# On Solvable Semiprimitive Permutation Groups

Áron Bereczky and Attila Maróti

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**Abstract**

We study permutation groups in which all normal subgroups are transitive or semiregular. The motivation for such investigations comes from universal algebra. This paper focuses on solvable groups.

## 1 Introduction

All groups in this paper are finite unless otherwise stated.

A transitive permutation group is primitive if its point-stabilizer is a maximal subgroup. Every proper normal subgroup of a primitive permutation group is transitive. The O'Nan-Scott theorem describes finite primitive permutation groups. Let $V$ be a finite vector space and let $H$ be a linear group acting faithfully and irreducibly on $V$. Then the semidirect product $VH$ can be considered as a primitive permutation group with point-stabilizer $H$ (and regular normal subgroup $V$). Such primitive groups are said to be of affine type. All finite solvable primitive permutation groups are of affine type.

A permutation group is called quasiprimitive if all of its non-trivial normal subgroups are transitive. Every primitive permutation group is quasiprimitive. Praeger [9] gave an O'Nan-Scott type theorem classifying (finite) quasiprimitive permutation groups. All (finite) solvable quasiprimitive permutation groups are primitive.

A (finite) permutation group is called innately transitive if it has a transitive minimal normal subgroup. It is easy to see that all normal subgroups of an innately transitive permutation group are transitive or semiregular. These groups are known since 2002 when an O'Nan-Scott type classification theorem for innately transitive permutation groups was given by Bamberg and Praeger [1]. A solvable innately transitive permutation group is primitive.

In this paper we will investigate an even broader class of permutation groups, the class of semiprimitive groups. We call a transitive permutation group (of finite degree) *semiprimitive* if it is not regular and if all of its normal subgroups are transitive or semiregular. Frobenius groups are semiprimitive (see Lemma 2.1). Most of the time we will restrict our consideration to the class of solvable semiprimitive groups.

Our original motivation for the study of semiprimitive groups comes from universal algebra. A permutation group $G$ on a finite set $A$ is collapsing if there is exactly one clone on $A$ with unary part $G$. (We refer the reader to the Appendix at the end of this paper for a self-contained definition. Note that

Kearnes and Szendrei [6] uses the slightly more general notion of a collapsing pair of groups.)

Pálfy and Szendrei [8] proved that if $G$ is a transitive permutation group on a set $A$, there is a $G$-invariant equivalence relation $\theta$ different from the universal relation on $A$ and that $G$ acts regularly on $A/\theta$, then $G$ cannot be collapsing. In particular, a regular permutation group is never collapsing. In the same paper it is also shown that a collapsing permutation group is necessarily transitive. Pálfy and Szendrei [8] also proved that (finite) non-regular quasiprimitive permutation groups are collapsing.

Later, Kearnes and Szendrei [6] gave a description of the normal subgroups of a collapsing permutation group. Let $G$ be a transitive permutation group on the finite set $A$, and let $H$ be a point stabilizer. For an arbitrary ($G$-) conjugate $S$ of $H$ consider the equivalence relation on $G$ whose blocks are the left cosets $gS$ ($g \in G$) of $S$. Let us denote this equivalence relation by $\alpha_S$. We say that a normal subgroup $N$ of $G$ has property (*) if the constant function with value 1 is the only function $f : G \longrightarrow N$ so that $f(1) = 1$ and $f$ preserves all equivalence relations $\alpha_S$ for all conjugates $S$ of $H$. Kearnes and Szendrei proved in [6] that, with the notations above, a (finite) non-regular transitive permutation group with no proper normal subgroup containing $H$ is collapsing, if and only if, all of its normal subgroups are transitive, semiregular, or have property (*).

By [9] and [1], solvable quasiprimitive and solvable innately transitive permutation groups are primitive. However solvable collapsing permutation groups are not necessarily primitive. In [6] Kearnes and Szendrei gave an infinite sequence of collapsing, solvable permutation groups with a non-transitive, non-semiregular normal subgroup having property (*).

We will attempt to give some kind of a structural insight to the so called semiprimitive groups, with a special attention to solvable semiprimitive groups. We hope that researchers from both group theory and universal algebra will find this work interesting.

# 2 General results on semiprimitive groups

**Definition 2.1.** *A transitive permutation group $G$ of finite degree is semiprimitive if it is not regular and if every normal subgroup of $G$ is transitive or semiregular.*

Primitive permutation groups have transitive normal subgroups, so non regular primitive groups are semiprimitive. More generally, non regular quasiprimitive groups are semiprimitive as well. The following important observation was long known (see [4] for example).

