Permutation capacities of families of oriented infinite paths

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Abstract

Körner and Malvenuto asked whether one can find $\binom{n}{\lfloor n/2 \rfloor}$ linear orderings (i. e., permutations) of the first n natural numbers such that any pair of them place two consecutive integers somewhere in the same position. This led to the notion of graph-different permutations. We extend this concept to directed graphs, focussing on orientations of the semi-infinite path whose edges connect consecutive natural numbers. Our main result shows that the maximum number of permutations satisfying all the pairwise conditions associated with all of the various orientations of this path is exponentially smaller, for any single orientation, than the maximum number of those permutations which satisfy the corresponding pairwise relationship. This is in sharp contrast with a result of Gargano, Körner, and Vaccaro concerning the analogous notion of Sperner capacity of families of finite graphs. We improve the exponential lower bound for the original problem, and list a number of open questions.

1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers and let D be an arbitrary loopless directed graph (digraph) with vertex set $\mathbb{N}$. We will say that two permutations σ and τ of the first n natural numbers are *D–different* if there is an $i \in [n] = \{1, \ldots, n\}$ such that the ordered couple of its images under these two permutations satisfies $(\sigma(i), \tau(i)) \in E(D)$. We write $N(G, n)$ for the largest cardinality of a set of pairwise D–different permutations of $[n]$. (This means that every couple is D–different in both orders.) Our main concern in this paper will be the behaviour of $N(D, n)$ in the special cases when D is an orientation of the semi-infinite path L containing as edges the pairs of consecutive positive integers.

The above definitions naturally extend to digraphs the notion of graph-different permutations investigated in [13, 14, 17] in the undirected case. In fact, if we identify undirected graphs with their *symmetrically directed* equivalent, i.e., with digraphs having two oppositely oriented edges in place of all of their undirected edges, then the undirected notion becomes a special case of the directed one. This relationship is analogous to that between the Shannon capacity of graphs [22] and its generalization to digraphs called Sperner capacity (cf. [10, 16] for its origins and [1, 4, 6, 11, 12, 15, 19, 20] for some further results about Sperner capacity). The close connection of Shannon capacity and the notion of graph-different permutations for undirected graphs is explored on a quantitative level in [17] and one could easily formulate a similar statement for the directed case.

The motivating example for introducing graph-different permutations was the puzzle presented in [13] that asks for the value of $N(L, n)$, i.e., the maximum size of a set of permutations of the elements in $[n]$ satisfying that if σ and τ are two distinct permutations in this set, then there is some $i \in [n]$ for which $|\sigma(i) - \tau(i)| = 1$, that is, $\{\sigma(i), \tau(i)\} \in E(L)$. The natural upper bound $N(L, n) \leq \binom{n}{\lfloor n/2 \rfloor}$ was presented, and conjectured to be sharp, in [13]. It is still an open problem to decide whether $N(L, n)$ is always equal to this upper bound. Indeed even the weaker conjecture that $\rho(L) := \lim_{n\to\infty} \frac{1}{n} \log N(L, n) = 1$ remains open; later in the paper we show that $\rho(L) \geq 0.8604$. (The base of logarithms is always taken to be 2.)

In this paper we will mainly focus on the various orientations of L. Our main result exhibits an exponential gap betwen the maximum size of a set of permutations that are pairwise $\vec{L}$-different for any fixed orientation $\vec{L}$ of L and the maximum size of a set of such permutations that are pairwise $\vec{L}$-different simultaneously for *all* orientations $\vec{L}$ of L. This is in sharp contrast with one of the main results about Sperner capacity proven in [11].

2 Fixed orientations: a lower bound

Given an undirected graph G, an *orientation* of G is a digraph obtained from G by replacing each edge xy with one directed edge, either from x to y or from y to x.

Let $\vec{L}$ be any fixed orientation of the semi-infinite path L, that is, the edge set of $\vec{L}$

contains, for every $i \in \mathbb{N}$, exactly one of the ordered pairs $(i, i+1)$ and $(i+1, i)$.

Let

$$R(\vec{L}) = \limsup_{n\to\infty} \frac{1}{n} \log N(\vec{L}, n)$$

be the *permutation capacity* of $\vec{L}$. Denoting by $\mathcal{L}$ the set of all orientations of L, we also define

$$R_{\min}(L) = \inf_{\vec{L}\in\mathcal{L}} R(\vec{L}) \qquad \text{and} \qquad R_{\max}(L) = \sup_{\vec{L}\in\mathcal{L}} R(\vec{L}).$$

It is clear from these definitions that $R_{\min}(L) \le R_{\max}(L) \le \rho(L) \le 1$. The last inequality follows from noting (see [13]) that, for two L-different permutations, the set of positions of odd (even) numbers must differ. (Here we use the notion of being L-different again in the sense of our definitions, identifying L with the symmetrically directed equivalent of its originally undirected version.)

Our first result is the following lower bound.

Theorem 1

$$R_{\min}(L) \ge \log \frac{1+\sqrt{5}}{2} \approx 0.694.$$

An improved lower bound will also be given for $R_{\min}(L)$ in Section 5; the above statement is included here because it has a simpler proof, and the lower bound is already large enough for our main conclusion in the next section.

