academy of Sciences
Pf. 127, Budapest, H-1364 Hungary
e-mail: tardos@cs.elte.hu

Abstract

Fix two distinct parallel lines e and f. A 2-interval is the union of an interval on e and an interval on f. We study the *transversal number* $\tau(\mathcal{H})$ of families of 2-intervals $\mathcal{H}$. This is the cardinality of the smallest set which intersects every 2-interval in $\mathcal{H}$. A. Gyárfás and J. Lehel [6] proved that $\tau(\mathcal{H}) = O(\nu(\mathcal{H})^2)$ where $\nu(\mathcal{H})$ is the maximum number of disjoint 2-intervals in $\mathcal{H}$. In the present paper we prove the tight bound $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$.

Our result has applications in the estimation of the transversal number of other types of set systems.

The method we use is topological. We associate a simplicial complex K with our system of 2-intervals and prove that a given subcomplex is contractible in K unless the required transversal exists. Then we construct a cocycle of (another subcomplex of) K to prove that the subcomplex is not contractible in K. We hope that this approach will be applicable to a wider variety of combinatorial optimization problems.

1. Introduction

For any set system $\mathcal{H}$ we define the *transversal number* $\tau(\mathcal{H})$ of $\mathcal{H}$ to be the cardinality of the smallest set intersecting every element of $\mathcal{H}$. Such a set is said to be a *transversal* of $\mathcal{H}$ or it said to *cover* $\mathcal{H}$. The *packing number* $\nu(\mathcal{H})$ is the maximum number of disjoint sets in $\mathcal{H}$.

Every set system $\mathcal{H}$ trivially satisfies the inequality

$$\nu(\mathcal{H}) \leq \tau(\mathcal{H}).$$

In general no inequality holds in the opposite direction. Gallai showed (see Hajnal and Surányi [7]) that equality holds for finite sets $\mathcal{H}$ consisting of intervals of a line. Finding other classes of set systems where $\tau = \nu$ and more generally bounding the transversal number in terms of the packing number for various set systems is a central problem of combinatorial duality theory. Here we are mainly interested in families of d-intervals (below) especially with $d = 2$. We allow infinite set systems, but when proving upper bounds on τ we naturally assume that ν is finite.

Let us fix d non-intersecting straight lines. Take I_i to be a closed interval of the ith line then $I = \cup_{i=1}^{d} I_i$ is said to be a *d-interval.* I_i is called the ith *component* of I.

---

* Supported by the NSF grant No. CCR-92-00788 and the (Hungarian) National Scientific Research Fund (OTKA) grant No. T4271. The author was visiting the Computation and Automation Institute of the Hungarian Academy of Sciences while part of this research was done.

A. Gyárfás and J. Lehel [6] proved that the transversal number of a family of d-intervals can be bounded in terms of its packing number. This is the same as stating the existence of the following function f:

$$f(d, k) = \max\{\tau(\mathcal{H})|\nu(\mathcal{H}) = k\}$$

where the maximum ranges over families of d-intervals. The following bound is implicit in [6].

Lemma 1.1. *[6] For any fixed value d we have $f(d, k) = O(k^{d!})$.* ∎

In this paper we give the exact value of f for the case $d = 2$:

Theorem 1.2. $f(2, k) = 2k$.

The lower bound $f(d, k) \geq dk$ is proved in [8]. This is not tight in general as [6] observes that $f(3, 1) = 4$.

Although [8] largely improves upon Lemma 1.1, for $d \geq 3$ the upper and lower bounds on $f(d, k)$ are still wide apart. Conjecture 6.3 would imply a linear upper bound for any fixed d.

Theorem 1.2 yields tighter connection between the transversal number and the packing number of set systems arising from geometrical objects, such as rectangles in the plane (see [8]).

The proof technique of our results is topological in nature. Our main result (Theorem 3.1) infers from the non-existence of a transversal for a family of d-intervals that a certain subcomplex of a complex is contractible in the larger complex. (See the definitions and the precise formulation of the theorem below.) In some cases we can show that the subcomplex (a triangulated sphere) is not homologous to 0 much less contractible in the larger complex.

2. Topological prerequisites

In this section we recall the definitions and results from topology we need. For a more detailed introduction see Spanier [9]. For an excellent survey of topological methods in combinatorics see Björner [2].

We do not use any deep results but mainly the terminology of the homology theory of simplicial complexes. For simplicity, namely to avoid the introduction of directed faces we use only chain groups over the two element group $\mathbb{Z}_2$.

A *(finite simplicial) complex* is a nonempty family K of subsets of a finite set V of vertices with the descending property: $S \subset T \in K$ implies $S \in K$. An element S of K is called a face of the simplicial complex and the dimension of S is $|S| - 1$. K^i denotes the set of the i-dimensional faces of K ($0 \leq i \in \mathbb{Z}$). A *subcomplex* of a complex is a subset which is a complex.

For the geometric realization we identify each vertex $v \in V$ with a standard basis vector in $\mathbb{R}^{|V|}$. The *body* $\overline{A}$ of a face A is a geometric simplex, the convex hull of the vertices $v \in A$. The *body* of the complex K is $\overline{K} = \cup_{S\in K}\overline{S}$. With this definition the body of a complex contains the body of any subcomplex.

