LARGE DEVIATIONS FOR FUNCTIONS OF TWO RANDOM PROJECTION MATRICES

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Abstract. In this paper two independent and unitarily invariant projection matrices $P(N)$ and $Q(N)$ are considered and the large deviation is proven for the eigenvalue density of all polynomials of them as the matrix size N converges to infinity. The result is formulated on the tracial state space $TS(\mathcal{A})$ of the universal C^*-algebra $\mathcal{A}$ generated by two selfadjoint projections. The random pair $(P(N), Q(N))$ determines a random tracial state $\tau_N \in TS(\mathcal{A})$ and τ_N satisfies the large deviation. The rate function is in close connection with Voiculescu's free entropy defined for pairs of projections.

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Introduction

Large deviation results for the empirical eigenvalue density of random matrices started with the paper of Ben Arous and Guionnet [2] in which generalized Wigner theorem concerning Gaussian symmetric (or selfadjoint) matrices was proven. The paper was followed by large deviation results for several other kind of random matrices (as Wishart, etc); see the monograph [9] for a detailed discussion and the survey [8] for more recent developments.

Up to now the typical large deviation results on random matrices have dealt with the empirical eigenvalue density of a certain sequence of matrices; occasionally these matrices were algebraically expressed from two (as in [12]). In this paper two independent projection matrices are considered and the large deviation is proven for all polynomials (even for more general functions) of them. More precisely, the main result is a C^*-algebraic formulation of large deviations for the sequence of two random selfadjoint projection matrices $P(N)$ and $Q(N)$ having independent and unitarily invariant distribution provided moreover $\alpha := \lim_N \mathrm{rank}(P(N))/N$ and $\beta := \lim_N \mathrm{rank}(Q(N))/N$ exist. The main theorem is formulated on the tracial state space $TS(\mathcal{A})$ of the universal C^*-algebra $\mathcal{A} := C^*(\mathbb{Z} \star \mathbb{Z})$ generated by two selfadjoint projections e and f. The random pair $(P(N), Q(N))$ determines a random tracial state $\tau_N \in TS(\mathcal{A})$ as follows:

$$\tau_N(h) = \frac{1}{N}\mathrm{Tr}(\psi(h)), \qquad h \in \mathcal{A},$$

where $\psi : \mathcal{A} \to M_N(\mathbb{C})$ is the unique $*$-homomorphism such that $\psi(e) = P(N)$ and $\psi(f) = Q(N)$. The random τ_N induces a measure ν_N on $TS(\mathcal{A})$ and the sequence ν_N satisfies the large deviation principle in the scale $1/N^2$ with a rate function $\mathcal{I} : TS(\mathcal{A}) \to [0, \infty]$ in the ordinary sense. It is very remarkable that the rate function $\mathcal{I}$ is in close relation with

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Voiculescu's free entropy $\chi(p,q)$ defined for a pair of projections in a W^*-probability space. Namely, the GNS-construction from $(\mathcal{A},\tau)$ yields a W^*-probability space $(\pi_\tau(\mathcal{A})'',\tilde{\tau})$ and for the projections $p=\pi_\tau(e)$ and $q=\pi_\tau(f)$, we have $\mathcal{I}(\tau)=-\chi(p,q)$.

The result includes a bunch of traditional large deviation results for the eigenvalue density of different polynomials of $P(N)$ and $Q(N)$. The corresponding rate function can be obtained from $\mathcal{I}$ by the contraction principle and computed explicitly in some examples as $P(N)Q(N)+Q(N)P(N)$ and $aP(N)+bQ(N)$.

The paper is organized as follows. First we establish a large deviation theorem for the empirical eigenvalue density of the random matrix $P(N)Q(N)P(N)$. This result is obtained via the joint eigenvalue density and the cases $\alpha+\beta\le 1$ and ≥ 1 are somewhat separated but treated parallel. A few facts about the Jacobi ensemble are used here. Since polynomials of two projections are easily controlled by the powers of $P(N)Q(N)P(N)$, we can move to the C*-algebraic formulation mentioned above. The tracial state space $TS(\mathcal{A})$ has a convenient representation in terms of four numbers and a measure on $(0,1)$. The large deviation theorem or more precisely the rate function is first identified in terms of the representation of tracial states and the description à la Voiculescu comes afterwards. The last section is the application of the contraction principle and contains very concrete computations.

1. Joint distribution of two projections

Let $M_N(\mathbb{C})$ be the algebra of $N\times N$ complex matrices. By an $N\times N$ *random projection matrix* P we always mean a random orthogonal (or selfadjoint) projection matrix, and the *unitary invariance* of P means that the distribution of VPV^* is equal to that of P for any unitary $V\in M_N(\mathbb{C})$.

The aim of this section is to analyze the joint distribution of two independent and unitarily invariant random projection matrices P,Q in $M_N(\mathbb{C})$, when their ranks $\mathrm{rank}(P)=k$ and $\mathrm{rank}(Q)=l$ are fixed; we may assume that $0\le k\le l\le N$. Throughout this section, we keep these assumptions on P and Q.

The joint eigenvalue distribution of PQP is related to the Jacobi ensemble. Let (A,B) be an independent pair of $N\times N$ complex *Wishart matrices* of p degrees of freedom and of q degrees of freedom, respectively, that is, $A=YY^*$ and $B=ZZ^*$ with complex $N\times p$ and $N\times q$ random matrices Y and Z such that $\mathrm{Re}\,Y_{ij}$, $\mathrm{Im}\,Y_{ij}$, $\mathrm{Re}\,Z_{ij}$ and $\mathrm{Im}\,Z_{ij}$ are independent standard Gaussians. Assume here that $p,q\ge N$. Then the random positive semidefinite matrix

$$(A+B)^{-1/2}A(A+B)^{-1/2}$$

is called an $N\times N$ *Jacobi ensemble* of parameter $(p-N,q-N)$. It has the probability distribution

$$\text{Constant}\times\mathrm{Det}(X)^{p-N}\mathrm{Det}(I-X)^{q-N}\mathbf{1}_{\{0\le X\le I\}}(X)\,dX, \tag{1.1}$$

(on the space of $N\times N$ selfadjoint matrices, see [4, Lemma 2.1]), where $\mathbf{1}_{\{0\le X\le I\}}$ denotes the characteristic function of $\{X\in M_N(\mathbb{C}):0\le X\le I\}$. The density formula (1.1) implies the joint distribution of the eigenvalues

$$\text{Constant}\times\prod_{i=1}^N x_i^{p-N}(1-x_i)^{q-N}\prod_{1\le i<j\le N}(x_i-x_j)^2\prod_{i=1}^N\mathbf{1}_{[0,1]}(x_i)\,dx_i,$$

see also [5] or [7, Chapter 2].

The next lemma is from [4, Theorem 2.2].

Lemma 1.1. *Assume that $k+l \le N$. Then PQP, when considered as a random matrix in $M_k(\mathbb{C}) = PM_N(\mathbb{C})P$, has the distribution of a Jacobi ensemble of parameter $(l-k, N-k-l)$. Hence, the joint eigenvalue distribution of the nonzero eigenvalues of PQP is given by*

$$\frac{1}{Z_{N,k,l}} \prod_{i=1}^{k} x_i^{l-k}(1-x_i)^{N-k-l} \prod_{1\le i<j\le k} (x_i-x_j)^2 \prod_{i=1}^{k} \mathbf{1}_{[0,1]}(x_i)\, dx_i \tag{1.2}$$

with a normalization constant $Z_{N,k,l}$.

Let (A,B) and (A',B') be pairs of selfadjoint $N\times N$ random matrices. We say that they have the same *joint distribution* if

$$\mathrm{tr}_N(h(A,B)) = \mathrm{tr}_N(h(A',B')) \quad \text{almost surely}$$

for any polynomial h of two non-commuting variables, where tr_N denotes the normalized trace on $M_N(\mathbb{C})$.

Our strategy is to modify the pair (P,Q) of projections in such a way that they are easy to be handled but their joint distribution does not change. As the first step, we may assume that (P,Q) are of the forms

$$P = I_k \oplus 0_{N-k}, \quad Q = U(I_l \oplus 0_{N-l})U^*,$$

where $I_k \oplus 0_{N-k}$ denotes the diagonal matrix whose k first diagonal entries are 1 and the remaining are 0, and U is an $N\times N$ Haar-distributed random unitary matrix. In this way, randomness belongs to only Q, while P is a constant projection matrix.

Proposition 1.2.

(a) *If $k+l \le N$, then the joint distribution of (P,Q) coincides with that of the pair*

$$P \quad \text{and} \quad \begin{bmatrix} X & \sqrt{X(I_k-X)} & 0 & 0 \\ \sqrt{X(I_k-X)} & I_k - X & 0 & 0 \\ 0 & 0 & I_{l-k} & 0 \\ 0 & 0 & 0 & 0_{N-k-l} \end{bmatrix},$$

where $X := \mathrm{Diag}(x_1,\dots,x_k)$ and $(x_1,\dots,x_k)\in[0,1]^k$ is distributed under the distribution (1.2).

(b) *If $k+l > N$, then the joint distribution of (P,Q) coincides with that of the pair*

$$P \quad \text{and} \quad \begin{bmatrix} I_{k+l-N} & 0 & 0 & 0 \\ 0 & X & \sqrt{X(I_{N-l}-X)} & 0 \\ 0 & \sqrt{X(I_{N-l}-X)} & I_{N-l}-X & 0 \\ 0 & 0 & 0 & I_{l-k} \end{bmatrix},$$

where $X := \mathrm{Diag}(x_1,\dots,x_{N-l})$ and $(x_1,\dots,x_{N-l})$ in $[0,1]^{N-l}$ is distributed under

$$\frac{1}{Z_{N,k,l}} \prod_{i=1}^{N-l} x_i^{l-k}(1-x_i)^{k+l-N} \prod_{1\le i<j\le N-l} (x_i-x_j)^2 \prod_{i=1}^{N-l} \mathbf{1}_{[0,1]}(x_i)\, dx_i. \tag{1.3}$$

Proof. (a) Assume $k+l \le N$. By the structure theorem of two projections (see [14, pp. 306–308]), after a (random) unitary conjugation, (P,Q) can represented as

$$\begin{aligned} P &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \oplus I \oplus I \oplus 0 \oplus 0, \\ Q &= \begin{bmatrix} X & \sqrt{X(I-X)} \\ \sqrt{X(I-X)} & I-X \end{bmatrix} \oplus I \oplus 0 \oplus I \oplus 0, \end{aligned}$$

where $0 \le X \le I$ with $\ker X = \{0\}$ and $\ker(I-X) = \{0\}$ on $\mathcal{H}_0$, under a decomposition

$$\mathbb{C}^N = (\mathcal{H}_0 \otimes \mathbb{C}^2) \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4.$$

(Note that $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ are the ranges of $P \wedge Q$, $P \wedge Q^\perp$, $P^\perp \wedge Q$ and $(P \vee Q)^\perp$, respectively, and some of them may be zero spaces.) Since $PQP|_{P\mathbb{C}^N}$ is $X \oplus I \oplus 0$ on $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, it follows from Lemma 1.1 that $\mathcal{H}_1$ and $\mathcal{H}_2$ are zero spaces almost surely. This shows that there exists an $N \times N$ random unitary matrix V such that

$$\begin{aligned} VPV^* &= P = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \oplus 0_{l-k} \oplus 0_{N-k-l}, \\ VQV^* &= \begin{bmatrix} X & \sqrt{X(I_k-X)} \\ \sqrt{X(I_k-X)} & I_k-X \end{bmatrix} \oplus I_{l-k} \oplus 0_{N-k-l}, \end{aligned}$$

where $X = \mathrm{Diag}(x_1,\dots,x_k)$ and $(x_1,\dots,x_k) \in [0,1]^k$ is distributed under (1.2) by Lemma 1.1. Hence we have the desired conclusion.