To prove Theorem 1 we need some preparation. For an arbitrary digraph D on $\mathbb{N}$ let $\Gamma_D(n)$ be the digraph defined as follows. The vertex set of $\Gamma_D(n)$ consists of all the different permutations of the elements of $[n]$. An ordered pair (σ, τ) of permutations is an edge of $\Gamma_D(n)$ if there exists an $i \in [n]$ for which $(\sigma(i), \tau(i)) \in E(D)$. We denote by $\Gamma_{D_{(j)}}(n)$ the similarly defined graph on the permutations of numbers $j, j+1, \ldots, j+n-1$.

For an arbitrary digraph D, its symmetric clique number $\omega_s(D)$ is the maximum number of vertices of D that form a symmetric clique, i.e., a subgraph in which every ordered pair of distinct nodes forms an edge. In particular, it follows from the definitions that $N(D, n) = \omega_s(\Gamma_D(n))$. The transitive clique number $\omega_{tr}(D)$ of a digraph D is the largest number of vertices in D that form a *transitive clique*, i.e., a subgraph in which the vertices could be labelled by numbers $1, 2, \ldots, k$ so that each label appears only once and all ordered pairs (u, v) form edges where u is labelled with a smaller number than v. Clearly, $\omega_s(D) \le \omega_{tr}(D)$ holds for every digraph D. For the clique number of an undirected graph G we use the usual notation $\omega(G)$.

We need the following technical lemma relating the value

$$t_{\mathcal{L}}(n) := \min_{\vec{L}\in\mathcal{L}} \{\omega_{tr}(\Gamma_{\vec{L}}(n))\}$$

to the permutation capacity of graphs in $\mathcal{L}$.

Lemma 2

$$R(\vec{L}) \geq \frac{1}{n} \log t_{\mathcal{L}}(n)$$

for any fixed orientation $\vec{L}$ of the semi-infinite path L and any natural number n.

Proof. Fix $n \in \mathbb{N}$ and $\vec{L} \in \mathcal{L}$. For every $j \in \mathbb{N}$, let L_j denote the oriented n-vertex path induced by $\vec{L}$ on the vertices $(j-1)n+1, (j-1)n+2, \ldots, jn$. It follows from the definition of $t := t_{\mathcal{L}}(n)$ that, for every j, there exist t permutations of the vertices of L_j which form a transitive clique in $\Gamma_{L_{((j-1)n+1)}}(n)$, i.e., they can be labelled by $\sigma_{j,1}, \ldots, \sigma_{j,t}$ so that for every $k < \ell$ there is an $1 \leq r \leq n$ for which we have $(\sigma_{j,k}(r), \sigma_{j,\ell}(r)) \in E(L_j)$. Fix such a set of permutations M_j together with the above type of labelling for every $j < h$, where h is some appropriately large natural number. Now consider all permutations in S_{hn} that can be written in the form of $\sigma_{1,i_1}\sigma_{2,i_2}\ldots\sigma_{h,i_h}$, where $\sigma_{j,i_j} \in M_j$ for each j. There are t^h such permutations, and there is an edge from $\sigma_{1,i_1}\sigma_{2,i_2}\ldots\sigma_{h,i_h}$ to $\sigma_{1,j_1}\sigma_{2,j_2}\ldots\sigma_{h,j_h}$ in $\Gamma_{\vec{L}}(hn)$ whenever $i_k < j_k$ for some index k. Therefore, the subset S^K of all these permutations for which the sum $\sum_{j=1}^{h} i_j$ is a fixed number K forms a symmetric clique in $\Gamma_{\vec{L}}(hn)$. Since the above sum can take fewer than $h \cdot t$ different values, this implies that $N(\vec{L}, hn) = \omega_s(\Gamma_{\vec{L}}(hn)) \geq \frac{t^h}{h \cdot t}$. Taking the (hn)-th root, the logarithm, and the limit in h, we arrive at the stated inequality. □

Lemma 3 *We have*

$$t_{\mathcal{L}}(n) \geq F_{n+1},$$

where F_n denotes the n-th element of the Fibonacci sequence defined by $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$ for $n \geq 3$.

Proof. We use induction on n. We obviously have $t_{\mathcal{L}}(1) = 1 = F_2$ and $t_{\mathcal{L}}(2) = 2 = F_3$. Assuming the validity of the stated inequality for all $n \leq k$, we show it for $n = k+1$. Fix an arbitrary orientation $\vec{L} \in \mathcal{L}$. For $i = k-1$ and $i = k$, let M_i be a set of permutations of $1, \ldots, i$ forming a transitive clique of size F_{i+1} in $\Gamma_{\vec{L}}(i)$. Extend the permutations in M_{k-1} to permutations of $[k+1]$ by putting k in the last position and $k+1$ in the next to last position thus obtaining the set

$$M_{k-1}(k+1)k = \{\sigma(1)\ldots\sigma(k-1)(k+1)k : \sigma \in M_{k-1}\}.$$

Similarly, define the set

$$M_k(k+1) = \{\sigma(1)\ldots\sigma(k)(k+1) : \sigma \in M_k\}.$$

The set $M_{k+1} := (M_{k-1}(k+1)k) \cup (M_k(k+1))$ then forms a transitive clique in $\Gamma_{\vec{L}}(k+1)$ (depending on the orientation of the edge $\{k, (k+1)\}$ we have the first or the second set dominating the other) and has size $F_{k-1} + F_k = F_{k+1}$. Since $\vec{L}$ was an arbitrary orientation of L, this implies the statement. □.