The ith *chaingroup* $C_i(K) = C_i(K, \mathbb{Z}_2)$ of a complex K is an Abelian group consisting of all (formal) linear combinations $\sum_{S\in K^i} \epsilon_S S$ of the i-dimensional faces S of K where the coefficients $\epsilon_i \in \{0,1\}$ are taken from the two element group $\mathbb{Z}_2$. These linear combinations form a group under addition. The *boundary operation* ∂ is a homomorphism from $C_i(K)$ to $C_{i-1}(K)$ and is defined by $\partial(S) = \sum_{v \in S} S \setminus \{v\}$ for any face S. An element in the chain group is called a *boundary element* if it is in the image of ∂. It is easy to see that composing the boundary operation from $C_{i-1}(K)$ with the boundary operation from $C_i(K)$ we get $\partial \circ \partial = 0$.

A homomorphism $\phi : C_i(K) \to \mathbb{Z}_2$ is called a *cocycle* if $\phi(\partial(S)) = 0$ for all $S \in C_{i+1}(K)$. Of course it is enough to require the above equality for faces $S \in K^{i+1}$ then it follows for linear combinations.

Let X and Y be topological spaces. Two continuous maps $g_0, g_1 : X \to Y$ are called *homotopic* if there exists a continuous map (*homotopy*) $g : X \times [0,1] \to Y$ such that the restriction of g to $K \times \{0\}$ is g_0 and restriction to $K \times \{1\}$ is g_1. A map $g : X \to Y$ is called *contractible* if it is homotopic to a constant map.

A subspace Y of a topological space X is said to be *contractible* in X if the inclusion map $\iota : Y \to X$ is contractible. A subcomplex L of the complex K is *contractible* if $\overline{L}$ is contractible in $\overline{K}$. A topological space or a complex is called *contractible* if it is contractible in itself.

A *triangulated i-sphere* is a complex whose body is homeomorphic to the i-sphere, i. e. to the set of unit norm vectors in $\mathbb{R}^{i+1}$.

A subspace Y is said to be a *retract* of the topological space X if there is a continuous map $f : X \to Y$ that is the identity on Y. A subcomplex L is said to be a *retract* of the complex K if $\overline{L}$ is a retract of $\overline{K}$.

We are ready now to formulate the simple results from topology we need. We start with a few observations that are trivial from the definitions above.

Observation 2.1.

a. *Homotopy is transitive.*
b. *Let $g_0, g_1 : X \to Y \subset \mathbb{R}^n$ be continues maps such that the interval $[g_0(x), g_1(x)]$ is contained in Y for any point $x \in X$. Then g_0 and g_1 are homotopic.*
c. *A convex subset of $\mathbb{R}^n$ is contractible.*
d. *A continues map from a contractible space, or a restriction of it is contractible.*
e. *Being a retract is transitive.*
f. *Let Y be a retract of the topological space X and Z a subspace of Y. Z is contractible in Y if and only if Z is contractible in X.* ∎

The following rather simple theorem connects the homotopy statements of section 3 to the homology statements of section 4. It can be found in any textbook e.g. [1, VIII §5. Satz III, page 340]

Fact 2.2. *Let the triangulated i-sphere L be subcomplex of K. If L is contractible in K then $\sum_{S \in L^i} S$ is a boundary element of $C_i(K)$.* ∎

The following theorem is the basis of our reasoning in section 5. It can be proven via standard methods and in fact a number of very similar results can be found in any book on the subject. To be self-contained we give a short proof here.

Fact 2.3. *Let K_1 and K_2 be complexes and suppose $K_1 \cap K_2$ is contractible. Then K_1 is a retract of $K_1 \cup K_2$.* ■

PROOF: We claim first that any continues function $f : \dot{S} \to X$ from the boundary of the geometrical simplex S can be extended to S if X is contractible. Indeed let $h : X \times [0,1] \to X$ be a homotopy connecting the identity on X to a constant map and let P_0 be an internal point of S. The formula $g(\alpha P_0 + (1-\alpha)Q) = h(f(Q), \alpha)$ where $\alpha \in [0,1]$ and $Q \in \dot{S}$ defines a continuous extension $g : S \to X$ of f.

Next we claim that any continuous map $\overline{L} \to X$ can be extended to $\overline{K}$ if L is a subcomplex of K and X is contractible. This extension can be constructed by adding faces to L one by one until K is reached and extending the map to the new face by the claim above.

Finally we apply the last claim to extend the identity map of $\overline{K_1 \cap K_2}$ to a retraction of $\overline{K_2}$ to $\overline{K_1 \cap K_2}$ and extend this identically on $K_1 \backslash K_2$. ■

3. Homotopy

A transversal of a system of d-intervals is said to be of *type* $\mathbf{t} = (t_1, \ldots, t_d)$ if it contains t_i points of the ith line for $i = 1, \ldots, d$. We denote by $|\mathbf{t}|$ the cardinality of any transversal of type $\mathbf{t}$, i. e. $|\mathbf{t}| = \sum_{i=1}^{d} t_i$.