(b) Next, assume $k+l > N$; then since $N-l < k$ and $(N-l)+k \le N$, one can apply the above case (a) to $(I-Q,P)$ instead of (P,Q). Thus, the joint distribution of $(I-Q,P)$ is equal almost surely to that of the pair

$$\begin{bmatrix} I_{N-l} & 0 \\ 0 & 0 \end{bmatrix} \oplus 0_{k+l-N} \oplus 0_{l-k}$$

and

$$\begin{bmatrix} X & \sqrt{X(I_{N-l}-X)} \\ \sqrt{X(I_{N-l}-X)} & I_{N-l}-X \end{bmatrix} \oplus I_{n+m-N} \oplus 0_{m-n}$$

so that (P,Q) has the same joint distribution almost surely as the pair

$$\begin{bmatrix} X & \sqrt{X(I_{N-l}-X)} \\ \sqrt{X(I_{N-l}-X)} & I_{N-l}-X \end{bmatrix} \oplus I_{k+l-N} \oplus 0_{l-k}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & I_{N-l} \end{bmatrix} \oplus I_{k+l-N} \oplus I_{l-k}.$$

Here, $X = \mathrm{Diag}(x_1,\dots,x_{N-l})$ and $(x_1,\dots,x_{N-l}) \in [0,1]^{N-l}$ is distributed under

$$\frac{1}{Z_{N,N-l,k}} \prod_{i=1}^{N-l} x_i^{k+l-N}(1-x_i)^{l-k} \prod_{1\le i<j\le N-l} (x_i-x_j)^2 \prod_{i=1}^{N-l} \mathbf{1}_{[0,1]}(x_i)\,dx_i. \tag{1.4}$$

Since $\begin{bmatrix} X & \sqrt{X(I_{N-l}-X)} \\ \sqrt{X(I_{N-l}-X)} & I_{N-l}-X \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & I_{N-l} \end{bmatrix}$ are respectively transformed into $\begin{bmatrix} I_{N-l} & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} I_{N-l}-X & \sqrt{X(I_{N-l}-X)} \\ \sqrt{X(I_{N-l}-X)} & X \end{bmatrix}$ by a conjugation by the unitary matrix

$\begin{bmatrix} \sqrt{X} & \sqrt{I_{N-l}-X} \\ -\sqrt{I_{N-l}-X} & \sqrt{X} \end{bmatrix}$, the conclusion follows after the coordinate change $X \mapsto I_{N-l} - X$ so that (1.4) is transformed into (1.3). □

From Proposition 1.2 we can readily obtain joint eigenvalue distributions of some polynomials of P and Q. For example, we have:

Corollary 1.3.

(i-a) *When $k+l \le N$, the eigenvalues of PQP (or PQ) are given as*

$$\underbrace{0,\dots,0}_{N-k\ \text{times}}, x_1,\dots,x_k$$

and the joint distribution of $(x_1,\dots,x_k)$ is (1.2).

(i-b) *When $k+l > N$, the eigenvalues of PQP (or PQ) are given as*

$$\underbrace{0,\dots,0}_{N-k\ \text{times}}, \underbrace{1,\dots,1}_{k+l-N\ \text{times}}, x_1,\dots,x_{N-l},$$

and the joint distribution of $(x_1,\dots,x_{N-l})$ is (1.3).

(ii-a) *When $k+l \le N$, the eigenvalues of $PQ+QP$ are given as*

$$\underbrace{0,\dots,0}_{N-2k\ \text{times}}, x_1 \pm \sqrt{x_1},\dots,x_k \pm \sqrt{x_k}$$

and the joint distribution of $(x_1,\dots,x_k)$ is (1.2).

(ii-b) *When $k+l > N$, the eigenvalues of $PQ+QP$ are given as*

$$\underbrace{0,\dots,0}_{l-k\ \text{times}}, \underbrace{2,\dots,2}_{k+l-N\ \text{times}}, x_1 \pm \sqrt{x_1},\dots,x_{N-l} \pm \sqrt{x_{N-l}},$$

and the joint distribution of $(x_1,\dots,x_{N-l})$ is (1.3).

(iii-a) *When $k+l \le N$ and $a,b \in \mathbb{R}\setminus\{0\}$, the eigenvalues of $aP+bQ$ are given as*

$$\underbrace{0,\dots,0}_{N-2k\ \text{times}}, \underbrace{b,\dots,b}_{l-k\ \text{times}}, x_1,\dots,x_k, a+b-x_1,\dots,a+b-x_k,$$

and the joint distribution of $(x_1,\dots,x_k)$ is

$$\frac{2^k}{|ab|^{k(N-k)} Z_{N,k,l}} \prod_{i=1}^k \left| x_i - \frac{a+b}{2} \right| \bigl|(x_i-a)(x_i-b)\bigr|^{l-k} \bigl|x_i(a+b-x_i)\bigr|^{N-k-l}$$
$$\times \prod_{1\le i<j\le k} (x_i-x_j)^2 (a+b-x_i-x_j)^2 \prod_{i=1}^k \mathbf{1}_{[A,B]}(x_i)\,dx_i, \tag{1.5}$$

where $Z_{N,k,l}$ is the normalization constant in (1.2) *and A,B are the first two smallest numbers of $0,a,b,a+b$.*

(iii-b) *When $k+l > N$ and $a,b \in \mathbb{R}\setminus\{0\}$, the eigenvalues of $aP+bQ$ are given as*

$$\underbrace{b,\dots,b}_{l-k\ \text{times}}, \underbrace{a+b,\dots,a+b}_{k+l-N\ \text{times}}, x_1,\dots,x_{N-l}, a+b-x_1,\dots,a+b-x_{N-l},$$

and the joint distribution of $(x_1,\dots,x_{N-l})$ *is*

$$\frac{2^{N-l}}{|ab|^{l(N-l)}Z_{N,k,l}}\prod_{i=1}^{N-l}\left|x_i-\frac{a+b}{2}\right|\bigl|(x_i-a)(x_i-b)\bigr|^{l-k}\bigl|x_i(a+b-x_i)\bigr|^{k+l-N}$$

$$\times\prod_{1\le i<j\le N-l}(x_i-x_j)^2(a+b-x_i-x_j)^2\prod_{i=1}^{N-l}\mathbf{1}_{[A,B]}(x_i)\,dx_i, \tag{1.6}$$

where $Z_{N,k,l}$ *is the normalization constant in* (1.3) *and* A,B *are as in* (iii-a).

Proof. (i-a) is Lemma 1.1 and (i-b) is immediate from Proposition 1.2 (b).

(ii-a) By Proposition 1.2 (a) we may assume that

$$PQ+QP=\begin{bmatrix}2X & \sqrt{X(I_k-X)}\\ \sqrt{X(I_k-X)} & 0\end{bmatrix}\oplus 0_{N-2k},$$

where X is as in Proposition 1.2 (a). Then the result immediately follows because the eigenvalues of the 2×2 matrix $\begin{bmatrix}2x & \sqrt{x(1-x)}\\ \sqrt{x(1-x)} & 0\end{bmatrix}$ for $0\le x\le 1$ are $x\pm\sqrt{x}$. The proof of (ii-b) is similar by Proposition 1.2 (b).

(iii-a) By Proposition 1.2 (a) we may assume that

$$aP+bQ=\begin{bmatrix}aI_k+bX & b\sqrt{X(I_k-X)} & 0 & 0\\ b\sqrt{X(I_k-X)} & b(I_k-X) & 0 & 0\\ 0 & 0 & bI_{l-k} & 0\\ 0 & 0 & 0 & 0_{N-k-l}\end{bmatrix}.$$

The eigenvalues of the 2×2 matrix $\begin{bmatrix}a+bx & b\sqrt{x(1-x)}\\ b\sqrt{x(1-x)} & b(1-x)\end{bmatrix}$ for $0\le x\le 1$ are

$$\frac{a+b\pm\sqrt{(a-b)^2+4abx}}{2}.$$

Set $t_i:=\frac{a+b\pm\sqrt{(a-b)^2+4abx_i}}{2}$ for $1\le i\le k$. Then the eigenvalues of $aP+bQ$ are

$$\underbrace{0,\dots,0}_{N-2k\ \text{times}},\underbrace{b,\dots,b}_{l-k\ \text{times}},t_1,\dots,t_k,a+b-t_1,\dots,a+b-t_k,$$

and $(t_1,\dots,t_k)$ is supported in $[A,B]^k$. By noting that

$$x_i=\frac{(t_i-a)(t_i-b)}{ab},\quad 1-x_i=\frac{t_i(a+b-t_i)}{ab},\quad \frac{dx_i}{dt_i}=2\left(t_i-\frac{a+b}{2}\right),$$

the form (1.5) of the joint distribution of $(x_1,\dots,x_k)$ can be directly computed from (1.2). The proof of (iii-b) is similar. □

2. Large deviation for PQP

From now on, for each $N\in\mathbb{N}$ let $(P(N),Q(N))$ be a pair of independent and unitarily invariant random projection matrices in $M_N(\mathbb{C})$ with non-random ranks $k(N):=\mathrm{rank}(P(N))$ and $l(N):=\mathrm{rank}(Q(N))$. Throughout what follows, we assume that $k(N)/N\to\alpha$ and $l(N)/N\to\beta$ as $N\to\infty$ for some $\alpha,\beta\in[0.1]$. Our goal is to obtain a large deviation theorem for the empirical eigenvalue density of $P(N)Q(N)P(N)$. Concerning large deviation theory, our general reference is [6], but [9] contains many matrix examples.