Proof of Theorem 1. Combining Lemma 2 with Lemma 3 gives $R(\vec{L}) \geq \limsup_{n\to\infty} \frac{1}{n} \log F_{n+1}$; thus the well-known explicit form of the Fibonacci numbers implies the statement. □

3 Robust capacity: an upper bound

One of the main results about Sperner capacity is a "bottleneck theorem" [11] concerning digraph families; see also the discussion in the next section. In this section, we prove that an analogous statement does not hold for the permutation capacity of the infinite family of graphs formed by all orientations of the semi-infinite path L.

Let $\Gamma_{\mathcal{L}}(n)$ denote the following graph on the common vertex set of the graphs $\Gamma_{\vec{L}}(n)$ with $\vec{L} \in \mathcal{L}$. The edge set of $\Gamma_{\mathcal{L}}(n)$ is

$$E(\Gamma_{\mathcal{L}}(n)) := \cap_{\vec{L}\in\mathcal{L}} E(\Gamma_{\vec{L}}(n)).$$

We would like to understand the asymptotic behaviour of $\omega_s(\Gamma_{\mathcal{L}}(n))$. In other words, we are interested in the size of the largest set of permutations of $[n]$, any two elements σ and τ of which satisfy that, for any $\vec{L} \in \mathcal{L}$, there is an i and a j such that $(\sigma(i), \tau(i)) \in E(\vec{L})$ and $(\tau(j), \sigma(j)) \in E(\vec{L})$.

First we observe that, if permutations τ and σ are in the above relation, then there must be a k and $i \neq j$ such that $(\sigma(i), \tau(i)) = (\tau(j), \sigma(j)) = (k, k+1)$. Indeed, if no such i, j, k would exist, then for every $k \in \mathbb{N}$ at most one of the ordered pairs $(k, k+1)$ and $(k+1, k)$ appears among the ordered pairs $(\sigma(i), \tau(i))$, and this means that L has an orientation $\hat{L}$ for which we never have $(\tau(i), \sigma(i)) \in E(\hat{L})$. This observation motivates the following definition.

Definition 4 *Let G be an undirected graph with vertex set $\mathbb{N}$. We will say that the permutations σ and τ of $[n]$ are* robustly G–different *if there are two elements $i \in [n]$ and $j \in [n]$ such that $(\sigma(i), \tau(i)) = (\tau(j), \sigma(j))$ and $\{\sigma(i), \tau(i)\} \in E(G)$.*

Let $NN(G, n)$ be the maximum cardinality of a set of pairwise robustly G–different permutations of $[n]$. We call

$$RR(G) = \limsup_{n\to\infty} \frac{1}{n} \log NN(G, n)$$

the robust permutation capacity *of G.*

We are interested in the value of $RR(L)$. First we prove the following easy fact.

Proposition 5 *For the semi-infinite path L we have*

$$NN(L, n) \geq 2^{\lfloor \frac{n}{2} \rfloor},$$

implying

$$RR(L) \geq \frac{1}{2}.$$

Proof. Consider the set of permutations that can be obtained as a product of some or all of the inversions $(2k-1, 2k)$, where $k \leq n/2$. It is straightforward to check that these permutations are pairwise robustly L-different and their number is $2^{\lfloor \frac{n}{2} \rfloor}$, which implies the statement. □

We conjecture that the above lower bound is tight. Our main result in this section is a weaker upper bound on $RR(L)$ which is nevertheless smaller than the lower bound proven on $R_{\min}(L)$ in Theorem 1.

Theorem 6

$$RR(L) \leq \log \frac{\pi}{2} \approx 0.651.$$

For an undirected graph G, let $\hat{\Gamma}_G(n)$ be the robust analogue of the graph $\Gamma_D(n)$ defined for digraphs D: the vertex set of $\hat{\Gamma}_G(n)$ is the set of permutations of $[n]$, and two vertices are adjacent in $\hat{\Gamma}_G(n)$ if they are robustly G-different. It follows from the definitions that $NN(G, n) = \omega(\hat{\Gamma}_G(n))$. (The discussion preceding Definition 4 shows that $\omega_s(\Gamma_{\mathcal{L}}(n)) = \omega(\hat{\Gamma}_L(n))$. Note that $\Gamma_{\mathcal{L}}(n)$ is a digraph, while $\hat{\Gamma}_L(n)$ is undirected.) Notice that $\hat{\Gamma}_G(n)$ (just like $\Gamma_D(n)$ for directed D) is a vertex-transitive graph, as for any two of its vertices there is a permutation that can take one to the other.

We will use the standard notation $\alpha(F)$ for the independence number and $\chi_f(F)$ for the fractional chromatic number of a graph F. We will make use of the basic inequality $\omega(F) \leq \chi_f(F)$, for any graph F. We will also use the fact that, if F is vertex-transitive, then $\chi_f(F) = |V(F)|/\alpha(F)$. For these and other basic facts about the fractional chromatic number, see [21].