For the rest of this section we fix an integer $d \geq 1$ a type $\mathbf{t} = (t_1, \ldots, t_d)$ and a number $k \geq 1$. We search for a transversal of type $\mathbf{t}$ for families of d-intervals with packing number at most k. The definition of the sets D, U and the complexes K_0, L, L_0 (below) depend on our choice of d and $\mathbf{t}$. The definition of K depends also on k.

Although we present the definitions and proofs in the general case the reader can first consider the special case $d = 2$ and $\mathbf{t} = (k, k)$. This is the case used in the proof of our main result, Theorem 1.2. In brackets we point out the possible simplifications in the argument when only considering this case.

Let $D = \{1, \ldots, d\}$ and let U be the set of the functions $e : D \to \mathbb{Z}$ satisfying $0 \leq e(i) \leq t_i$ for $i \in D$. We call two elements e and e' of U disjoint if $e(i) \neq e'(i)$ for any $i \in D$.

[In the special case consider the complete bipartite graph G whose color classes are $\{v_0, \ldots, v_k\}$ and $\{v'_0, \ldots, v'_k\}$. A function $e \in U$ can be identified with the edge from $v_{e(1)}$ to $v'_{e(2)}$, and thus U is the set of the edges of G.]

We define a complex K_0 on the set of vertices U. A set $S \subset U$ is a face of K_0 if and only if there is an $i \in D$ and a $0 \leq j \leq t_i$ such that for all $e \in S$ $e(i) \neq j$.

Let the complex K consist of the sets $S \subset U$ satisfying one of the following two conditions

(i) S does not contain $k+1$ pairwise disjoint elements of U or
(ii) $S \in K_0$.

[In the special case subsets of U are bipartite graphs (subgraphs of G). K_0 consists of graphs with isolated vertices, while (i) covers graphs without perfect matchings. Therefore (i) contains (ii) and K is the complex of bipartite graphs on twice $k+1$ vertices without perfect matchings.]

Another complex on the same set of vertices U is L whose faces are those sets $S = \{e_0, \ldots, e_m\} \subset U$ that satisfy $e_{j-1}(i) \le e_j(i)$ for all $1 \le j \le m$ and $i \in D$ (the dimension m is arbitrary).

Let $L_0 = L \cap K_0$, this is a subcomplex of both K and L.

[In the special case L consists of the graphs which do not have intersecting edges when drawn in the usual way. L_0 consists of the graphs that in addition have an isolated vertex.]

Theorem 3.1. *Let $\mathcal{H}$ be a family of d-intervals with $\nu(\mathcal{H}) \le k$. If there is no type $\mathbf{t}$ transversal of $\mathcal{H}$ then L_0 is contractible in K.*

PROOF: Without loss of generality we may suppose that $\mathcal{H}$ covers only a part of a finite interval (A_i, B_i) of the ith line ($i \in D$). (We need this assumption for compactness. In case the infinite family $\mathcal{H}$ covers an unbounded part of the lines then take monotonous one to one mappings of the lines to intervals and consider the equivalent but bounded family $\mathcal{H}'$ in the image.) Consider the sets

$$Z_i = \{(x_1, \ldots, x_{t_i}) | A_i \le x_1 \le \ldots \le x_{t_i} \le B_i\} \subset \mathrm{I\!R}^{t_i} \tag{1}$$

for $i \in D$ and let

$$Z = \mathop{\times}_{i=1}^{d} Z_i \subset \mathrm{I\!R}^{|\mathbf{t}|}. \tag{2}$$

When referring to a point $\mathbf{x} = (x_{11}, \ldots, x_{1t_1}; \ldots; x_{d1}, \ldots, x_{dt_d}) \in Z$ we are going to use the notation $x_{i0} = A_i$ and $x_{it_i+1} = B_i$ for $i \in D$.

A point $\mathbf{x} = (x_{11}, \ldots, x_{1t_1}; \ldots; x_{d1}, \ldots, x_{dt_d}) \in Z$ represents the set $S_\mathbf{x}$ consisting of the points x_{ij} on the ith line for $1 \le j \le t_i$ and $i \in D$. As this set would be of type $\mathbf{t}$ (or smaller if some of the points coincide) it is not a transversal of $\mathcal{H}$. Therefore there exists a d-interval $I \in \mathcal{H}$ disjoint from $S_\mathbf{x}$. Thus for every $i \in D$ the ith component I_i of I is disjoint from $\{x_{i1}, \ldots, x_{it_i}\}$ therefore $I_i \subset (x_{ij}, x_{ij+1})$ for some $0 \le j \le t_i$.

An equivalent formulation of the statement in the last paragraph is that the sets

$$H_e = \{\mathbf{x} = (x_{11}, \ldots, x_{1t_1}; \ldots; x_{d1}, \ldots, x_{dt_d}) \in Z | \; \nexists I \in \mathcal{H} \, \forall i \in D \, I_i \subset (x_{ie(i)}, x_{ie(i)+1})\}$$

do not have a common point for all $e \in U$, i. e. $\cap_{e \in U} H_e = \emptyset$.