We have already observed that the two cases $\alpha+\beta \le 1$ and $\alpha+\beta \ge 1$ are slightly different. To treat them parallel, we set

$$n_0(N) := N - \min\{k(N), l(N)\}, \quad n_1(N) := \max\{k(N) + l(N) - N, 0\},$$
$$n(N) := N - n_0(N) - n_1(N) \ \big(= \min\{k(N), l(N), N - k(N), N - l(N)\}\big).$$

Then one can combine (i-a) and (i-b) of Corollary 1.3 to see that the eigenvalues of the $N \times N$ selfadjoint random matrix $P(N)Q(N)P(N)$ are

$$\underbrace{0, \dots, 0}_{n_0(N) \text{ times}}, \ \underbrace{1, \dots, 1}_{n_1(N) \text{ times}}, \ x_1, \dots, x_{n(N)}$$

and the joint distribution of $(x_1, \dots, x_{n(N)})$ is

$$\frac{1}{Z(N)} \prod_{i=1}^{n(N)} x_i^{|k(N)-l(N)|}(1 - x_i)^{|k(N)+l(N)-N|} \prod_{1 \le i < j \le n(N)} (x_i - x_j)^2 \prod_{i=1}^{n(N)} \mathbf{1}_{[0,1]}(x_i)\, dx_i \tag{2.1}$$

with a normalization constant $Z(N)$.

When $\mathcal{X}$ is a Polish space, let $\mathcal{M}(\mathcal{X})$ denote the set of all probability measures on $\mathcal{X}$, which becomes a Polish space with respect to weak topology. For $\mu \in \mathcal{M}(\mathbb{R})$ let $\Sigma(\mu)$ be Voiculescu's *free entropy* (or the minus of the *logarithmic energy*) of μ defined by

$$\Sigma(\mu) := \iint \log|x - y|\, d\mu(x)\, d\mu(y)$$

(see [16] and [9, §5.3]). In particular, when μ is compactly supported, $\Sigma(\mu) \in [-\infty, +\infty)$ is well defined.

We first prove large deviation for the sequence of distributions (2.1) with slight modifications of notation.

Proposition 2.1. *For each $N \in \mathbb{N}$ consider the distribution*

$$\frac{1}{Z(N)} \prod_{i=1}^{n(N)} x_i^{\kappa(N)}(1 - x_i)^{\lambda(N)} \prod_{1 \le i < j \le n(N)} (x_i - x_j)^2 \prod_{i=1}^{n(N)} \mathbf{1}_{[0,1]}(x_i)\, dx_i \tag{2.2}$$

on $[0,1]^{n(N)}$ with $n(N) \in \mathbb{N}$, $\kappa(N), \lambda(N) \in [0, \infty)$ and a normalization constant $Z(N)$. Assume that $n(N)/N \to \rho$, $\kappa(N)/N \to \kappa$ and $\lambda(N)/N \to \lambda$ as $N \to \infty$ for some $\rho \in (0, \infty)$ and $\kappa, \lambda \in [0, \infty)$. Then:

(1) *The limit $\lim_{N\to\infty} \frac{1}{N^2} \log Z(N)$ exists and it equals $\rho^2 B(\kappa/\rho, \lambda/\rho)$, where*

$$\begin{aligned} B(s,t) \ := \ & \frac{(1+s)^2}{2}\log(1+s) - \frac{s^2}{2}\log s + \frac{(1+t)^2}{2}\log(1+t) - \frac{t^2}{2}\log t \\ & - \frac{(2+s+t)^2}{2}\log(2+s+t) + \frac{(1+s+t)^2}{2}\log(1+s+t) \end{aligned}$$

for $s, t \ge 0$.

(2) *When $(x_1, \dots, x_{n(N)})$ is distributed under (2.2), the empirical measure*

$$\frac{\delta_{x_1} + \cdots + \delta_{x_{n(N)}}}{n(N)} \tag{2.3}$$

satisfies the large deviation principle in the scale $1/N^2$ *with the rate function*

$$I(\mu) := -\rho^2\Sigma(\mu) - \rho\int_0^1 \bigl(\kappa\log x + \lambda\log(1-x)\bigr)\,d\mu(x) + \rho^2 B\left(\frac{\kappa}{\rho},\frac{\lambda}{\rho}\right) \tag{2.4}$$

for $\mu \in \mathcal{M}([0,1])$*. Moreover, there exists a unique minimizer* $\mu_0 \in \mathcal{M}([0,1])$ *of* $I(\mu)$ *with* $I(\mu_0) = 0$*.*

Proof. (1) The Selberg integral formula (see [11, §17.1]) gives

$$\begin{aligned} Z(N) &= \int_{[0,1]^{n(N)}} \prod_{i=1}^{n(N)} x_i^{\kappa(N)}(1-x_i)^{\lambda(N)} \prod_{1\le i<j\le n(N)} (x_i-x_j)^2 \prod_{i=1}^{n(N)} dx_i \\ &= \prod_{j=1}^{n(N)} \frac{\Gamma(j+1)\Gamma(j+\kappa(N))\Gamma(j+\lambda(N))}{\Gamma(2)\Gamma(j+n(N)+\kappa(N)+\lambda(N))}. \end{aligned}$$

By using the Stirling formula, under neglecting the small order $o(N)$, we compute

$$\begin{aligned} &\frac{1}{N^2}\log Z(N) \\ &= \frac{1}{N^2}\Biggl\{\sum_{j=1}^{n(N)} j\log j + \sum_{j=1}^{n(N)} (j+\kappa(n))\log(j+\kappa(n)) + \sum_{j=1}^{n(N)} (j+\rho(n))\log(j+\rho(n)) \\ &\qquad - \sum_{j=1}^{n(N)} (j+n+\kappa(n)+\rho(n))\log(j+n+\kappa(n)+\rho(n))\Biggr\} \\ &= \frac{n(N)}{N^2}\Biggl\{\sum_{j=1}^{n(N)} \frac{j}{n(N)}\log\frac{j}{n(N)} + \sum_{j=1}^{n(N)} \left(\frac{j}{n(N)}+\frac{\kappa}{\rho}\right)\log\left(\frac{j}{n(N)}+\frac{\kappa}{\rho}\right) \\ &\qquad + \sum_{j=1}^{n(N)} \left(\frac{j}{n(N)}+\frac{\lambda}{\rho}\right)\log\left(\frac{j}{n(N)}+\frac{\lambda}{\rho}\right) \\ &\qquad - \sum_{j=1}^{n(N)} \left(\frac{j}{n(N)}+1+\frac{\kappa}{\rho}+\frac{\lambda}{\rho}\right)\log\left(\frac{j}{n(N)}+1+\frac{\kappa}{\rho}+\frac{\lambda}{\rho}\right)\Biggr\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N^2}\log Z(N) \\ &= \rho^2\Biggl\{\int_0^1 x\log x\,dx + \int_0^1 \left(x+\frac{\kappa}{\rho}\right)\log\left(x+\frac{\kappa}{\rho}\right)dx + \int_0^1 \left(x+\frac{\lambda}{\rho}\right)\log\left(x+\frac{\lambda}{\rho}\right)dx \\ &\qquad - \int_0^1 \left(x+1+\frac{\kappa}{\rho}+\frac{\lambda}{\rho}\right)\log\left(x+1+\frac{\kappa}{\rho}+\frac{\lambda}{\rho}\right)dx\Biggr\} \\ &= \rho^2 B\left(\frac{\kappa}{\rho},\frac{\lambda}{\rho}\right). \end{aligned}$$

(2) Denote the distribution (2.2) by $\nu_{n(N)}$ and define the probability measure P_N on $\mathcal{M}([0,1])$ by

$$P_N(\Lambda) := \nu_{n(N)}\bigl(\{x\in[0,1]^{n(N)} : \mu_x \in \Lambda\}\bigr)$$

for Borel subsets Λ of $\mathcal{M}([0,1])$, where μ_x denotes the empirical measure (2.3) for $x = (x_1, \dots, x_{n(N)})$. Define the kernel functions on $[0,1]^2$ as follows:

$$F(x,y) := -\log|x-y| - \frac{\kappa}{2\rho}(\log x + \log y) - \frac{\lambda}{2\rho}(\log(1-x) + \log(1-y)),$$

$$F_R(x,y) := \min\{F(x,y), R\} \quad \text{for } R > 0.$$

Furthermore, for each $N \in \mathbb{N}$ we define

$$\begin{aligned}\widetilde{F}_N(x,y) \quad &:= \quad -\log|x-y| - \delta_{\kappa>0}\frac{\kappa(N)}{2n(N)}(\log x + \log y) \\ &\qquad -\delta_{\lambda>0}\frac{\lambda(N)}{2n(N)}(\log(1-x) + \log(1-y)),\end{aligned}$$

$$\widetilde{F}_{N,R}(x,y) := \min\{\widetilde{F}_N(x,y), R\} \quad \text{for } R > 0,$$

where $\delta_{\kappa>0} = 1$ if $\kappa > 0$, $\delta_{\kappa>0} = 0$ if $\kappa = 0$, and $\delta_{\lambda>0}$ is similar. Then we observe the following:

(i) $\widetilde{F}_{N,R}(x,y) \le -\log|x-y| - \frac{\kappa(n)}{2n(N)}(\log x + \log y) - \frac{\lambda(n)}{2n(N)}(\log(1-x) + \log(1-y))$ for all $x, y \in [0,1]$.
(ii) For any $R > 0$, $\widetilde{F}_{N,R}(x,y)$ converges to $F_R(x,y)$ uniformly for $x, y \in [0,1]$ as $N \to \infty$.

In fact, (i) is obvious by definition of $\widetilde{F}_{N,R}(x,y)$. For (ii) assume that $\kappa, \lambda > 0$ (the proof is similar for other cases). For $\delta > 0$ set

$$T_\delta := \{(x,y) \in [0,1]^2 : \delta \le x \le 1-\delta,\ \delta \le y \le 1-\delta,\ |x-y| \ge \delta\}.$$

For any $R > 0$ there exist $\delta > 0$ and $N_0 \in \mathbb{N}$ such that $F(x,y) \ge R$ and $\widetilde{F}_N(x,y) \ge R$ for all $(x,y) \in [0,1]^2 \setminus T_\delta$ and $N \ge N_0$. Obviously, $\widetilde{F}_N(x,y)$ converges to $F(x,y)$ uniformly on T_δ as $N \to \infty$, and the assertion follows.

According to general theory of large deviations ([6]), the stated large deviation is shown when we prove the following two inequalities for every $\mu \in \mathcal{M}([0,1])$:

$$\inf_G \left[\limsup_{N\to\infty} \frac{1}{N^2} \log P_N(G)\right] \le -\rho^2 \iint F(x,y)\, d\mu(x)\, d\mu(y) - C, \tag{2.5}$$

$$\inf_G \left[\liminf_{N\to\infty} \frac{1}{N^2} \log P_N(G)\right] \ge -\rho^2 \iint F(x,y)\, d\mu(x)\, d\mu(y) - C, \tag{2.6}$$

where $C := \rho^2 B(\kappa/\rho, \lambda/\rho)$ and G runs over neighborhoods of μ.