Proof of Theorem 6. First we find a large independent set in the graph $\hat{\Gamma}_L(n)$. Let

$$I_n = \{\sigma \in S_n : \forall k \in [\lfloor n/2 \rfloor] \;\; \sigma^{-1}(2k) < \sigma^{-1}(2k-1) \text{ and } \sigma^{-1}(2k) < \sigma^{-1}(2k+1)\}.$$

In other words, I_n is the collection of all those permutations of $[n]$ that place each even number in an earlier position than either of its at most two neighboring odd numbers. We show that the permutations in I_n form an independent set in the graph $\hat{\Gamma}_L(n)$.

Let σ and τ be two arbitrary elements of I_n, and suppose that they form an edge in $\hat{\Gamma}_L(n)$. Then there is some edge $\{\ell, \ell+1\}$ of L for which there exists i and j such that $\sigma(i) = \tau(j) = \ell$ and $\sigma(j) = \tau(i) = \ell + 1$. We may assume without loss of generality that $i < j$. Then $\sigma \in I_n$ implies that ℓ is even, while $\tau \in I_n$ implies that ℓ is odd. This contradiction proves that I_n is indeed an independent set in $\hat{\Gamma}_L(n)$.

By the vertex-transitivity of $\hat{\Gamma}_L(n)$, we have that

$$\chi_f(\hat{\Gamma}_L(n)) = \frac{|V(\hat{\Gamma}_L(n))|}{\alpha(\hat{\Gamma}_L(n))} \leq \frac{n!}{|I_n|}.$$

The size of the set I_n is a well-investigated quantity. The permutations in the set I_n are called *alternating*, and the problem of determining their number, called André's problem,

was already considered in [2]. Some more recent references where the asymptotics of this sequence appears are [25] (cf. the Note on page 455) and [3] (cf. page 3); see also [24] for the vast literature on this sequence. The asymptotic behavior of the sequence is given by $|I_n| \sim 2^{(n+2)} n!/\pi^{(n+1)}$.

Substituting this value into the above bound on $\chi_f(\hat{\Gamma}_L(n))$, and using that $NN(L,n) = \omega(\hat{\Gamma}_L(n)) \leq \chi_f(\hat{\Gamma}_L(n))$, we obtain that

$$RR(L) \leq \lim_{n\to\infty} \frac{1}{n} \log \frac{\pi^{n+1}}{2^{n+2}} = \log \frac{\pi}{2}$$

as stated. □

The following is an immediate consequence of Theorems 1 and 6.

Corollary 7

$$RR(L) < R_{\min}(L).$$

□

It is rather frustrating that, for $R_{\min}(L)$ itself, we do not have any better upper bound than the trivial value 1. A modest improvement on the best known upper bound in the undirected case is that we at least know $N(\vec{L},n) < \binom{n}{\lfloor n/2 \rfloor}$ for some orientations of L.

Proposition 8 *If $\vec{L}$ is an orientation of L that has at least two vertices of $[n]$ which have different parity and either both have zero outdegree or both have zero indegree, then*

$$N(\vec{L},n) < \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. Assume $\vec{L}$ is as in the statement and let $i = 2k$ and $j = 2\ell+1$ be the two vertices satisfying the conditions therein. We may assume without loss of generality that they both have outdegree zero. Let M_n be a set of pairwise $\vec{L}$-different permutations. We may assume that the identity permutation is in M_n. Now consider an arbitrary permutation σ of $[n]$ that puts odd elements in the odd positions and even elements in the even positions, except that there is an even number in position j and an odd number in position i. Thus the parity pattern of σ is different from that of the identity permutation. Hence, if $M_n = \binom{n}{\lfloor n/2 \rfloor}$, then one such permutation σ should appear in M_n. However, since the identity permutation (which is in M_n) has a sink at both of those places where it has an element of different parity from σ, there is no position with an arc in $\vec{L}$ from the element in the identity permutation to the element of σ in the same position. This implies that our $\vec{L}$-different set of permutations cannot contain such a σ, and therefore $|M_n| < \binom{n}{\lfloor n/2 \rfloor}$. This proves the statement. □

It should be clear that if there are many sources and sinks in both parity classes, then the difference $\binom{n}{\lfloor n/2 \rfloor} - N(\vec{L},n)$ can be made large. Unfortunately this is still not enough to prove an exponential gap.

4 On bottlenecks

As stated in the Introduction, Corollary 7 is in sharp contrast with the main result about Sperner capacity proven in [11]. For the sake of completeness, we state this result here. This needs some definitions.

Definition 9 *The* nth co-normal power *of a digraph* D *is the digraph* D^n *with vertex set* $V(D^n) = V(D)^n$*, i.e., the* n*-length sequences of vertices of* D*, and edge set*

$$E(D^n) = \{(\boldsymbol{x}, \boldsymbol{y}) : \exists i \ (x_i, y_i) \in E(D)\}.$$

Definition 10 ([10]) *The* Sperner capacity *of a digraph* D *is defined as*

$$\Sigma(D) = \limsup_{n\to\infty} \frac{1}{n} \log \omega_s(D^n).$$

If $\mathcal{D} = \{D_1, \ldots, D_k\}$ *is a family of digraphs on the same (finite) vertex set* V*, then the* Sperner capacity *of this family is defined as*

$$\Sigma(\mathcal{D}) = \limsup_{n\to\infty} \frac{1}{n} \log \omega_s(\cap_{D_i\in\mathcal{D}} D_i^n),$$

where $\cap_{D_i\in\mathcal{D}} D_i^n$ *denotes the graph on vertex set* V^n *with edge set* $\cap_{D_i\in\mathcal{D}} E(D_i^n)$*.*

Csiszár and Körner [8] introduced a "within a fixed type" version of Shannon capacity, which has a natural and straightforward extension for Sperner capacity. To introduce this notion we need the concept of *types*.