Our goal is to collect enough topological observations about the sets H_e ($e \in U$) to be able to finish the proof using these observations and not referring to the family of d-intervals again. This last part of the proof is stated separately in Lemma 3.2.

First we observe that as all components of the d-intervals in $\mathcal{H}$ are closed so are all the sets H_e for $e \in U$.

Next we consider the boundary of Z. Z_i is a t_i dimensional simplex for each $i \in D$, its $(t_i - 1)$-dimensional faces are characterized by having equality at one of the $t_i + 1$ inequalities in its definition. Thus their product Z is a convex polytope, Z has $\sum_{i=1}^{d} (t_i + 1) = |\mathbf{t}| + d$ maximal dimensional faces, for any $i \in D$ and $0 \le j \le t_i$ we have a face

$$X_{ij} = \{\mathbf{x} = (x_{11}, \ldots, x_{1t_1}; \ldots; x_{d1}, \ldots, x_{dt_d}) \in Z \,|\, x_{ij} = x_{ij+1}\}. \tag{3}$$

For any $\mathbf{x} = (x_{11}, \ldots, x_{1t_1}; \ldots; x_{d1}, \ldots, x_{dt_d}) \in X_{ij}$ we have $x_{ij} = x_{ij+1}$ therefore the ith component of no d-interval can be between x_{ij} and x_{ij+1}. Thus $\mathbf{x} \in H_e$ for all $e \in U$ with $e(i) = j$. Therefore $H_e \supset X_{ie(i)}$ for all $e \in U$ and $i \in D$.

Finally consider a fixed $\mathbf{x} \in Z$. For any $e \in U$ by definition $\mathbf{x} \notin H_e$ means the existence of a special d-interval $I_e \in \mathcal{H}$. It is easy to verify that if e and e' are disjoint elements of U then I_e and $I_{e'}$ are also disjoint. Thus from $\nu(\mathcal{H}) \leq k$ we have that $\mathbf{x} \notin H_e$ cannot happen simultaneously for more than k pairwise disjoint elements $e \in U$. Thus if $S \subset U$ consists of $k+1$ pairwise disjoint elements then $\cup_{e \in S} H_e = Z$.

An application of Lemma 3.2 (below) concludes the proof. ∎

[In the special case of $d = 2$ and $\mathbf{t} = (k, k)$ a set S of $k+1$ pairwise disjoint elements of U is a perfect matching.]

The following lemma refers to the polytope Z as defined in (1) and (2) and its faces X_{ij} ($i \in D$, $0 \leq j \leq t_i$) as defined in (3).

Lemma 3.2. *For $e \in U$ let H_e be a closed subset of Z satisfying $H_e \supset \cup_{i \in D} X_{ie(i)}$. Suppose we have $\cup_{e \in S} H_e = Z$ for each set S consisting of $k+1$ pairwise disjoint elements of U. If $\cap_{e \in U} H_e = \emptyset$ then L_0 is contractible in K.*

PROOF: We identify the elements of U with the standard basis vectors of $\mathbb{R}^{|U|}$ as in the definition of the body of a complex on U. Let $g' : Z \to \mathbb{R}^{|U|}$ be defined by $g'(p) = \sum_{e \in U} \rho(p, H_e)e$ where ρ denotes the Euclidean distance. Here g' is continuous and for any $p \in Z$ all coordinates of $g'(p)$ are non-negative. As all the sets H_e are closed the coefficient of e in $g'(p)$ is zero if and only if $p \in H_e$. As $\cap_{e \in U} H_e = \emptyset$ the image of g' does not contain the origin. Thus we can define the normalized map $g : Z \to \mathbb{R}^{|U|}$ by $g(p) = g'(p)/(\sum_{e \in U} \rho(p, H_e))$. This map is also continuous and because of the normalization $g(p)$ is a convex combination of the points in U. Moreover we know that the coefficient of e in this convex combination is zero if $p \in H_e$. Thus $g(p)$ is in the convex hull of the points $e \in U$ for which $p \notin H_e$. This set of vertices form a face of the complex K as condition (i) in the definition must be satisfied. This means that $g(p) \in \overline{K}$ so g maps Z to $\overline{K}$.

In order to approximate g we study the geometry of Z. For $e \in U$ let us denote by V_e the vertex of Z that is *not* contained in $X_{ie(i)}$ for any $i \in D$. There is exactly one such vertex for each $e \in U$, these vertices are different for different elements of U and all vertices of Z are among $\{V_e | e \in U\}$. We define a mapping $h : \overline{L} \to Z$ by $h(\sum \alpha_e e) = \sum \alpha_e V_e$. It is easy to see (see e.g. [5, II, Lemma 8.9, page 68]) that h is a homeomorphism. (L is called a *triangulation* of Z.)

Let $Z_0 = \cup_{ij} X_{ij}$ be the boundary of Z. The inverse image of X_{ij} in $\overline{L}$ consists of the bodies of the faces $S \in L$ that contain no vertices $e \in U$ with $e(i) = j$. Thus the inverse image of Z_0 is the body of the complex L_0, as L_0 consists of those faces of L for which such i and j exist. Therefore $\overline{L_0}$ is homeomorphic to Z_0, the boundary of the convex $|\mathbf{t}|$ dimensional polytope Z, thus L_0 is a triangulated $(|\mathbf{t}| - 1)$-sphere.