**Lemma 2.1.** *Frobenius groups are semiprimitive.*

As it was probably known to Frobenius himself, this lemma can be derived from standard facts about Frobenius groups as follows. Let $G$ be a Frobenius group with Frobenius kernel $K$. If $\psi$ is a nonprincipal irreducible character of $K$, then the induced character $Ind_K^G(\psi)$ is an irreducible character of $G$. The kernel of $Ind_K^G(\psi)$ is contained in $K$ since $Ind_K^G(\psi)$ vanishes off $K$. Moreover, if $\chi$ is an irreducible character of $G$ whose kernel does not contain $K$, then $\chi = Ind_K^G(\psi)$ for some nonprincipal irreducible character $\psi$ of $K$. Finally, if $\chi$ is an irreducible character of $G$ whose kernel contains $K$, then the restricted character $Res_K^G(\chi)$ is a multiple of the trivial character of $K$. Since every normal subgroup $N$ of $G$ is the intersection of the kernels of some of the irreducible characters of $G$, we conclude that $N$ contains or is contained in (the regular permutation group) $K$. This proves that $G$ is semiprimitive.

But as we will see, there are plenty of semiprimitive groups that are not quasiprimitive and not Frobenius. Another simple observation to continue with is the following.

**Lemma 2.2.** *A nilpotent group cannot be semiprimitive.*

*Proof.* Suppose $G$ is a nilpotent non-regular permutation group and $H$ is a point stabilizer in it. Since $G$ is nilpotent, a maximal subgroup $N$ that contains $H$ has to be normal in $G$. Now $N$ is not semiregular because it contains $H$, and $N$ cannot be transitive either, because the only transitive subgroup of $G$ containing a point stabilizer is $G$ itself. □

**Lemma 2.3.** *Let $N$ be a semiregular normal subgroup in a semiprimitive group $G$. Then the quotient group $G/N$ is faithful and semiprimitive on the orbits of $N$.*

*Proof.* Let $\psi$ be the homomorphism which maps each element of $G$ to its action on the orbits of $N$. Clearly, $N \leq \ker(\psi)$. Since $\ker(\psi)$ is not transitive, it must be semiregular on the same set of orbits as $N$. Therefore $\ker(\psi) = N$, and this means that $G/N$ acts faithfully on the orbits of $N$.

Let $M/N$ be a normal subgroup of $G/N$. By hypothesis, the normal subgroup $M$ in $G$ is transitive or semiregular. If $M$ is transitive, then $M/N$ is also transitive on the orbits of $N$. Now suppose that $M$ is semiregular. In order to prove that $M/N$ is semiregular on the orbits of $N$, all we need to show is that $M \cap Stab(\Delta) = N$ for each orbit $\Delta$ of $N$. Indeed, if $\Delta$ is an orbit of $N$ and $\alpha$ is an arbitrary element of $\Delta$, then $M \cap Stab(\Delta)$ is transitive on $\Delta$ (since it contains $N$), and hence

$$M \cap Stab(\Delta) = (M \cap Stab(\alpha)) \cdot N = \{1\} \cdot N = N.$$

This completes the proof. □

A solvable primitive, quasiprimitive or a Frobenius permutation group $G$ has a unique regular normal subgroup $K$ with the property that every normal subgroup of $G$ contains $K$ or it is contained in $K$. This happens to be true for solvable semiprimitive groups in general.

**Theorem 2.1.** *A solvable semiprimitive group $G$ has a unique regular normal subgroup $K$ with the property that every normal subgroup of $G$ is contained in $K$ or contains $K$ itself. In particular, $G$ is a split extension of $K$ by $H$ where $H$ is a point-stabilizer of $G$.*

*Proof.* Take the derived series

$$G > G' = G^{(1)} > G^{(2)} > \ldots > G^{(n)} = \{1\}$$

of $G$ where $G^{(k+1)} = [G^{(k)}, G^{(k)}]$ for all $k$. Each subgroup in this series is a normal subgroup of $G$, obviously the first is transitive and the last is not, so there is a maximal integer $k$ such that $G^{(k)}$ is transitive but $G^{(k+1)}$ is not. Take $K = G^{(k)}$. We claim that $K$ is regular. For this it is sufficient to show that $K$ is semiregular. By Lemma 2.3, the quotient group $G/G^{(k+1)}$ is faithful and semiprimitive on the orbits of $G^{(k+1)}$. In this permutation representation the normal subgroup $K/G^{(k+1)}$ is transitive and Abelian, hence regular. Let $\alpha$ be an arbitrary element of the original permuted set of $G$, and let $\Delta$ be the orbit of $G^{(k+1)}$ containing $\alpha$. Suppose that $g$ is an element of $K$ fixing $\alpha$. Then $gG^{(k+1)}$ fixes $\Delta$, and so $gG^{(k+1)} = G^{(k+1)}$ by the regular property of $K/G^{(k+1)}$. This means that $g \in G^{(k+1)}$. Since $G^{(k+1)}$ is semiregular, we conclude that $g = 1$. This proves the claim.