Definition 11 *The* type *of a sequence* $\boldsymbol{x} \in V^n$ *is the probability distribution* $P_{\boldsymbol{x}}$ *on* V *defined by*

$$P_{\boldsymbol{x}}(a) = \frac{|\{i : x_i = a\}|}{n}, \text{ for all } a \in V.$$

For a fixed distribution P *on* V *and* $\varepsilon > 0$*, we say that* $\boldsymbol{x} \in V^n$ *is* (P, ε)-typical *if, for all* $a \in V$*, we have* $|P_{\boldsymbol{x}}(a) - P(a)| < \varepsilon$*.*

Definition 12 (cf. [8]) *The* Sperner capacity within type P *of a (finite) family* $\mathcal{D}$ *of (finite) digraphs on the common vertex set* V *is*

$$\Sigma(\mathcal{D}, P) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \omega_s(\cap_{D\in\mathcal{D}}(D^n(P, \varepsilon))),$$

where $D^n(P, \varepsilon)$ *denotes the digraph induced by* D^n *on the* (P, ε)*-typical sequences in* V^n*. We write* $\Sigma(D, P)$ *for* $\Sigma(\mathcal{D}, P)$ *if* $\mathcal{D} = \{D\}$*.*

The main result in [11] is the following statement.

Theorem 13 ([11]) *For any two (finite) families of (finite) digraphs $\mathcal{C}$ and $\mathcal{D}$ on the same common vertex set V, we have*

$$\Sigma(\mathcal{C} \cup \mathcal{D}, P) = \min\{\Sigma(\mathcal{C}, P), \Sigma(\mathcal{D}, P)\}.$$

Since the number of different types is only polynomial in n (cf. Lemma 2.2 in [9]), this immediately implies its main corollary.

Corollary 14 ([11]) *For any (finite) family of (finite) digraphs $\mathcal{D}$ on a common vertex set, we have*

$$\Sigma(\mathcal{D}) = \max_P \min_{D \in \mathcal{D}} \Sigma(D, P).$$

This theorem is sometimes referred to informally as the Bottleneck Theorem. This result was the key in the solution of several extremal set theoretic problems, including a longstanding open problem by Rényi on the maximum possible number of pairwise so-called qualitatively 2-independent partitions of an n-element set, cf. [11]. It also has non-trivial consequences in information theory, see [7, 11, 19, 23] for examples of the latter.

Note that, if the family $\mathcal{D}$ consists of all possible orientations of the same undirected graph G, then Theorem 13 states that, within any type P, the Sperner capacity of the family $\mathcal{D}$ is the same as that of the most restrictive single digraph (called the *bottleneck*) in the family. Note also that, for such a family $\mathcal{D}$, if $(\boldsymbol{x}, \boldsymbol{y})$ is an edge of $\cap_{D_i \in \mathcal{D}} D_i$, then there are coordinates i and j and an edge $\{a, b\} \in E(G)$ such that $(x_i, y_i) = (y_j, x_j) = (a, b)$. This follows analogously to the similar statement for permutation capacities that we described right before the introduction of robust capacity in Definition 4.

In view of the obvious analogy between Sperner capacity and the notions investigated in this paper, one naturally asks whether a statement analogous to the consequence of Theorem 13 described in the previous paragraph is true in the present context. Note that when dealing with permutation capacities, we always have sequences "of the same type" as every natural number appears now exactly once in each sequence. In fact, the main role of types in the proof of Theorem 13 is that the elements of any such sequence can be permuted so that we get an arbitrarily chosen other sequence of the same type. This property also holds for our current sequences representing permutations. Therefore, the methods of [11] can be used, but there are serious limitations due to the fact that, in the present context, we are dealing with infinite families of digraphs. Corollary 7 of the previous section indicates that these limitations are essential, as they lead to the nonexistence of a bottleneck theorem here.

If we consider only finitely many orientations of L, then the methods of [11] seem to work. By this we mean that defining, for every $\mathcal{F} \subseteq \mathcal{L}$, the quantity $R(\mathcal{F}) := \limsup_{n\to\infty} \frac{1}{n} \log \omega_s(\cap_{\vec{L} \in \mathcal{F}} \Gamma_{\vec{L}}(n))$, which is the asymptotic exponent of the maximum size of a set of permutations that are pairwise $\vec{L}$-different simultanously for all $\vec{L} \in \mathcal{F}$ (so, in particular, $R(\mathcal{L}) = RR(L)$) we have $R(\mathcal{F}) \geq R_{\min}(L)$ whenever $\mathcal{F}$ is finite. This

statement is somewhat weaker than the more direct analogue of Corollary 14 stating that $R(\mathcal{F}) = \min_{\vec{L}\in\mathcal{F}} R(\vec{L})$, which is perhaps also true; however, it already shows that the main reason for a different behavior in the present case is that the digraph family we consider here has infinitely many elements.