We claim that the identity inclusion $\iota : \overline{L}_0 \to \overline{K}$ is homotopic to $g_0 \circ h_0$ where g_0 is the restriction of g to Z_0, and h_0 is the restriction of h to $\overline{L}_0$. By Observation 2.1.b it is enough to prove that the interval $[\iota(p), g_0(h_0(p))]$ is contained in $\overline{K}$ for all $p \in \overline{L}_0$. Here $h(p) \in Z_0$. Thus $h(p) \in X_{ij}$ for some $i \in D$ and $0 \leq j \leq t_i$. Then p is in the body of a face S of L_0 containing no vertex $e \in U$ with $e(i) = j$. Using the condition in the lemma we have $h(p) \in H_e$ for all these vertices e and thus $g(h(p)) \in \overline{S}_{ij}$ where $S_{ij} = \{e \in U | e(i) \neq j\}$. Here S_{ij} is a face of (K_0 and) K containing S therefore both $\iota(p) = p$ and $g_0(h_0(p)) = g(h(p))$ are in the body of the same face $S_{ij} \in K$ so the interval connecting them is also contained in $\overline{S}_{ij} \subset \overline{K}$.

Being homeomorphic to the convex polytope Z, $\overline{L}$ is contractible (Observation 2.1.c). Thus $g \circ h$ is contractible and so is its restriction $g_0 \circ h_0$ (Observation 2.1.d). Since homotopy is transitive (Observation 2.1.a) ι is also contractible. By definition this means that L_0 is contractible in K as claimed. ■

Notice that type (ii) faces in the definition of the complex K were needed to allow for the homotopy between $g_0 \circ h_0$ and ι at the end of the proof above. We close this section with observing that using these faces of K we can relax somewhat the conditions of Theorem 3.1.

[In the special case $d = 2$ and $\mathbf{t} = (k, k)$ Theorem 3.3 is not stronger than Theorem 3.1.]

Theorem 3.3. *Let $\mathcal{H}$ be a family of d-intervals. Suppose there is no subset $\mathcal{H}_1 \subset \mathcal{H}$ satisfying the following two properties:*

— *$H_i = \cup_{I \in \mathcal{H}_1} I_i$ is the union of more than t_i connected components (intervals) for each $i \in D$ (here I_i is the ith component of the d-interval I) and*

— *there is a subset $\mathcal{H}_2 \subset \mathcal{H}_1$ of more than k d-intervals such that the ith components of them are all in separate connected components of H_i for all $i \in D$.*

If there is no type $\mathbf{t}$ transversal of $\mathcal{H}$ then L_0 is contractible in K.

PROOF: The proof of Theorem 3.1 goes through in this case with hardly any modification. We need to replace the $\cup_{e \in S} H_e = Z$ condition for sets S consisting of $k+1$ pairwise disjoint elements of U in Lemma 3.2 with the same condition for sets $S \subset U$ with $S \notin K$. ■

4. Homology

Our goal is to prove that L_0 is not contractible in K for some values of d, $\mathbf{t}$ and k. Theorem 3.1 yields $f(d, k) \leq |\mathbf{t}|$ in each of these cases.

Let $z = z(d, \mathbf{t}, k) = \sum_{S \in L_0^{|\mathbf{t}|-1}} S \in C_{|\mathbf{t}|-1}(K)$.

In a previous draft of this paper we finished the proof of our main result (Theorem 1.2) by considering the $d = 2$, $\mathbf{t} = (k, k)$ case and constructing a cocycle of K not vanishing on z. This proved that z is not a boundary element of $C_{|\mathbf{t}|-1}(K)$ and by Fact 2.2 that L_0 is not contractible in K. (Note that L_0 is a triangulated $|\mathbf{t}| - 1$ sphere as it was pointed out in the proof of Lemma 3.2.) This approach made the concept of a retract and Fact 2.3 unnecessary to use. We illustrate this direct method in the proof of Proposition 4.1, a simple special case related to 3-intervals. But we prove our main result differently. The first step is to prove that L_0 is not contractible in K_0 (Theorem 4.4). This is true for every set of the parameters. We finish the proof by showing that in the special case $d = 2$ and $\mathbf{t} = (k, k)$ K_0 is a retract of K (Theorem 5.1). We find this approach more intuitive and hope that it will be simpler to extend for higher values of d. We remark that K_0 is also a retract of K in the case handled by Proposition 4.1.

We note that even for fixed d and k, an upper bound on $f(d, k)$ covers an infinite number of cases (families of d-intervals). We have just reduced the proof to showing that z is not a boundary element, a finite problem since $C_{|\mathbf{t}|-1}(K)$ is a finite group.