Proof of (2.5). For every neighborhood G of $\mu \in \mathcal{M}([0,1])$, setting $\widetilde{G} := \{x \in [0,1]^{n(N)} : \mu_x \in G\}$, by the above (i) we have

$$\begin{aligned}
P_N(G) &= \nu_{n(N)}(\widetilde{G}) \\
&= \frac{1}{Z(N)} \int_{\widetilde{G}} \prod_{i=1}^{n(N)} x_i^{\kappa(N)}(1-x_i)^{\lambda(N)} \prod_{1\le i<j\le n(N)} (x_i - x_j)^2 \prod_{i=1}^{n(N)} dx_i \\
&\le \frac{1}{Z(N)} \int_{\widetilde{G}} \prod_{i=1}^{n(N)} x_i^{\kappa(N)/n(N)}(1-x_i)^{\lambda(N)/n(N)} \\
&\qquad\qquad \times \exp\Biggl(-2 \sum_{1\le i<j\le n(N)} \widetilde{F}_{N,R}(x_i,x_j)\Biggr) \prod_{i=1}^{n(N)} dx_i \\
&\le \frac{1}{Z(N)} \biggl(\int_0^1 x^{\kappa(N)/n(N)}(1-x)^{\lambda(N)/n(N)}\, dx\biggr)^{n(N)} \\
&\qquad \times \exp\biggl(-n(N)^2 \inf_{\mu'\in G} \iint \widetilde{F}_{N,R}(x,y)\, d\mu'(x)\, d\mu'(y) + n(N)R\biggr).
\end{aligned}$$

Since the above fact (ii) implies that

$$\lim_{N\to\infty} \biggl(\inf_{\mu'\in G} \iint \widetilde{F}_{N,R}(x,y)\, d\mu'(x)\, d\mu'(y)\biggr) = \inf_{\mu'\in G} \iint F_R(x,y)\, d\mu'(x)\, d\mu'(y),$$

we get

$$\lim_{N\to\infty} \frac{1}{N^2} \log P_N(G) \le -\rho^2 \inf_{\mu'\in G} \iint F_R(x,y)\, d\mu'(x)\, d\mu'(y) - C$$

thanks to (1). Furthermore, appealing to the continuity of $\mu' \mapsto \iint F_R(x,y)\, d\mu'(x)\, d\mu'(y)$, we obtain

$$\inf_G \biggl[\limsup_{N\to\infty} \frac{1}{N^2} \log P_N(G)\biggr] \le -\rho^2 \iint F_R(x,y)\, d\mu(x)\, d\mu(y) - C$$

so that (2.5) follows by letting $R \to +\infty$.

Proof of (2.6). If μ has an atom at 0 or 1, then $\iint F(x,y)\, d\mu(x)\, d\mu(y) = +\infty$ so that we have nothing to do. Otherwise, letting $d\mu_\delta(x) := \mu([\delta, 1-\delta])^{-1} \mathbf{1}_{[\delta,1-\delta]}(x)\, d\mu(x)$, we get

$$\iint F(x,y)\, d\mu(x)\, d\mu(y) = \lim_{\delta\searrow 0} \iint F(x,y)\, d\mu_\delta(x)\, d\mu_\delta(y).$$

Also it is immediate to see that

$$\mu \in \mathcal{M}([0,1]) \mapsto \inf\biggl\{\liminf_{N\to\infty} \frac{1}{N^2} \log P_N(G) : G \text{ is a neighborhood of } \mu\biggr\}$$

is upper semicontinuous. Hence we may assume that μ is supported in $[a,b]$ with $0 < a < b < 1$. For $\varepsilon > 0$ let $\phi_\varepsilon \ge 0$ be a C^∞-function supported in $[-\varepsilon, \varepsilon]$ such that $\int \phi_\varepsilon(x)\, dx = 1$. Then we get $\Sigma(\phi_\varepsilon * \mu) \ge \Sigma(\mu)$ (see [9, p. 216]) as well as

$$\lim_{\varepsilon\searrow 0} \int \log x\, d(\phi_\varepsilon * \mu)(x) = \int \log x\, d\mu(x),$$

$$\lim_{\varepsilon\searrow 0} \int \log(1-x)\, d(\phi_\varepsilon * \mu)(x) = \int \log(1-x)\, d\mu(x)$$

so that μ may be assumed to have a continuous density. Furthermore, by the concavity of $\Sigma(\mu)$, it suffices to prove (2.6) for $(1-\varepsilon)\mu+\varepsilon m$ for each $0<\varepsilon<1$, where m is the uniform measure on an interval including the support $\operatorname{supp}\mu$. After all, we can assume that μ has a continuous density $f>0$ on $\operatorname{supp}\mu=[a,b]$ with $0<a<b<1$ and $\delta\le f(x)\le\delta^{-1}$ on $[a,b]$ for some $\delta>0$.

For each $N\in\mathbb{N}$ let

$$a<a_1^{(N)}<b_1^{(N)}<a_2^{(N)}<\cdots<a_{n(N)}^{(N)}<b_{n(N)}^{(N)}$$

be such that

$$\int_a^{a_i^{(N)}} f(x)\,dx=\frac{i-\frac12}{n(N)},\qquad \int_a^{b_i^{(N)}} f(x)\,dx=\frac{i}{n(N)},\qquad 1\le i\le n(N);$$

then

$$b_i^{(N)}-a_i^{(N)}\ge\frac{\delta}{2n(N)},\qquad 1\le i\le n(N).$$

Define

$$\Delta_{n(N)}:=\bigl\{x=(x_1,\dots,x_{n(N)})\in[0,1]^{n(N)}:a_i^{(N)}\le x_i\le b_i^{(N)},\ 1\le i\le n(N)\bigr\}.$$

For any neighborhood G of μ, whenever N is large enough, we have

$$\Delta_{n(N)}\subset\widetilde{G}:=\bigl\{x\in[0,1]^{n(N)}:\mu_x\in G\bigr\}$$

so that

$$\begin{aligned}
P_N(G)&=\nu_{n(N)}(\widetilde{G})\\
&\ge\frac{1}{Z(N)}\int_{\Delta_{n(N)}}\prod_{i=1}^{n(N)}x_i^{\kappa(N)}(1-x_i)^{\lambda(N)}\prod_{1\le i<j\le n(N)}(x_i-x_j)^2\prod_{i=1}^{n(N)}dx_i\\
&\ge\frac{1}{Z(N)}\left(\frac{\delta}{2n(N)}\right)^{n(N)}\prod_{i=1}^{n(N)}\bigl(a_i^{(N)}\bigr)^{\kappa(N)}\bigl(1-b_i^{(N)}\bigr)^{\lambda(N)}\prod_{1\le i<j\le n(N)}\bigl(a_j^{(N)}-b_i^{(N)}\bigr)^2.
\end{aligned}$$

With $g:[0,1]\to[a,b]$ being the inverse function of $t\in[a,b]\mapsto\int_a^t f(x)\,dx$, since $a_i^{(N)}=g\bigl((i-\frac12)/n(N)\bigr)$ and $b_i^{(N)}=g\bigl(i/n(N)\bigr)$, we have

$$\lim_{N\to\infty}\frac{\kappa(N)}{N^2}\sum_{i=1}^{n(N)}\log a_i^{(N)}=\rho\kappa\int_0^1\log g(t)\,dt=\rho\kappa\int\log x\,d\mu(x),$$

$$\lim_{N\to\infty}\frac{\kappa(N)}{N^2}\sum_{i=1}^{n(N)}\log\bigl(1-b_i^{(N)}\bigr)=\rho\kappa\int_0^1\log(1-g(t))\,dt=\rho\kappa\int\log(1-x)\,d\mu(x),$$

$$\begin{aligned}
&\lim_{N\to\infty}\frac{2}{N^2}\sum_{1\le i<j\le n(N)}\log\bigl(a_j^{(N)}-b_i^{(N)}\bigr)\\
&\qquad=2\rho^2\iint_{0\le s<t\le t\le 1}\log(g(t)-g(s))\,ds\,dt=\rho^2\Sigma(\mu).
\end{aligned}$$

These estimates altogether imply (2.6).

The proof of the large deviation is now completed, and the existence of a unique minimizer of the rate function is known as a general result on weighted logarithmic energy functionals (see [13, I.1.3]). $\square$

Now, the large deviation theorem for the random matrix $P(N)Q(N)P(N)$ can be easily shown from Proposition 2.1. Set

$$\rho := \min\{\alpha, \beta, 1-\alpha, 1-\beta\},$$
$$C := \rho^2 B\biggl(\frac{|\alpha-\beta|}{\rho}, \frac{|\alpha+\beta-1|}{\rho}\biggr)$$

(meant zero if $\rho = 0$), and denote by $\mathcal{M}((0,1))$ the set of all probability measures on $[0,1]$ with no atoms at 0 and 1.

Theorem 2.2. *The empirical eigenvalue density of $P(N)Q(N)P(N)$ satisfies the large deviation principle in the scale $1/N^2$ with the rate function $\tilde{I}(\tilde{\mu})$ for $\tilde{\mu} \in \mathcal{M}([0,1])$ given as follows: If*

$$\tilde{\mu} = (1-\min\{\alpha,\beta\})\delta_0 + \max\{\alpha+\beta-1, 0\}\delta_1 + \rho\mu$$

with $\mu \in \mathcal{M}((0,1))$, then

$$\begin{aligned} \tilde{I}(\tilde{\mu}) \;\; &:= \;\; -\rho^2\Sigma(\mu) - \rho|\alpha-\beta| \int_0^1 \log x \, d\mu(x) \\ &\qquad -\rho|\alpha+\beta-1| \int_0^1 \log(1-x)\, d\mu(x) + C; \end{aligned} \tag{2.7}$$

otherwise $\tilde{I}(\tilde{\mu}) = +\infty$. Moreover, a unique minimizer of $\tilde{I}(\tilde{\mu})$ is given by

$$\tilde{\mu}_0 := (1-\min\{\alpha,\beta\})\delta_0 + \max\{\alpha+\beta-1,0\}\delta_1 + \frac{\sqrt{(x-\xi)(\eta-x)}}{2\pi x(1-x)} \mathbf{1}_{(\xi,\eta)}(x)\, dx \tag{2.8}$$

where

$$\xi, \eta := \alpha + \beta - 2\alpha\beta \pm \sqrt{4\alpha\beta(1-\alpha)(1-\beta)}. \tag{2.9}$$

In particular, when $\rho = 0$, $\tilde{I}(\tilde{\mu})$ is identically $+\infty$ except at only $\tilde{\mu}_0 = (1-\min\{\alpha,\beta\})\delta_0 + \max\{\alpha+\beta-1,0\}\delta_1$.