Now it is clear that every subgroup containing $K$ is transitive and every subgroup contained in $K$ is semiregular, but we still need to show that every normal subgroup of $G$ has one of these two properties. So let $N$ be a normal subgroup of $G$ and assume that $K$ is not contained in $N$. Then $G_1 = K \cap N$ is a proper normal subgroup of $K$. Let $G_2/G_1$ be the derived subgroup of $K/G_1$. Now $G_2/G_1$ is characteristic in $K/G_1$, so it is normal in $G/G_1$, hence $G_2$ is a normal subgroup of $G$ with the property that $K/G_2$ is a non-trivial Abelian group. Notice that $G_2$ is semiregular.

Let $\Omega$ be the set of orbits of $G_2$, and let $\psi : G \to S_\Omega$ be the permutation representation of $G$ on $\Omega$. As before with $G^{(k+1)}$, we can again see that $\ker(\psi) = G_2$, and $K/G_2 = \psi(K)$ is Abelian and transitive. In particular, $\psi(K)$ is self-centralizing in $S_\Omega$. On the other hand $\psi(N) \cap \psi(K) = 1$, because $\psi(N) = NG_2/G_2$, $\psi(K) = K/G_2$, and $NG_2 \cap K = G_2$ (for if $ng \in K$ with $n \in N$ and $g \in G_2$, then $n \in Kg^{-1} = K$, so $n \in N \cap K = G_1 \leq G_2$ and $ng \in G_2$).

We saw that $\psi(K)$ is the centralizer of itself in $S_\Omega$, but $\psi(N)$ is another normal subgroup of $\psi(G)$ that intersects trivially with $\psi(K)$. Hence they centralize each other and therefore we must have $\psi(N) \leq \psi(K)$, that is, $NG_2/G_2 \leq K/G_2$. This implies that $N \leq K$, as we wanted. □

We note here that Theorem 2.1 can be generalized for all (not necessarily finite) solvable transitive permutation groups in which all normal subgroups are transitive or semiregular. The proof of the above theorem naturally carries over to infinite permutation groups.

From now on let us call $K$ the *kernel* of the solvable semiprimitive group $G$. Unfortunately, a non-solvable semiprimitive group does not necessarily have such a kernel. Take for example a non-solvable matrix group $G = \mathrm{GL}_d(q)$ acting on the non-zero vectors of a $d$ dimensional vector space $V$ over the field of $q$ elements. Then every normal subgroup of $G$ either contains $\mathrm{SL}_d(q)$ in which case it is transitive, or is contained in the subgroup of scalar matrices, then it is semiregular. But there is no regular normal subgroup in $G$. The above motivates the following definition.

**Definition 2.2.** *Let $G$ be a permutation group with a unique regular normal subgroup $K$ with the property that every normal subgroup of $G$ is contained in $K$ or contains $K$. We say that $K$ is the kernel of $G$.*

**Lemma 2.4.** *Let $G$ be a permutation group with kernel $K$ and stabilizer subgroup $H$. Then the set of transitive normal subgroups of $G$ forms a lattice isomorphic to the lattice of normal subgroups in $H$. Moreover the set of semiregular normal subgroups of $G$ forms a lattice isomorphic to a sublattice of the lattice of subgroups $L$ so that $H \leq L \leq G$.*

*Proof.* The first statement follows from the fact that each transitive normal subgroup of $G$ has the form $NK$ for some normal subgroup $N$ of $H$. For the second statement notice that the set of all subgroups of the form $HN$ where $N$ is a semiregular normal subgroup of $G$ forms a lattice isomorphic to the lattice of subgroups $L$ so that $H \leq L \leq G$. $\square$

In the next theorem, using character theory, we give a mild restriction on the structure of the lattice of semiregular normal subgroups of a (finite) solvable semiprimitive group.

**Theorem 2.2.** *Let $G$ be a permutation group with kernel $K$ and stabilizer subgroup $H$. Let $k$ be the number of $G$-conjugacy classes in $K$. Then there exist (not necessarily distinct) normal subgroups $N_1, \ldots, N_{k-1}$ of $K$ such that every semiregular normal subgroup of $G$ can be expressed as the intersection of some of the $N_i$'s.*