Proposition 4.1. *Let $d = 3$, $\mathbf{t} = (1, 1, 2)$, and $k = 1$. Then there exists a cocycle $\phi : C_3(K) \to \mathbb{Z}_2$ with $\phi(z) \neq 0$.*

PROOF: We define ϕ by listing the six faces $S \in K^3$ with $\phi(S) = 1$. On the rest of the 3-dimensional faces of K ϕ vanishes. Then ϕ extends to linear combinations and it is easy to check that ϕ is a cocycle and $\phi(z) = 1$. The list consists of S_i and S'_i for $i = 0, 1, 2$. Here $S_i = \{e_{000}, e_{001}, e_{002}, e_{01i}\}$ and $S'_i = (S_i \cup \{e_{10i}\}) \setminus \{e_{00i}\}$ where $e \in U$ was denoted by $e_{e(1)e(2)e(3)}$. ∎

The following corollary was implicit in [6]. It was also showed there that $f(3, 1) = 4$.

Corollary 4.2. *[6] Every pairwise intersecting family of 3-intervals has a transversal of type $(1, 1, 2)$. Thus $f(3, 1) \leq 4$.*

PROOF: Fact 2.2, Theorem 3.1, and Proposition 4.1 yield the proof. ∎

With our method we can prove the same conclusion from a somewhat weaker assumption.

Corollary 4.3. *Suppose a family $\mathcal{H}$ of 3-intervals does not contain 3-intervals I_1, I_2, and I_3 such that I_1 and I_2 are disjoint, the first component of the three 3-intervals are pairwise disjoint, and both the second and the third components of I_3 is disjoint from either I_1 or I_2. Then $\mathcal{H}$ has a transversal of type $(1, 1, 2)$.*

PROOF: Fact 2.2, Theorem 3.3, and Proposition 4.1 yield the proof. ∎

The next observation holds for any set of the parameters d and $\mathbf{t}$.

Theorem 4.4. *L_0 is not contractible in K_0.*

PROOF: By Fact 2.2 it is enough to show that z is not a boundary element of $C_{|\mathbf{t}|-1}(K_0)$. We prove this by constructing a cocycle ϕ of K_0 not vanishing on z.

Let us say that a vertex $e \in U$ is of type (i, j) with $i \in D$ and $1 \leq j \leq t_i$ if $e(i) = j$ and $e(i') = 0$ for $1 \leq i' < i$. If we say that the constant 0 function $e_0 \in U$ is of type $(d, 0)$ we get a partitioning of U into these types.

We define ϕ on faces $S \in K_0^{|\mathbf{t}|-1}$ then it extends to linear combinations. We take $\phi(S) = 1$ if S contains a vertex of each type except for the type $(1, t_1)$. We take $\phi(S) = 0$ otherwise.

To prove that ϕ is a cocycle of K_0 take a face $S \in K_0^{|\mathbf{t}|}$. It is easy to see that ϕ vanishes on each face in the sum $\partial(S)$ unless S contains a vertex of each type except possibly the type $(1, t_1)$. If S contains a vertex of each other type but no vertex of type $(1, t_1)$ then it has to contain two vertices of one type and ϕ does not vanish on exactly two faces of the sum $\partial(S)$ thus $\phi(\partial(S)) = 0$. This finishes the proof since no face $S \in K_0$ contains a vertex of each type. Indeed if $S \in K_0$ then there is a $i \in D$ and $0 \leq j \leq t_i$ such that $e(i) \neq j$ holds for all vertex $e \in S$. Thus if $j \neq 0$ S contains no vertex of type (i, j) and if $j = 0$ S does not contain the only vertex of type $(d, 0)$.

It is left to prove that $\phi(z) \neq 0$. For this we need to find the faces $S \in L_0^{|\mathbf{t}|-1}$ for which $\phi(S) = 1$. Such a face contains a vertex of each type except for the type $(1, t_1)$. Downward induction on i shows that the vertex of type (i, j) must be e_{ij} defined by $e_{ij}(i') = 0$ if $1 \leq i' < i$, $e_{ij}(i) = j$, and $e_{ij}(i') = t_{i'}$ if $i < i' \leq d$. Thus this face is unique and therefore $\phi(z) = 1$. ∎

Let us remark here that an easy application of the Nerve Theorem (see Borsuk [4] and Björner, Korte and Lovász [3]) shows that K_0 is in fact homotopy equivalent to the $(|\mathbf{t}|-1)$-sphere, so z is essentially the unique cycle in K_0 not homologous to 0.

5. Retracts

Proposition 5.1. *If $d=2$ and $\mathbf{t}=(k,k)$ then K_0 is a retract of K.*

PROOF: This is the special case considered in brackets throughout section 3. As it was pointed out there U can be considered the edge set of a complete bipartite graph G with color classes V and W both consisting of $k+1$ vertices. The complex K consists of the subgraphs of G (considered as edge sets) that contain no perfect matching. K_0 is the complex of subgraphs with isolated vertices.

Let us consider all the pairs (A,B) with $A\subset V$, $B\subset W$ and $|A|+|B|>k+1$. Note that neither A nor B is empty. Arrange these pairs in a sequence $(A_1,B_1),\ldots,(A_n,B_n)$ such that for $i=1,\ldots,n-1$ we have $|A_i|+|B_i|\geq|A_{i+1}|+|B_{i+1}|$ and in case of equality here we have $|A_i|\leq|A_{i+1}|$.