Proof. From the fact mentioned at the beginning of the section, the empirical eigenvalue density of $P(N)Q(N)P(N)$ is given by

$$\widetilde{R}_N := \frac{n_0(N)}{N}\delta_0 + \frac{n_1(N)}{N}\delta_1 + \frac{n(N)}{N} R_N,$$

where $R_N := \frac{1}{n(N)}(\delta_{x_1} + \cdots + \delta_{x_{n(N)}})$ and the joint distribution of $(x_1, \dots, x_{n(N)})$ is (2.1). First, assume that $\rho > 0$. Proposition 2.1 says that (R_N) satisfies the large deviation in the scale $1/N^2$ with the rate function $I(\mu)$ for $\mu \in \mathcal{M}([0,1])$ given in (2.4) with $\kappa := |\alpha-\beta|$ and $\lambda := |\alpha+\beta-1|$. We now proceed as in the proof of [9, 5.5.11]. Let P_N and $\widetilde{P}_N$ be the distributions on $\mathcal{M}([0,1])$ of R_N and $\widetilde{R}_N$, respectively; then

$$\widetilde{P}_N(\Lambda) = P_N\biggl(\biggl\{\mu \in \mathcal{M}([0,1]) : \frac{n_0(N)}{n}\delta_0 + \frac{n_1(N)}{N}\delta_1 + \frac{n(N)}{N}\mu \in \Lambda\biggr\}\biggr)$$

for $\Lambda \subset \mathcal{M}([0,1])$. Let $\mathcal{D}$ denote the set $\{\rho_0\delta_0 + \rho_1\delta_1 + \rho\mu : \mu \in \mathcal{M}([0,1])\}$, where $\rho_0 := 1 - \min\{\alpha,\beta\}$ and $\rho_1 := \max\{\alpha+\beta-1, 0\}$. If $\tilde{\mu} \notin \mathcal{D}$, then $\tilde{\mu}(\{0\}) < \rho_0$ or $\tilde{\mu}(\{1\}) < \rho_1$ so that letting $\tilde{\mu}(\{0\}) < \varepsilon < \rho_0$ (or $\tilde{\mu}(\{1\}) < \varepsilon < \rho_1$) we have a neighborhood $\widetilde{G} := \{\mu' \in \mathcal{M}([0,1]) : \mu'(\{0\}) < \varepsilon$ (or $\mu'(\{1\}) < \varepsilon)\}$ of μ. Since $\widetilde{P}_N(\widetilde{G}) = 0$ for large N, we get $\lim_{N\to\infty} \frac{1}{N^2} \log \widetilde{P}_N(\widetilde{G}) = -\infty$. Next, assume that $\tilde{\mu} \in \mathcal{D}$ and $\tilde{\mu} = \rho_0\delta_0 + \rho_1\delta_1 + \rho\mu$. For any neighborhood $\widetilde{G}$ of $\tilde{\mu}$ there exists a neighborhood G of μ such that $\frac{n_0(N)}{N}\delta_0+\frac{n_1(N)}{N}\delta_1+\frac{n(N)}{N}G \subset \widetilde{G}$ for large N and hence

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \widetilde{P}_N(\widetilde{G}) \ge \liminf_{N\to\infty} \frac{1}{N^2} \log P_N(G) \ge -I(\mu).$$

On the other hand, for any neighborhood G of μ there exists a neighborhood $\widetilde{G}$ of $\tilde{\mu}$ such that

$$\left(\frac{N}{n(N)}\widetilde{G} - \frac{n_0(N)}{n(N)}\delta_0 - \frac{n_1(N)}{n(N)}\delta_1\right) \cap \mathcal{M}([0,1]) \subset G,$$

that is,

$$\left\{\mu \in \mathcal{M}([0,1]) : \frac{n_0(N)}{n}\delta_0 + \frac{n_1(N)}{n}\delta_1 + \frac{n(N)}{n}\mu \in \widetilde{G}\right\} \subset G$$

for large N. Therefore,

$$\inf_{\widetilde{G}}\left[\limsup_{N\to\infty} \frac{1}{n^2} \log \widetilde{P}_N(\widetilde{G})\right] \le \inf_{G}\left[\limsup_{N\to\infty} \frac{1}{n^2} \log P_N(G)\right] \le -I(\mu).$$

Noting that $\Sigma(\mu) = -\infty$ if $\mu \in \mathcal{M}([0,1])$ has an atom at 0 or 1, we obtain the desired large deviation for $(\widetilde{R}_N)$ when $\rho > 0$. The proof in the case $\rho = 0$ is similar to the above argument for $\tilde{\mu} \notin \mathcal{D}$.

Finally, the existence of a unique minimizer of $\tilde{I}(\tilde{\mu})$ is already known by Proposition 2.1. To obtain the explicit form of the minimizer, we may apply a standard method in free probability theory. In fact, by the *asymptotic freeness* due to Voiculescu [15, Theorem 3.11] (see also [9, 4.3.5]), the joint distribution of $(P(N), Q(N))$ converges to that of (p,q) where p and q are free projections in a tracial W^*-probability space $(\mathcal{M}, \tau)$ with $\tau(p) = \alpha$ and $\tau(q) = \beta$. The computation by use of S-transform in [18] says that the measure (2.8) is the distribution measure of pqp; hence it is the minimizer of $\tilde{I}(\tilde{\mu})$. □

Note that the rate function $\tilde{I}(\tilde{\mu})$ is indeed lower semicontinuous and convex on $\mathcal{M}([0,1])$, which is of course a good rate function because of the compactness of $\mathcal{M}([0,1])$.

3. C^*-ALGEBRA FORMULATION

The two-dimensional commutative C^*-algebra $\mathbb{C}\oplus\mathbb{C} = C^*(\mathbb{Z}_2)$ is the universal C^*-algebra generated by a single orthogonal projection; hence the universal C^*-algebra generated two orthogonal projections is

$$(\mathbb{C}\oplus\mathbb{C}) \star (\mathbb{C}\oplus\mathbb{C}) = C^*(\mathbb{Z}\star\mathbb{Z})$$

with projection generators $(1,0)$'s in two components. As was pointed out in [3, p. 14], one can see from the structure theorem for two projections ([14, pp. 306–308]) that $C^*(\mathbb{Z}\star\mathbb{Z})$ is isomorphic to an algebra of $M_2(\mathbb{C})$-valued continuous functions on $[0,1]$, namely

$$\mathcal{A} := \{a \in C([0,1]; M_2(\mathbb{C})) : a(0) \text{ and } a(1) \text{ are diagonal}\},$$

where the corresponding two projection generators are represented as

$$e(t) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f(t) := \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix} \quad \text{for } 0 \le t \le 1.$$

We thus consider the above C^*-algebra $\mathcal{A}$ with generators e, f as the universal C^*-algebra generated by two projections. We denote by $TS(\mathcal{A})$ the set of all tracial states on $\mathcal{A}$, which becomes a Polish space with respect to w*-topology. The following lemma is a concrete description of $TS(\mathcal{A})$, the details are left to the reader.

Lemma 3.1. *For each $\tau \in TS(\mathcal{A})$ there exist $\alpha_{11}, \alpha_{10}, \alpha_{01}, \alpha_{00} \geq 0$ with $\sum_{i,j=0}^{1} \alpha_{ij} \leq 1$ and $\mu \in \mathcal{M}((0,1))$ such that*

$$\begin{aligned}\tau(a) &= \alpha_{10}a_1(0) + \alpha_{01}a_2(0) + \alpha_{11}a_1(1) + \alpha_{00}a_2(1) \\ &\quad + \left(1 - \sum_{i,j=0}^{1} \alpha_{ij}\right) \int_0^1 \mathrm{tr}(a(t))\, d\mu(t)\end{aligned}$$

for all $a \in \mathcal{A}$ with $a(0) = \mathrm{Diag}(a_1(0), a_2(0))$ and $a(1) = \mathrm{Diag}(a_1(1), a_2(1))$.

In this way, the set $TS(\mathcal{A})$ is parameterized by the set of all $(\{\alpha_{ij}\}_{i,j=0}^1, \mu)$ of $\alpha_{ij} \geq 0$, $\sum_{i,j=0}^{1} \alpha_{ij} \leq 1$ and $\mu \in \mathcal{M}((0,1))$, and we write $\tau = (\{\alpha_{ij}\}_{i,j=0}^1, \mu)$ under this parameterization. But, note that μ is irrelevant if $\sum_{i,j=0}^{1} \alpha_{ij} = 1$. For $\tau = (\{\alpha_{ij}\}_{i,j=0}^1, \mu)$ we have

$$\tau(e) = \frac{1}{2}(1 + \alpha_{11} + \alpha_{10} - \alpha_{01} - \alpha_{00}), \tag{3.1}$$

$$\tau(f) = \frac{1}{2}(1 + \alpha_{11} - \alpha_{10} + \alpha_{01} - \alpha_{00}). \tag{3.2}$$

Furthermore, let π_τ be the GNS representation of $\mathcal{A}$ associated with τ and $\tilde{\tau}$ be the normal extension of τ to $\pi_\tau(\mathcal{A})''$. Then, for $p := \pi_\tau(e)$ and $q := \pi_\tau(f)$ in $\pi_\tau(\mathcal{A})''$ we have

$$\tilde{\tau}(p \wedge q) = \alpha_{11}, \quad \tilde{\tau}(p \wedge q^\perp) = \alpha_{10}, \quad \tilde{\tau}(p^\perp \wedge q) = \alpha_{01}, \quad \tilde{\tau}(p^\perp \wedge q^\perp) = \alpha_{00}. \tag{3.3}$$

For any two projections p, q in a tracial W^*-probability space $(\mathcal{M}, \tau)$, the universality property of $\mathcal{A}$ shows that there exists a (unique) $*$-homomorphism $\psi_{p,q} : \mathcal{A} \to \mathcal{M}$ such that $\psi_{p,q}(e) = p$ and $\psi_{p,q}(f) = q$. We simply write $h(p,q)$ for $\psi_{p,q}(h)$ for each $h \in \mathcal{A}$, which may be regarded as a sort of "noncommutative functional calculus." Then a tracial state $\tau_{p,q} \in TS(\mathcal{A})$ is defined by $\tau_{p,q}(h) := \tau(h(p,q))$ for $h \in \mathcal{A}$. In particular, for $N \times N$ projection matrices P, Q, we have $\tau_{P,Q} \in TS(\mathcal{A})$ given by $\tau_{P,Q}(h) = \mathrm{tr}_N(h(P,Q))$ for $h \in \mathcal{A}$. When P, Q are random projection matrices, $\tau_{P,Q}$ is a random tracial state on $\mathcal{A}$ regarded as the "noncommutative empirical measure" of the pair (P,Q). Its *distribution measure* on $TS(\mathcal{A})$ is defined by

$$\nu(\Lambda) := \mathrm{Prob}(\{\tau_{P,Q} \in \Lambda\})$$

for Borel subsets $\Lambda \subset TS(\mathcal{A})$, where Prob denotes probability measure of the underlying probability space where P, Q are defined.

We are now in a position to state our main large deviation result formulated on the tracial state space $TS(\mathcal{A})$.