5 Further lower bounds

In this section we improve upon the lower bound proven in Theorem 1, namely we prove the following.

Theorem 15 *Let $\gamma \approx 1.647$ be the largest root of the polynomial $x^4 - x^2 - x - 3$. Then*

$$R_{\min}(L) \geq \log \gamma \approx 0.7198.$$

We know by Lemma 2 that it is enough to give lower bounds on $t_{\mathcal{L}}(n)$. Here and in the sequel we will use the following notation. For $k < n$ positive integers, an n-length sequence containing each of the numbers $1, \ldots, k$ exactly once, and with a $*$ at the remaining $n-k$ positions, stands for a permutation of $[n]$ in which the place of the first k natural numbers is already fixed while the $*$'s can be substituted by $k+1, \ldots, n$ in an arbitrary manner (provided that the resulting sequence is a permutation of the elements of $[n]$).

We will also use the notation $\vec{L}_{(j)}$ for the orientation of the semi-infinite path L obtained from a given orientation $\vec{L}$ of L by deleting its first $j-1$ vertices, i.e., j will be its "starting" vertex. Accordingly, just as before, the vertices of $\Gamma_{\vec{L}_{(j)}}(n)$ are the permutations of the numbers $j, j+1, \ldots, j+n-1$, while adjacency is defined analogously as in $\Gamma_{\vec{L}}(n)$.

We prove the following lemma.

Lemma 16 *We have*

$$t_{\mathcal{L}}(n) \geq g_n,$$

where g_n is the sequence defined by: $g_n = F_{n+1}$ for $n \leq 5$, and $g_n = g_{n-2} + g_{n-3} + 3g_{n-4}$ for $n \geq 6$.

Proof. For $n \leq 5$ the statement follows from Lemma 3. Let us fix an arbitrary orientation $\vec{L}$ of L. For $n \geq 6$ we consider three cases according to how the first three edges of L are oriented.

Case 1:

If both vertices 2 and 3 have equal outdegree and indegree (that is all of the first three edges are oriented towards their larger, or all of them towards their smaller, endpoint), then the following permutations form a transitive clique in $\Gamma_{\vec{L}}(n)$. (According to the actual directions, the first sequence is the source or the sink in that transitive clique.)

132**...*
21***...*
3412*...*
3421*...*
4231*...*

(Note that the elements of the fourth column have no role in forming this transitive clique.)

Here the first sequence contains $n-3$ $*$'s, the second $n-2$, and the three others $n-4$. By the induction hypothesis, there exists a transitive tournament of size g_{n-4} in $\Gamma_{\vec{L}_{(5)}}(n-4)$: take any such transitive tournament, and substitute each of its vertices into (the stars of) a different copy of each of the last three sequences. Do the same with a transitive clique of size g_{n-3} in $\Gamma_{\vec{L}_{(4)}}(n-3)$ for the first sequence and with a transitive clique of size g_{n-2} in $\Gamma_{\vec{L}_{(3)}}(n-2)$ for the second sequence. It is now easy to see that the resulting $g_{n-2}+g_{n-3}+3g_{n-4}$ permutations of $[n]$ form a transitive tournament in $\Gamma_{\vec{L}}(n)$.

Case 2:

If one of the two vertices 2 and 3 has outdegree 0 while the other has outdegree 2 (that is, the directions of the first three edges in $\vec{L}$ "alternate"), then the same sequences as above form again a transitive tournament in $\Gamma_{\vec{L}}(n)$, except that their ordering is different. In the scheme below, either all edges go "downwards" or all go "upwards", depending on the direction of the first edge of the path:

132**...*
3412*...*
3421*...*
21***...*
4231*...*

The argument is completed in the same way as in Case 1.

Case 3:

If we are neither in Case 1 or Case 2, then we may assume without loss of generality that 2 has outdegree 0 and 3 has outdegree 1, i.e., that $(1,2),(3,2),(4,3) \in E(\vec{L})$: all other cases not covered so far are equivalent to this one, so the following construction can be modified accordingly. The following scheme gives a transitive tournament in $\Gamma_{\vec{L}}(n)$:

132*4*...*
1234**...*
12*34*...*
1*2*3*...*
21****...*

Once again the argument is completed in the same way as in Case 1.

This concludes the proof of the lemma. □

Proof of Theorem 15. It is immediate from Lemma 16 that $R_{\min}(L) \geq \limsup_{n\to\infty} \frac{1}{n} \log g(n)$ where the right hand side is equal to γ by virtue of the recursion satisfied by the sequence g_n. □

For the special orientations of L where all vertices except 1 have equal outdegree and indegree (there are two such orientations that are equivalent for our purposes), we have a slightly better lower bound. The oriented L in which all edges are oriented towards their larger endpoint will be referred to as the "thrupath". The following proposition for this orientation is clearly valid also for its reverse.

Proposition 17 *Let L_t denote the thrupath. We have*

$$R(L_t) \geq \log \gamma' \approx 0.7413,$$

where γ' is the largest root of the polynomial $x^3 - x - 3$.