Let us consider the complex K'_i consisting of the graphs having no edge connecting A_i to B_i $(i=1,\ldots,n)$. We define K_i by recursion for $i=1,\ldots,n$, let $K_i=K_{i-1}\cup K'_i$. As the complexes K'_i contain exactly the graphs violating the Hall-criterion for perfect matching their union K_n coincides with K.

Fix an index $1\leq i\leq n$. We claim that K_{i-1} is a retract of K_i. If $A_i=V$ then every vertex of the nonempty set B_i is isolated in each face of K'_i thus $K'_i\subset K_0\subset K_{i-1}$, therefore $K_i=K_{i-1}$. The same equality holds if $B_i=W$ thus we may suppose that there exist vertices $v\in V\backslash A_i$ and $w\in W\backslash B_i$. Let us call e the edge of U connecting v to w. To prove the claim by Fact 2.3 we need to show that $K^*=K'_i\cap K_{i-1}$ is contractible. We claim that the identity map on K^* and the constant map to e are homotopic by Observation 2.1.b therefore K^* is contractible. To see that for every point $p\in\overline{K}^*$ the interval $[p,e]$ is also contained in $\overline{K}^*$ it is enough to show that for every face $S\in K^*$ we have $S\cup\{e\}\in K^*$.

Take therefore a face $S\in K^*$. As $S\cup\{e\}\in K'_i$ trivially holds it is enough to show $S\cup\{e\}\in K_{i-1}$. As $S\in K_{i-1}$ we have $S\in K_0$ or $S\in K'_j$ for some $1\leq j<i$.

Suppose first that $S\in K_0$. Then S has an isolated vertex. If this is neither v nor w then $S\cup\{e\}\in K_0\subset K_{i-1}$. Thus by symmetry we may suppose that v is isolated in S. The pair $(A_i\cup\{v\},B_i)=(A_j,B_j)$ must precede (A_i,B_i) since $|A_j|+|B_j|=|A_i|+|B_i|+1$, so $1\leq j<i$. Thus $S\cup\{e\}\in K'_j\subset K_{i-1}$.

Suppose now that $S\in K'_j$ for some $1\leq j<i$. If $v\notin A_j$ or $w\notin B_j$ then $S\cup\{e\}\in K_j\subset K_{i-1}$. Otherwise consider the pairs $(A,B)=(A_i\cup A_j,B_i\cap B_j)$ and $(A',B')=(A_i\cap A_j,B_i\cup B_j)$. As $|A|+|B|+|A'|+|B'|=|A_i|+|B_i|+|A_j|+|B_j|\geq 2(|A_i|+|B_i|)$ either one of the pairs is on the list and precedes (A_i,B_i) or both are on the list and $|A|+|B|=|A'|+|B'|=|A_i|+|B_i|$. In this last case $|A|>|A'|$ ensures by our tie-braking rule that (A,B) precedes (A_i,B_i). So one of the two pairs is (A_l,B_l) with $1\leq l<i$ and $S\cup\{e\}\in K_l\subset K_{i-1}$.

We finish the proof of this theorem by using the transitivity (Observation 2.1.e) to infer that K_0 is a retract of $K_n=K$. ∎

Just as Proposition 4.1 yields Corollary 4.2 the above theorem yields this:

Corollary 5.2. *Any family $\mathcal{H}$ of 2-intervals has a transversal of type $(\nu(\mathcal{H}), \nu(\mathcal{H}))$. Thus $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$.*

PROOF: Observation 2.1.f, Theorem 4.4, Theorem 3.1, and Theorem 5.1 yield the proof. ■

PROOF OF THEOREM 1.2: We have just proved the upper bound $f(2,k) \leq 2k$.

The lower bound $f(2,k) \geq 2k$ follows from the general observation $f(d,k) \geq dk$ found in [8]. Another way to derive the bound is to use the construction of three pairwise intersecting 2-intervals with empty intersection in [6]. It shows $f(2,1) \geq 2$. Taking k far away isomorphic copies of these three 2-intervals one gets the desired bound. ■

6. Further results and open problems

The following is an easy consequence of Corollary 5.2:

Corollary 6.1. *Let $k_1 \geq k_2 > 0$ be integers. Suppose the family $\mathcal{H}$ of 2-intervals has the following property: if $k_1 + 1$ elements of $\mathcal{H}$ have disjoint first components then at most k_2 of them have disjoint second components. Then $\mathcal{H}$ has a transversal of type (k_1, k_2).*

PROOF: Find a set T of $k_1 - k_2$ pairwise disjoint intervals on the second line that are disjoint from all 2-intervals in $\mathcal{H}$. Let $\mathcal{H}'$ consist of the 2-intervals $I^1 \cup I^2$ where I^1 is the first component of a 2-interval in $\mathcal{H}$ and $I^2 \in T$. Apply Corollary 5.2 to $\mathcal{H} \cup \mathcal{H}'$. ■

The condition on $\mathcal{H}$ can be relaxed a little here too, but then the proof has to use the topological proof techniques of this paper rather then Corollary 5.2:

Proposition 6.2. *Let $k_1 \geq k_2 > 0$ be integers and suppose the family $\mathcal{H}$ of 2-intervals has the following property: if $k_1 + 1$ elements of the family have pairwise disjoint first components then the union of their second components is the union of at most k_2 intervals. Then $\mathcal{H}$ has a transversal of type (k_1, k_2).*

PROOF: We construct K, K_0 and L_0 as in Section 3 for $d = 2$ $\mathbf{t} = (k_1, k_2)$ and $k = k_2$. By Theorem 3.3 and Theorem 4.4 it is enough to show that K_0 is a retract of K. It is easy show this by direct analogy to the proof of proposition 5.1. ■

We believe that Theorem 3.1 can be used to bound $f(d,k)$ for higher values of d too. Let us state here the following conjecture:

Conjecture 6.3. *For any fixed d there is a function $t(k) = O(k)$ such that when K_0 and K are defined for d, k, and $\mathbf{t} = (t(k), \ldots, t(k))$ then K_0 is a retract of K.*

This conjecture would imply $f(d,k) \leq dt(k) = O(k)$ for any fixed d. The best current bound is $f(d,k) = O_d(k^{d-1})$in [8]. (The proof uses Theorem 1.2 to improve the earlier $f(d,k) = O_d(k^d)$ bound.)

Finally we show how to apply Theorem 1.2 to prove linear dependence between τ and ν for another model of 2-intervals.

Let us fix a single line. We call the union of d intervals of the line a *homogeneous d-interval.*

Theorem 6.4. *For any family $\mathcal{H}$ of homogeneous d-intervals we have $\tau(\mathcal{H}) \leq f(d, 2d(d-1)\nu(\mathcal{H}))$.*

PROOF: We put the components of the homogeneous d-intervals to d parallel lines by perpendicular translation to get d-intervals. Let $\mathcal{H}'$ be the set of the d-intervals so obtained from the elements of $\mathcal{H}$.

We claim that $\nu(\mathcal{H}') \leq 2d(d-1)\nu(\mathcal{H})$. Using the trivial observation $\tau(\mathcal{H}) \leq \tau(\mathcal{H}')$ this proves the statement of our theorem.

To prove the bound on $\nu(\mathcal{H}')$ take a maximum set S' of pairwise disjoint d-intervals from $\mathcal{H}'$. Let S be the set of the corresponding homogeneous d-intervals from $\mathcal{H}$. Let us consider the graph G whose vertices are the intervals constituting the homogeneous d-intervals in S; two vertices are adjacent if the corresponding intervals intersect. The graph G is d-partite as all edges connect intervals that are moved to distinct parallel lines. Therefore it is the union of $\binom{d}{2}$ bipartite subgraphs. It is easy to see that none of these subgraphs contain a cycle. Therefore the average degree of any subgraph of G is less than $2(d-1)$. Consider now the graph G' whose vertices are the homogeneous d-intervals of S; two vertices are adjacent if the corresponding elements of S intersect. As G' can be obtained from G by identifying d-tuples of vertices the average degree of any subgraph of G' is less than $2d(d-1)$. Thus the greedy strategy finds an independent set of size $\lceil |S|/(2d(d-1)) \rceil$ in G'. As this is a set of disjoint elements of $\mathcal{H}$ the bound on $|S| = \nu(\mathcal{H}')$ is proved. ∎

Corollary 6.5. *For any family $\mathcal{H}$ of homogeneous 2-intervals we have $\tau(\mathcal{H}) \leq 8\nu(\mathcal{H})$.*

PROOF: Theorem 1.2 and Theorem 6.4 yield the result. ∎

Acknowledgment

This research was initiated by discussions with Gyula Károlyi. Lots of fruitful discussions with László Lovász, Gábor Moussong, András Gyárfás, and Miklós Ruszinkó helped the author. Many thanks to László Babai for his comments on a previous draft of this paper.

References

1. P. Alexandroff und H. Hopf, Topologie, Springer Verlag, 1935
2. A. Björner, Topological methods, in *Handbook of Combinatorics*, ed. R. Graham, M. Grötschel, and L. Lovász
3. A. Björner, B. Korte and L. Lovász, Homotopy properties of greedoids, *Advances in Appl. Math.*, **6** (1985) 447–494
4. K. Borsuk, On the imbedding of compacta in simplicial complexes, *Fundamenta Math.*, **35** (1948) 217–234
5. S. Eilenberg and N. Steenrod, Foundations of algebraic topology, Princeton University Press, 1952
6. A. Gyárfás and L. Lehel, A Helly-type theorem in trees, in *Combinatorial Theory and its Applications*, Balatonfüred (Hungary), Colloquia Mathematica Societatis János Bolyai **4**, 1969, pp. 571–584

7. A. Hajnal and J. Surányi: Über die Auflösung von Graphen in vollständige Teilgraphen, *Annales Univ. Sci. Budapest*, **1** (1958) 113–121
8. Gy. Károlyi and G. Tardos, On point covers of multiple intervals and axis-parallel rectangles, manuscript, 1994
9. E. Spanier, Algebraic topology, McGraw-Hill, 1966