Theorem 3.2. *For each $N \in \mathbb{N}$ let $(P(N), Q(N))$ be a pair of independent and unitarily invariant random projection matrices in $M_N(\mathbb{C})$ such that $\mathrm{rank}(P(N))/N \to \alpha$ and $\mathrm{rank}(Q(N))/N \to \beta$ as $N \to \infty$. Let ν_N be the distribution measure of the random tracial state $\tau_N := \tau_{P(N),Q(N)}$ on $TS(\mathcal{A})$. Then (ν_N) satisfies the large deviation principle in the scale $1/N^2$ with rate function*

$$\begin{aligned}\mathcal{I}(\tau) &:= -\rho^2\Sigma(\mu) - \rho|\alpha - \beta| \int_0^1 \log x\, d\mu(x) \\ &\quad -\rho|\alpha + \beta - 1| \int_0^1 \log(1-x)\, d\mu(x) + C_{\alpha,\beta}\end{aligned}$$

evaluated at $\tau = (\{\alpha_{ij}\}_{i,j=0}^{1}, \mu) \in TS(\mathcal{A})$ *if*

$$\begin{cases} \alpha_{11} = \max\{\alpha + \beta - 1, 0\}, \\ \alpha_{00} = \max\{1 - \alpha - \beta, 0\}, \\ \alpha_{10} = \max\{\alpha - \beta, 0\}, \\ \alpha_{01} = \max\{\beta - \alpha, 0\}, \end{cases} \tag{3.4}$$

otherwise $\mathcal{I}(\tau) = +\infty$.

Moreover, the unique minimizer of $\mathcal{I}$ *is the tracial state* $\tau_{p,q}$ *corresponding to a pair* (p, q) *of free projections with trace values* α *and* β.

Proof. First we notice that all mixed moments of e, f with respect to τ are listed as $\tau(e)$, $\tau(f)$ and

$$\tau((ef)^k) = \tau((fe)^k) = \tau((efe)^k) = \tau((fef)^k), \quad k \geq 1. \tag{3.5}$$

Since the moments $\tau((efe)^k)$, $k \geq 1$, determine the distribution of efe with respect to τ, one can define an affine homeomorphism Ψ of $TS(\mathcal{A})$ with w*-topology into $[0,1]\times[0,1]\times\mathcal{M}([0,1])$ with product topology by $\Psi(\tau) := (\tau(e), \tau(f), \tilde{\mu})$ where $\tilde{\mu}$ is the distribution measure of efe with respect to τ. For each $\tau = (\{\alpha_{ij}\}_{i,j=0}^{1}, \mu) \in TS(\mathcal{A})$ let $p := \pi_\tau(e)$ and $q := \pi_\tau(f)$ in $(\pi_\tau(\mathcal{A})'', \tilde{\tau})$, and let $e_{pqp}(\cdot)$ be the spectral measure of pqp. From the structure theorem for two projections, we get

$$\begin{aligned} \tilde{\mu}(\{0\}) &= \tilde{\tau}(e_{pqp}(\{0\}) \\ &= \frac{1}{2}\tilde{\tau}(\mathbf{1} - p \wedge q - p \wedge q^\perp - p^\perp \wedge q - p^\perp \wedge q^\perp) \\ &\qquad + \tilde{\tau}(p \wedge q^\perp + p^\perp \wedge q + p^\perp \wedge q^\perp) \\ &= \frac{1}{2}(1 - \alpha_{11} + \alpha_{10} + \alpha_{01} + \alpha_{00}) \end{aligned}$$

and

$$\tilde{\mu}(\{1\}) = \tilde{\tau}(e_{pqp}(\{1\}) = \tilde{\tau}(p \wedge q) = \alpha_{11}$$

thanks to (3.3). Hence it is straightforward to check that τ satisfies (3.4) if and only the following hold:

$$\begin{cases} \tau(e) = \alpha, \\ \tau(f) = \beta, \\ \tilde{\mu}(\{0\}) = 1 - \min\{\alpha, \beta\}, \\ \tilde{\mu}(\{1\}) = \max\{\alpha + \beta - 1, 0\}. \end{cases}$$

Furthermore, in this case we obviously have

$$\tilde{\mu} = (1 - \min\{\alpha, \beta\})\delta_0 + \max\{\alpha + \beta - 1, 0\}\delta_1 + \rho\mu,$$

where

$$\rho = \min\{\alpha, \beta, 1 - \alpha, 1 - \beta\} = \frac{1}{2}\left(1 - \sum_{i.j=0}^{1} \alpha_{ij}\right). \tag{3.6}$$

Based on Theorem 2.2 together with these facts, to show the theorem, it suffices to prove the following assertions:

(i) If $\tau \in TS(\mathcal{A})$ and $(\tau(e), \tau(f)) \neq (\alpha, \beta)$, then

$$\inf_G \left[\limsup_{N\to\infty} \frac{1}{N^2} \log \nu_N(G) \right] = -\infty.$$

(ii) If $\tau \in TS(\mathcal{A})$ and $\Psi(\tau) = (\alpha, \beta, \tilde{\mu})$, then

$$\inf_G \left[\limsup_{N\to\infty} \frac{1}{N^2} \log \nu_N(G) \right] \leq -\tilde{I}(\tilde{\mu}),$$

$$\inf_G \left[\liminf_{N\to\infty} \frac{1}{N^2} \log \nu_N(G) \right] \geq -\tilde{I}(\tilde{\mu}),$$

where $\tilde{I}(\tilde{\mu})$ is the rate function in Theorem 2.2 and G runs over neighborhoods of τ.

When $(\tau(e), \tau(f)) \neq (\alpha, \beta)$, choose $\varepsilon > 0$ such that $\varepsilon < |\tau(e) - \alpha|$ (or $\varepsilon < |\tau(f) - \beta|$), and set $G := \{\tau' \in TS(\mathcal{A}) : |\tau'(e) - \alpha| < \varepsilon$ (or $|\tau'(f) - \beta| < \varepsilon)\}$. Since $\tau_N(e) = \mathrm{tr}_N(P(N)) = k(N)/N \to \alpha$ and $\tau_N(f) = \mathrm{tr}_N(Q(N)) = l(N)/N \to \beta$ as $N \to \infty$, we get $\nu_N(G) = 0$ for large N so that (i) follows.

To prove (ii), assume that $\Psi(\tau) = (\alpha, \beta, \tilde{\mu})$. For any neighborhood $\widetilde{G}$ of $\tilde{\mu}$, note that $\Psi^{-1}([0,1] \times [0,1] \times \widetilde{G})$ is a neighborhood of τ and

$$\begin{aligned} \nu_N\big(\Psi^{-1}([0,1] \times [0,1] \times \widetilde{G})\big) &= \mathrm{Prob}\big(\{\Psi(\tau_{P(N),Q(N)}) \in [0,1] \times [0,1] \times \widetilde{G}\}\big) \\ &= \mathrm{Prob}\big(\{\widetilde{R}_N \in \widetilde{G}\}\big) = \widetilde{P}_N(\widetilde{G}), \end{aligned}$$

where $\widetilde{R}_N$ is the empirical eigenvalue distribution of $P(N)Q(N)P(N)$ and $\widetilde{P}_N$ is its distribution on $\mathcal{M}([0,1])$ (see the proof of Theorem 2.2). Hence we have

$$\inf_G \left[\limsup_{N\to\infty} \frac{1}{N^2} \log \nu_N(G) \right] \leq \inf_{\widetilde{G}} \left[\limsup_{N\to\infty} \frac{1}{N^2} \log \widetilde{P}_N(\widetilde{G}) \right] \leq -\tilde{I}(\tilde{\mu})$$

by Theorem 2.2. On the other hand, for any neighborhood G of τ, one can choose $\varepsilon > 0$ and a neighborhood $\widetilde{G}$ of $\tilde{\mu}$ such that $\Psi^{-1}\big((\alpha - \varepsilon, \alpha + \varepsilon) \times (\beta - \varepsilon, \beta + \varepsilon) \times \widetilde{G}\big) \subset G$, which implies that

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N^2} \log \nu_N(G) \\ &\quad \geq \liminf_{N\to\infty} \frac{1}{N^2} \log \nu_N\big(\Psi^{-1}\big((\alpha - \varepsilon, \alpha + \varepsilon) \times (\beta - \varepsilon, \beta + \varepsilon) \times \widetilde{G}\big)\big) \\ &\quad = \liminf_{N\to\infty} \frac{1}{N^2} \log \mathrm{Prob}\big(\big\{|\mathrm{tr}_N(P(N)) - \alpha| < \varepsilon,\ |\mathrm{tr}_N(Q(N)) - \beta| < \varepsilon,\ \widetilde{R}_N \in \widetilde{G}\big\}\big). \end{aligned}$$

Since $|\mathrm{tr}_N(P(N)) - \alpha| < \varepsilon$ and $|\mathrm{tr}_N(Q(N)) - \beta| < \varepsilon$ for large N (as in the proof of (i)), we have

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \nu_N(G) \geq \liminf_{N\to\infty} \frac{1}{N^2} \log \widetilde{P}_N(\widetilde{G}) \geq -\tilde{I}(\tilde{\mu})$$

by Theorem 2.2, and hence (ii) is proven. Finally, Theorem 2.2 proves the assertion on the minimizer as well (or this is a direct consequence of the asymptotic freeness of $(P(N), Q(N))$). $\square$

For $N \in \mathbb{N}$ and $k \in \{0, 1, \dots, N\}$ let $\mathcal{P}(N,k)$ denote the set of all $N \times N$ orthogonal projection matrices of rank k, and $\gamma_{N,k}$ be the unitarily invariant measure on $\mathcal{P}(N,k)$. We note that $\mathcal{P}(N,k)$ is identified with the homogeneous space $U(N)/(U(k) \oplus U(N-k))$ (or the *Grassmannian manifold* $G(N,k)$) and $\gamma_{N,k}$ corresponds to the measure on that space induced from the Haar probability measure on the unitary group $U(N)$. In fact, an $N \times N$

unitarily invariant random projection matrix of rank k we have treated is standardly realized by $P \in \mathcal{P}(N,k)$ distributed under $\gamma_{N,k}$.

Let (p,q) be a pair of projections in a tracial W^*-probability space $(\mathcal{M},\tau)$ and let $\alpha := \tau(p)$ and $\beta := \tau(q)$. The *free entropy* $\chi(p,q)$ of (p,q) proposed in [17, 14.2] by Voiculescu is defined as follows: Choose sequences $k(N)$ and $l(N)$ such that $k(N)/N \to \alpha$ and $l(N)/N \to \beta$ as $N \to \infty$. For each $m \in \mathbb{N}$ and $\varepsilon > 0$ set

$$\begin{aligned}
&\Gamma\big(p,q;k(N),l(N);N,m,\varepsilon\big) \\
&\quad := \Big\{(P,Q) \in \mathcal{P}(N,k(N)) \times \mathcal{P}(N,l(N)) : \big|\mathrm{tr}_N(P_1\cdots P_m) - \tau(p_1\cdots p_m)\big| < \varepsilon \\
&\qquad\qquad \text{for all } (P_j,p_j) \in \{(P,p),(Q,q)\},\ 1 \le j \le m\Big\},
\end{aligned}$$

and define

$$\chi(p,q) := \lim_{\substack{m\to\infty \\ \varepsilon\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\log\big(\gamma_{N,k(N)}\otimes\gamma_{N,l(N)}\big)\Big(\Gamma\big(p,q;k(N),l(N);N,m,\varepsilon\big)\Big). \tag{3.7}$$

Let $\mathcal{A}$ be the C^*-algebra with two projection generators e,f introduced in the previous section. The *free entropy* of a tracial state $\tau \in TS(\mathcal{A})$ is defined as $\chi(\pi_\tau(e),\pi_\tau(f))$ in the tracial W^*-probability space $(\pi_\tau(\mathcal{A})'',\tilde{\tau})$ obtained via the GNS construction associated with τ.