Proof. The proof goes along the same lines as the proof of Theorem 15 after realizing that the following permutations form a transitive clique for the thrupath.

2**1
321*
3*21
13*2

□

One of the most interesting open problems concerning Sperner capacity is whether every graph has an orientation, the Sperner capacity of which achieves the Shannon capacity of the underlying undirected graph which is simply the Sperner capacity of the symmetrically directed equivalent. (This question is explored in [20], where a positive answer was proven for a non-trivial special case. The same question is also treated in [12].)

The analogous question for us here is whether the permutation capacity of the undirected semi-infinite path L can be achieved as the permutation capacity of one of its orientations. Needless to say, we do not know the answer, as our best upper bound on $R(\vec{L})$ for any orientation $\vec{L}$ of L is just the trivial value 1. From the other side, Proposition 17 gives the best lower bound we know on any single orientation of L. For L itself, the best lower bound published so far is the one in [14] having value $\frac{1}{4} \log 10 \approx 0.83048$. Next we improve on this lower bound. (Unfortunately, the construction contained in Proposition 18 below is not very aesthetic. We supply a slightly weaker, but more appealing, construction in the remark following this proposition.)

Proposition 18 *The maximum number of pairwise L-different permutations $T(n)$ satisfies*

$$T(n) \geq 5T(n-4) + 9T(n-5) + 3T(n-6)$$

implying

$$\rho(L) \geq 0.8604.$$

Proof.

The value 0.8604 is an approximation of the logarithm of the largest root of the characteristic equation of the recurrence relation above, so it is enough to prove the validity of this recurrence relation.

This is done along similar lines to those in the proof of Theorem 15 by verifying that the following permutations are pairwise L-different (*colliding* in the terminology of [13]).

52314***
5*2314**
54*231**
514*23**
5314*2**
53241***
5*3241**
51*324**
541*32**
*241*3**
4**23*1*
43**2*1*
4**132**
43***12*
623*4*15
643**125
6251*3*4

□

Remark 1 The following construction is perhaps somewhat nicer than the one in Proposition 18. Consider the 14 cyclic permutations of the following two 7-length sequences:

1342***
35214**

It is straightforward to check that these 14 permutations are pairwise colliding and thus prove the validity of the recursive lower bound

$$T(n) \geq 7[T(n-4) + T(n-5)].$$

This implies $\rho(L) \geq 0.8599$. ◊

6 Finite graphs and digraphs

The paper [14] investigated the maximum number of pairwise G-different permutations of $[n]$ for finite graphs G with vertex set $[m]$, $m \leq n$. It was observed that, for a fixed

finite graph G, this number is constant if n is large enough. This eventual constant value $\kappa(G)$ was introduced as a new graph invariant: it is straightforward to note that $\kappa(G)$ does not depend on the actual labelling of the vertices of G by natural numbers. This invariant seems to be quite difficult to determine even for relatively small graphs, and the only infinite family of graphs for which we could determine the value of $\kappa(G)$ was that of the stars $K_{1,r}$.

Interestingly, we can say just a little more in the case of digraphs. As for undirected graphs, if D is a finite digraph, then the maximum number of pairwise D-different permutations of $[n]$ will also be a constant – which we denote $\kappa_d(D)$ – for large enough n. This immediately follows from the corresponding statement for undirected graphs, since $\kappa_d(D)$ is clearly bounded above by $\kappa(G)$, where G is the underlying undirected graph of D. While the value of $\kappa(G)$ is not known in general for complete bipartite graphs G, the directed parameter is, at least in the case of the most natural special orientation. The key to this is the simple observation that the answer is just a reincarnation of a well-known theorem of Bollobás.

We denote by $\binom{[n]}{r}$ the set of r-element subsets of n.

Theorem 19 ([5]) *Suppose that $A_1, \ldots, A_k \subseteq \binom{[n]}{p}$ and $B_1, \ldots, B_k \subseteq \binom{[n]}{q}$ are such that, for all i, $A_i \cap B_i = \emptyset$, while, for all $i \neq j$, $A_i \cap B_j \neq \emptyset$. Then*

$$k \leq \binom{p+q}{p}.$$

The bound in Theorem 19 is sharp: consider the sets in $\binom{[p+q]}{p}$ as the A_i's and let $B_i = [p+q] \setminus A_i$.

Corollary 20 *Let $\vec{K}_{p,q}$ denote the oriented complete bipartite graph with all edges having their heads in the q-element partition class. Then*

$$\kappa_d(\vec{K}_{p,q}) = \binom{p+q}{q}.$$

Proof. Let the two partition classes of $\vec{K}_{p,q}$ be A and B and consider a set M of pairwise $\vec{K}_{p,q}$-different permutations of $[n]$. For a permutation $\sigma \in M$, associate $A_\sigma := \{i : \sigma(i) \in A\}$ and $B_\sigma := \{i : \sigma(i) \in B\}$. It is easy to see that the system of set pairs $\{(A_\sigma, B_\sigma)\}_{\sigma \in M}$ satisfies the conditions in Theorem 19, and therefore we have $M \leq \binom{p+q}{p}$.