Next we identify the rate function in Theorem 3.2 as the free entropy $\chi(\tau)$ (up to a sign).

Proposition 3.3. *The rate function in Theorem 3.2 given for $\alpha = \tau(e)$ and $\beta = \tau(f)$ is*

$$\mathcal{I}(\tau) = -\chi(\tau).$$

Moreover $\limsup$ *can be replaced by* $\lim$ *in definition* (3.7).

Proof. Let $p := \pi_\tau(e)$, $q := \pi_\tau(f)$ and $\tilde{\mu}$ be the distribution of efe with respect to τ. In view of the form (3.5) of joint moments of e,f and the choices of $k(N), l(N)$ as above, one can easily see that for each $m \in \mathbb{N}$ and $\varepsilon > 0$

$$\begin{aligned}
&\Gamma(p,q;k(N),l(N);N,2m,\varepsilon) \\
&\quad = \Big\{(P,Q) \in \mathcal{P}(N,k(N)) \times \mathcal{P}(N,l(N)) : \big|\mathrm{tr}_N((PQP)^k) - \tau((efe)^k)\big| < \varepsilon,\ 1 \le k \le m\Big\}
\end{aligned}$$

whenever N is large enough. This implies that

$$\big(\gamma_{N,k(N)}\otimes\gamma_{N,l(N)}\big)\Big(\Gamma\big(p,q;k(N),l(N);N,2m,\varepsilon\big)\Big) = \widetilde{P}_N(\widetilde{G}(m,\varepsilon)),$$

where $\widetilde{P}_N$ is the distribution on $\mathcal{M}([0,1])$ mentioned in the proof of Theorem 3.2 and $\widetilde{G}(m,\varepsilon)$ is a neighborhood of $\tilde{\mu}$ given by

$$\widetilde{G}(m,\varepsilon) := \left\{\tilde{\mu}' \in \mathcal{M}([0,1]) : \left|\int x^k\,d\tilde{\mu}'(x) - \int x^k\,d\tilde{\mu}(x)\right| < \varepsilon,\ 1 \le k \le m\right\}.$$

Now, as in the proof of [9, 5.6.2] we have the limit

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N^2}\log\big(\gamma_{N,k(N)}\otimes\gamma_{N,l(N)}\big)\Big(\Gamma\big(p,q;k(N),l(N);N,2m,\varepsilon\big)\Big) \\
&\quad = \lim_{N\to\infty}\frac{1}{N^2}\log\widetilde{P}_N(\widetilde{G}(m,\varepsilon)),
\end{aligned}$$

and the conclusion follows from Theorem 3.2 and its proof. $\square$

Theorem 3.2 implies that the free entropy $\chi(p,q)$ of two projections p,q admits a maximal value, i.e., $\chi(p,q)=0$ if and only if p,q are free. Moreover, note by Proposition 3.3 that the definition (3.7) of $\chi(p,q)$ is independent of the choices of the sequences $k(N)$ and $l(N)$, but this fact is easy to directly verify.

A further study of the free entropy $\chi(p_1,\dots,p_n)$ for general n-tuples of projections as well as some related topics will be in a forthcoming paper [10].

4. Applications of the contraction principle

Let $(P(N),Q(N))$ be as before, and let $\mathcal{A}$ be the C^*-algebra of two projection generators introduced in the previous section. Our large deviation in Theorem 3.2 is formulated on the tracial state space of $\mathcal{A}$. The aim of this section is to exemplify how Theorem 3.2 implies, via the contraction principle, the large deviation for the empirical eigenvalue density of various random matrices made from $(P(N),Q(N))$.

For each selfadjoint element $h\in\mathcal{A}$ and $\tau\in TS(\mathcal{A})$, let $\lambda_h(\tau)$ denote the distribution measure of h with respect to τ. Fixing h we then have a map $\lambda_h: TS(\mathcal{A})\to\mathcal{M}(\mathbb{R})$; in fact, $\lambda_h(\tau)\in\mathcal{M}([-\|h\|,\|h\|])$ for every $\tau\in TS(\mathcal{A})$. It is straightforward to see that λ_h is continuous with respect to w*-topology on $TS(\mathcal{A})$ and weak topology on $\mathcal{M}(\mathbb{R})$. Let $\tau_N:=\tau_{P(N),Q(N)}$ be the random tracial state on $\mathcal{A}$ induced by $(P(N),Q(N))$ and ν_N the distribution on $TS(\mathcal{A})$ of τ_N (see Section 3). We then notice that

$$\nu_N\circ\lambda_h^{-1}(\Lambda)=\mathrm{Prob}(\{\tau_N\in\lambda_h^{-1}(\Lambda)\})=\mathrm{Prob}(\{\lambda_h(\tau_N)\in\Lambda\})$$

for Borel sets $\Lambda\subset\mathcal{M}(\mathbb{R})$. Since

$$\int x^m\,d\lambda_h(\tau_N)(x)=\tau_N(h^m)=\mathrm{tr}_N(h(P(N),Q(N))^m),\qquad m\in\mathbb{N},$$

it follows that $\lambda_h(\tau_N)$ is nothing but the empirical eigenvalue distribution of an $N\times N$ selfadjoint random matrix $h(P(N),Q(N))$ (via "noncommutative functional calculus" mentioned in Section 3). Therefore, by the *contraction principle* (see [6, 4.2.1]), Theorem 3.2 implies the following:

Theorem 4.1. *For every selfadjoint element $h\in\mathcal{A}$, the empirical eigenvalue distribution of $h(P(N),Q(N))$ satisfies the large deviation principle in the scale $1/N^2$ with the good rate function*

$$I_h(\mu):=\inf\{\mathcal{I}(\tau):\tau\in TS(\mathcal{A}),\ \lambda_h(\tau)=\mu\}$$

for $\mu\in\mathcal{M}(\mathbb{R})$, and $\mu_0:=\lambda_h(\tau_0)$ is a unique minimizer of I_h, where $\mathcal{I}$ and τ_0 are as in Theorem 3.2.

Remark 4.2. For any unitary $u\in\mathcal{A}$ define a map $\lambda_u: TS(\mathcal{A})\to\mathcal{M}(\mathbb{T})$, $\mathbb{T}$ being the unit circle, by letting $\lambda_u(\tau)$ the distribution of u with respect to τ. Then a similar large deviation is satisfied for the empirical eigenvalue distribution of the unitary random matrix $u(P(N),Q(N))$ and the rate function I_u is given in the same way as in Theorem 4.1.

In this way, for concrete applications, it remains only to find an explicit form of the rate function I_h (or I_u) as well as that of the minimizer μ_0. We present a few examples in the rest of the section.

Example 4.3. Consider $h = ef + fe \in \mathcal{A}$ and let $\tau = (\{\alpha_{ij}\}_{i,j=0}^1, \mu) \in TS(\mathcal{A})$ as in Section 3. Since $e(t)f(t) + f(t)e(t)$ has the eigenvalues $t \pm \sqrt{t}$, we get

$$\begin{aligned}\tau(\varphi(ef+fe)) &= (\alpha_{10}+\alpha_{01}+\alpha_{00})\varphi(0) + \alpha_{11}\varphi(2) \\ &\quad + \left(1 - \sum_{i,j=0}^{1} \alpha_{ij}\right) \int_0^1 \frac{\varphi(t+\sqrt{t}) + \varphi(t-\sqrt{t})}{2}\, d\mu(t)\end{aligned}$$

for every continuous function φ on $\mathbb{R}$. By this expression and (3.6), whenever τ satisfies (3.4), we have

$$\begin{aligned}\lambda_{ef+fe}(\tau) &= \max\{|\alpha-\beta|, 1-2\alpha, 1-2\beta\}\delta_0 + \max\{\alpha+\beta-1\}\delta_2 \\ &\quad + \rho(\mu\circ S^{-1} + \mu\circ T^{-1}),\end{aligned} \tag{4.1}$$

where $S : (0,1) \to (0,2)$ and $T : (0,1) \to [-1/4, 0)$ are given by $St := t + t\sqrt{t}$ and $Tt := t - \sqrt{t}$. Hence the empirical eigenvalue distribution of $P(N)Q(N) + Q(N)P(N)$ satisfies the large deviation in the scale $1/N^2$ and the good rate function $\tilde{I}(\tilde{\mu})$ for $\tilde{\mu} \in \mathcal{M}(\mathbb{R})$ is given by (2.7) if $\tilde{\mu}$ is of the form in the right-hand side of (4.1) with $\mu \in \mathcal{M}((0,1))$; otherwise $\tilde{I}(\tilde{\mu}) = +\infty$. The minimizer of $\tilde{I}(\tilde{\mu})$ is the right-hand side of (4.1) with $\mu = \mu_0$, where $\rho\mu_0$ is the continuous part of the measure (2.8).

Example 4.4. Consider $h = ae + bf$ with $a, b \in \mathbb{R}\setminus\{0\}$. Since $ae(t) + bf(t)$ has the eigenvalues $\frac{1}{2}(a + b \pm \sqrt{(a-b)^2 + 4abt})$, we get

$$\begin{aligned}\tau(\varphi(ae+bf)) &= \alpha_{00}\varphi(0) + \alpha_{10}\varphi(a) + \alpha_{01}\varphi(b) + \alpha_{11}\varphi(a+b) \\ &\quad + \left(1 - \sum_{i,j=0}^{1} \alpha_{ij}\right) \int_0^1 \frac{1}{2}\left(\varphi\left(\frac{a+b-\sqrt{(a-b)^2+4abt}}{2}\right)\right. \\ &\quad \left. + \varphi\left(\frac{a+b+\sqrt{(a-b)^2-4abt}}{2}\right)\right) d\mu(t)\end{aligned}$$

for every continuous function φ on $\mathbb{R}$ and $\tau = (\{\alpha_{ij}\}_{i,j=0}^1, \mu) \in TS(\mathcal{A})$. Let A, B be the first two smallest numbers of $0, a, b, a+b$, and define $S : (0,1) \to (A,B)$ and $T : (0,1) \to (a+b-B, a+b-A)$ by

$$St := \frac{a+b-\sqrt{(a-b)^2+4abt}}{2}, \quad Tt := \frac{a+b+\sqrt{(a-b)^2+4abt}}{2}.$$

When τ satisfies (3.4), the above expression shows that

$$\begin{aligned}\lambda_{ae+bf}(\tau) &= \max\{1-\alpha-\beta, 0\}\delta_0 + \max\{\alpha-\beta, 0\}\delta_a \\ &\quad + \max\{\beta-\alpha, 0\}\delta_b + \max\{\alpha+\beta-1, 0\}\delta_{a+b} \\ &\quad + \rho(\mu\circ S^{-1} + \mu\circ T^{-1}).\end{aligned} \tag{4.2}$$

Hence the empirical eigenvalue distribution of $aP(N) + bQ(N)$ satisfies the large deviation and the good rate function as well as its minimizer is determined similarly to the above example.

Let us express the rate function $\tilde{I}(\tilde{\mu})$ and the minimizer $\tilde{\mu}_0$ more explicitly. When $\mu \in \mathcal{M}((0,1))$, the measure $\nu := \frac{1}{2}(\mu\circ S^{-1} + \mu\circ T^{-1})$ is supported in $(A,B) \cup (a+b-B, a+b-A)$ and symmetric at $(a+b)/2$ so that $\mu = 2\nu\circ S|_{(A,B)} = 2\nu\circ T|_{(a+b-B,a+b-A)}$. Since $St = x$ (or $Tt = x$) implies $t = (x-a)(x-b)/ab$, we get

$$\int_0^1 \log t\, d\mu(t) = 2\int_A^B \log\frac{(x-a)(x-b)}{ab}\, d\nu(x) = 2\int_{a+b-B}^{a+b-A} \log\frac{(x-a)(x-b)}{ab}\, d\nu(x)$$

so that

$$\int_0^1 \log t\, d\mu(t) = \int_{(A,B)\cup(a+b-B,a+b-A)} \log \frac{(x-a)(x-b)}{ab}\, d\nu(x).$$

Similarly,

$$\int_0^1 \log(1-t)\, d\mu(t) = \int_{(A,B)\cup(a+b-B,a+b-A)} \log \frac{x(a+b-x)}{ab}\, d\nu(x).$$

On the other hand, we get

$$\begin{aligned}
\Sigma(\mu) &= 4\int_A^B \int_A^B \log \left| \frac{(x-a)(x-b)}{ab} - \frac{(y-a)(y-b)}{ab} \right| d\nu(x)\, d\nu(y) \\
&= 4\int_A^B \int_A^B \log \left| \frac{(x-a)(a+b-x-y)}{ab} \right| d\nu(x)\, d\nu(y) \\
&= 2\Sigma(\nu) - \log|ab|.
\end{aligned}$$

Consequently, the rate function $\tilde{I}(\tilde{\mu})$ is written as

$$\begin{aligned}
\tilde{I}(\tilde{\mu}) = & -2\rho^2\Sigma(\nu) - \rho|\alpha-\beta| \int_{(A,B)\cup(a+b-B,a+b-A)} \log|(x-a)(x-b)|\, d\nu(x) \\
& -\rho|\alpha+\beta-1| \int_{(A,B)\cup(a+b-B,a+b-A)} \log|x(a+b-x)|\, d\nu(x) \\
& +C + \rho\max\{\alpha,\beta,1-\alpha,1-\beta\} \log|ab|
\end{aligned}$$

if $\tilde{\mu} \in \mathcal{M}(\mathbb{R})$ is of the form

$$\begin{aligned}
\tilde{\mu} = & \max\{1-\alpha-\beta,0\}\delta_0 + \max\{\alpha-\beta,0\}\delta_a \\
& + \max\{\beta-\alpha,0\}\delta_b + \max\{\alpha+\beta-1,0\}\delta_{a+b} + 2\rho\nu
\end{aligned}$$

with $\nu \in \mathcal{M}((A,B)\cup(a+b-B,a+b-A))$ symmetric at $(a+b)/2$; otherwise $\tilde{I}(\tilde{\mu}) = +\infty$.

Moreover, by transforming the continuous part of (2.8), the explicit form of the minimizer $\tilde{\mu}_0$ can be easily computed as follows:

$$\begin{aligned}
\tilde{\mu}_0 = & \max\{1-\alpha-\beta,0\}\delta_0 + \max\{\alpha-\beta,0\}\delta_a \\
& + \max\{\beta-\alpha,0\}\delta_b + \max\{\alpha+\beta-1,0\}\delta_{a+b} \\
& + \frac{\left|x-\frac{a+b}{2}\right| \sqrt{-(x-A_0)(x-B_0)(x-a-b+B_0)(x-a-b+A_0)}}{\pi|x(x-a)(x-b)(x-a-b)|} \\
& \times \mathbf{1}_{(A_0,B_0)\cup(a+b-B_0,a+b-A_0)}(x)\, dx,
\end{aligned} \tag{4.3}$$

where

$$A_0 := \frac{a+b-\sqrt{(a-b)^2+4ab\eta}}{2}, \quad B_0 := \frac{a+b-\sqrt{(a-b)^2+4ab\xi}}{2}$$

(or exchange A_0, B_0 depending on the sign of ab) with ξ,η in (2.9). As is guaranteed by the asymptotic freeness ([15]) of $(P(N),Q(N))$, the minimizer $\tilde{\mu}_0$ is equal to the distribution of $ap+bq$ where (p,q) is a pair of free projections in a tracial W^*-probability space $(\mathcal{M},\tau)$ with $\tau(p)=\alpha$ and $\tau(q)=\beta$. In fact, the distribution was computed in [1] by use of R-transform.

Although one can prove the large deviation result for the empirical eigenvalue density of $aP(N)+bQ(N)$ (also $P(N)Q(N)+Q(N)P(N)$) based on the joint eigenvalue distributions given in Corollary 1.3, our stress is that this is just a particular case of grand Theorem 4.1 (or Theorem 3.2).

Example 4.5. For unitaries we consider a simple example $u = e^{\mathrm{i}\,\pi e}e^{-\mathrm{i}\,\pi f}$. Since the eigenvalues of $e^{\mathrm{i}\,\pi e(t)}e^{-\mathrm{i}\,\pi f(t)}$ are $2t-1\pm 2\mathrm{i}\sqrt{t(1-t)} = e^{\pm\mathrm{i}\,\theta(t)}$ where $\theta(t) := \cos^{-1}(2t-1)$ for $t\in(0,1)$, we get

$$\begin{aligned}\tau(\varphi(u)) &= (\alpha_{11}+\alpha_{00})\varphi(1) + (\alpha_{10}+\alpha_{01})\varphi(-1) \\ &\quad + \left(1-\sum_{i,j=0}^{1}\alpha_{ij}\right)\int_0^1 \frac{\varphi(e^{\mathrm{i}\,\theta(t)})+\varphi(e^{-\mathrm{i}\,\theta(t)})}{2}\,d\mu(t)\end{aligned}$$

for every continuous function φ on $\mathbb{T}$ and $\tau = (\{\alpha_{ij}\}_{i,j=0}^{1},\mu)\in TS(\mathcal{A})$. When τ satisfies (3.4), this implies that

$$\lambda_u(\tau) = |\alpha+\beta-1|\delta_1 + |\alpha-\beta|\delta_{-1} + \rho(\mu\circ\theta^{-1}+\mu\circ\tilde\theta^{-1}),$$

where $\tilde\theta(t) := -\theta(t)$ for $t\in(0,1)$. For $\mu\in\mathcal{M}((0,1))$ let $\nu := \frac{1}{2}(\mu\circ\theta^{-1}+\mu\circ\tilde\theta^{-1})$, which is a probability measure on $\mathbb{T}$ symmetric for the real axis. We then have

$$\int_0^1 \log t\,d\mu(t) = \int_{\mathbb{T}} \log\frac{1+\cos\theta}{2}\,d\nu(e^{\mathrm{i}\,\theta}),$$

$$\int_0^1 \log(1-t)\,d\mu(t) = \int_{\mathbb{T}} \log\frac{1-\cos\theta}{2}\,d\nu(e^{\mathrm{i}\,\theta}),$$

$$\Sigma(\mu) = \iint_{\mathbb{T}^2} \log|\cos\theta-\cos\psi|\,d\nu(e^{\mathrm{i}\,\theta})\,d\nu(e^{\mathrm{i}\,\psi}) - \log 2.$$

Hence we see by Remark 4.2 that the empirical eigenvalue distribution of $e^{\mathrm{i}\,\pi P(N)}e^{-\mathrm{i}\,\pi Q(N)}$ satisfies the large deviation in the scale $1/N^2$ and the rate function is given by

$$\begin{aligned}\tilde I(\tilde\mu) &= -\rho^2\iint_{\mathbb{T}^2}\log|\cos\theta-\cos\psi|\,d\nu(e^{\mathrm{i}\,\theta})\,d\nu(e^{\mathrm{i}\,\psi}) \\ &\quad -\rho|\alpha-\beta|\int_{\mathbb{T}}\log(1+\cos\theta)\,d\nu(e^{\mathrm{i}\,\theta}) - \rho|\alpha+\beta-1|\int_{\mathbb{T}}\log(1-\cos\theta)\,d\nu(e^{\mathrm{i}\,\theta}) \\ &\quad +C+\rho\max\{\alpha,\beta,1-\alpha,1-\beta\}\log 2\end{aligned}$$

if $\tilde\mu\in\mathcal{M}(\mathbb{T})$ is of the form $\tilde\mu = |\alpha+\beta-1|\delta_1+|\alpha-\beta|\delta_{-1}+2\rho\nu$ with $\nu\in\mathcal{M}(\mathbb{T})$ having no atoms at ±1 and symmetric for the real axis; otherwise $\tilde I(\tilde\nu) = +\infty$. The minimizer $\tilde\mu_0$ is also easy to compute as

$$\begin{aligned}\tilde\mu_0 &= |\alpha+\beta-1|\delta_1+|\alpha-\beta|\delta_{-1} \\ &\quad + \frac{\sqrt{-(\cos\theta+1-2\xi)(\cos\theta+1-2\eta)}}{|\sin\theta|}\mathbf{1}_{(\theta_1,\theta_2)\cup(-\theta_2,-\theta_1)}(\theta)\,\frac{d\theta}{2\pi},\end{aligned}\tag{4.4}$$

where $\theta_1 := \cos^{-1}(2\eta-1)$ and $\theta_2 := \cos^{-1}(2\xi-1)$. This measure is the distribution of $e^{\mathrm{i}\,\pi p}e^{-\mathrm{i}\,\pi q}$ for free projections p,q sometimes mentioned above. It may be natural that this distribution is rather different (except the same atomic parts) from that of $e^{\mathrm{i}\,\pi(p-q)}$ computed from (4.3). In particular, when $\alpha=\beta=1/2$ so that $\xi=0$ and $\eta=1$, the minimizer (4.4) is the uniform measure on $\mathbb{T}$ but (4.3) induces the arcsine law on the angular variable $(-\pi,\pi)$.

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