To prove that this upper bound is attainable, we assume without loss of generality that the vertices in A are labelled by $1, \ldots, p$ and those in B by $p+1, \ldots, p+q$. Take all possible p-element subsets of $[p+q]$ and, for each such subset S, take any permutation that puts the elements of A in the positions in S, and the elements of B in the positions of $[p+q] \setminus S$. It is easy to see that these $\binom{p+q}{q}$ permutations are pairwise $\vec{K}_{p,q}$-different. □

Remark 2 The undirected invariant $\kappa(K_{p,q})$ has a very similar "translation" to a problem in extremal set theory. Namely, it is the maximum possible m for which set pairs $\{(A_i, B_i) : |A_i| = p, |B_i| = q\}_{i=1}^m$ can be given with the property that, for all i, $A_i \cap B_i = \emptyset$, while for all $i \neq j$, $A_i \cap B_j \neq \emptyset$ *or* $A_j \cap B_i \neq \emptyset$. This problem was considered by Tuza in [26], where it is solved in the case when p or q is equal to 1. The result in [14] for $\kappa(K_{1,r})$ translates to this solution. As far as we know, the problem is unsolved for all other pairs of values p and q. $\Diamond$

It is observed in [14] that, if G is a finite graph with vertex disjoint subgraphs $G_1, \ldots G_s$ then $\kappa(G) \geq \prod_{i=1}^s \kappa(G_i)$. The proof of this result carries over immediately to the digraph parameter κ_d.

In particular, if the graph G is the disjoint union of components $G_1, \ldots, G_s$, then we have $\kappa(G) \geq \prod_{i=1}^s \kappa(G_i)$. In the undirected case, we know of no examples where we have strict inequality. For digraphs, however, the inequality can be strict. For example, let D_1 be the digraph on $\{1, 2, 3\}$ with directed edges $(1, 2)$ and $(2, 3)$: it is easy to check that $\kappa_d(D_1) = 2$. Now let D_2 be a copy of the same digraph on vertex set $\{4, 5, 6\}$, with directed edges $(4, 5)$ and $(5, 6)$. The following is a collection of eight $(D_1 \cup D_2)$-different permutations:

32*1456** ... *
32*154*6* ... *
231*456** ... *
231*54*6* ... *
*321*465* ... *
*3214*56* ... *
3*12*465* ... *
3*124*56* ... *

Here, the graph $D_1 \cup D_2$ is to be regarded as being a graph on $[n]$, for $n \geq 8$, and the $*$'s represent the natural numbers $7, \ldots, n$, in arbitrary order.

Thus we have $\kappa_d(D_1 \cup D_2) \geq 8 > 4 = \kappa_d(D_1)\kappa_d(D_2)$.

Returning to the undirected case, it seems even to be difficult to find $\kappa(tK_2)$, where tK_2 is the union of t disjoint edges: it is conjectured that the lower bound $\kappa(tK_2) \geq 3^t$ is tight in this case, and an upper bound of 4^t was given in [14].

Even checking that $\kappa(2K_2) = 9$ takes some work: we give a brief sketch of an argument. Let $\{1, 2\}$ and $\{3, 4\}$ be the two edges of $2K_2$, and let C be a set of $(2K_2)$-different permutations. First, assume that there are three permutations in C with, say, a 1 in the first position. By a case analysis involving how many different positions are occupied by the 2's in these three permutations, it can be shown that $|C| \leq 9$. On the other hand, if there is no instance of three permutations in C with the same element in the same position, then any element of C is adjacent to at most 8 others in C – two via each of the four positions where 1,2,3,4 occur – and so again $|C| \leq 9$. It is possible to use this result to improve the upper bound $\kappa(tK_2) \leq 4^t$ slightly, but not by an exponential factor.

Let $t\vec{K}_2$ be the disjoint union of t directed edges. It seems likely that $\kappa_d(t\vec{K}_2) = \kappa_d(\vec{K}_2)^t = 2^t$, but again there seems to be no immediate proof.

At the other extreme, the problem of finding κ_d for oriented complete graphs, e.g., those of transitive tournaments, is as open as for their undirected counterparts, i.e., the determination of the values $\kappa(K_r)$, cf. [14]. We do not know even whether $\hat{\kappa}(K_r) := \lim_{n\to\infty} NN(K_r)$ is superexponential in r.

7 Open problems

We conclude by collecting some of the open problems, some already mentioned, that are related to the topic of the present paper.

Problem 1: What is the value of $RR(L)$? In particular, is it equal to $\frac{1}{2}$?

Problem 2: Is $R_{\max}(L) > R_{\min}(L)$, i.e., are there two different orientations L_1 and L_2 of the semi-infinite path L for which $R(L_1) \neq R(L_2)$? Is $R_{\max}(L)$, or even $R_{\min}(L)$, equal to 1?

If $R_{\max}(L) = 1$, then that immediately solves the next problem. However, in case of a negative answer, the problem is still interesting.

Problem 3: Is $R_{\max}(L)$ equal to $\rho(L)$?

We repeat the asymptotic version of the conjecture by Körner and Malvenuto.

Problem 4: Is $\rho(L)$ equal to 1?

Finally, we put here again the problems mentioned at the end of the previous section.

Problem 5: Is $\kappa_d(t\vec{K}_2)$ equal to 2^t?

Problem 6: Is $\hat{\kappa}(K_r)$ superexponential in r?

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