*Proof.* Notice that $G$ is a split extension of $K$ by $H$. Consider the action by conjugation of $H$ on the set of irreducible characters of $K$. Let the number of orbits of this action be $k$. Brauer's permutation lemma tells us that $k$ is equal to the number of $G$-conjugacy classes of $K$. The trivial character $1_K$ of $K$ forms an orbit on its own. Let us label the other orbits by the numbers 1 through $k-1$ in any way we like. By Gallagher's theorem, the irreducible constituents of $Ind_K^G(1_K)$ (in $G$) can be considered to be the irreducible characters of $G/K \cong H$. Now let $\chi$ be any irreducible character of $G$. By Clifford's theorem, $Res_K^G(\chi)$ is a positive integer multiple of the sum of the characters in some $G$-orbit of the set of irreducible characters of $K$. The kernel of $\chi$ contains $K$ if and only if this $G$-orbit is $\{1_K\}$. For each $1 \leq i \leq k-1$, let $N_i$ be the kernel of the sum of the characters in orbit $i$. By Clifford's theorem, if $\chi$ is any irreducible character of $G$ with kernel $\ker(\chi)$ not containing $K$, then $\ker(\chi)$ contains a normal subgroup $N_i$ of $K$ for some $i$. Since $G$ is semiprimitive, we must also have $\ker(\chi) = N_i$. We conclude that if $N$ is a normal subgroup of $G$ not containing $K$, then $N$ can be expressed as the intersection of some of the $N_i$ normal subgroups of $K$. $\square$

The $k$ in Theorem 2.2 is best possible in the sense that there exist infinitely many transitive permutation groups $G$ with kernel $K$ so that there exist exactly $k$ distinct normal subgroups $N_1, \ldots, N_{k-1}$ of $K$ so that every semiregular normal subgroup of $G$ can be expressed as the intersection of some of the $N_i$'s where $k$ is the number of $G$-conjugacy classes of $K$. Indeed, let $K$ be an elementary Abelian group of odd order and consider $K$ as a vector space over the prime field of order $p$. Let $H$ be the cyclic group of order $p-1$ consisting of all non-zero scalar multiplications on $K$. Then the semidirect product $G$ of $H$ and $K$ is a Frobenius group with kernel $K$. The normal subgroups of $G$ in $K$ are precisely the subspaces of $K$, and every subspace of $K$ can be expressed as the intersection of some of the $k-1$ maximal subspaces of $K$.

As we see from Theorem 2.1, every solvable semiprimitive group is a semidirect product of its kernel with a point-stabilizer. Thus it seems natural to look for abstract group-theoretic conditions to decide whether a given action of a finite group on another one produces a semiprimitive semidirect product. There is the following useful characterization of solvable semiprimitive groups.

**Theorem 2.3.** *Let $K$ be a finite solvable group and $H$ another finite group acting on it. Then the semidirect product $G = H \ltimes K$ has a semiprimitive* *action on the right cosets of $H$ if and only if $H$ acts faithfully on every non-trivial $H$-invariant quotient group of $K$.*

*Proof.* To start with one direction, assume that $M < K$ is a normal subgroup in $G = H \ltimes K$, but the action of $H$ is not faithful on $K/M$. Let us denote the kernel of this action by $B$ and consider the subgroup $BM$. This subgroup is normalized by $H$, since $B$ is contained in $H$ as a normal subgroup and $M$ is normal in $G$. $BM$ is also normalized by $K$, for if $k \in K$, $b \in B$ and $m \in M$, then

$$(1, k) \cdot (b, m) \cdot (1, k^{-1}) = (b, k^b \cdot m \cdot k^{-1})$$

and $k^b \cdot m \cdot k^{-1} = (k^b \cdot k^{-1}) \cdot (k \cdot m \cdot k^{-1})$, and here $k^b \cdot k^{-1} \in M$ because of the trivial action of $B$ on $K/M$ and $k \cdot m \cdot k^{-1} \in M$ because $M$ is normal in $K$. This shows that $K$ normalizes $BM$. Before we saw that $H$ normalizes $BM$ as well, hence $BM$ is a normal subgroup in $G = H \ltimes K$. However, $BM$ is not semiregular, because $B$ is a non-trivial subgroup of a point stabilizer, and $BM$ is not transitive either, for the set $\{Hm \mid m \in M\}$ is an orbit of $BM$ on the right cosets of $H$. $(HM \cdot BM = H \cdot (BM) \cdot M = (HB) \cdot (MM) = HM.)$ This proves that if there is an $H$-invariant quotient of $K$, on which $H$ does not act faithfully, then $G$ is not semiprimitive on the right cosets of $H$.

To prove the other direction, assume that $H$ acts faithfully on every non-trivial $H$-invariant quotient of $K$ and let $N$ be a normal subgroup of $G$, which is not transitive on the right cosets of $H$. In particular, $N$ does not contain $K$. Take $M_1 = N \cap K$ which is then also normal in $G$, and a proper subgroup of $K$. Further, let $M/M_1$ be the derived subgroup of $K/M_1$, so $M$ is still normal in $G$ and a proper subgroup of $K$, and the quotient $K/M$ is Abelian.

For any elements $b \in H$ and $l \in K$ with $(b, l) \in N$ and for any $k \in K$ we have $(1, k)(b, l)(1, k)^{-1}(b, l)^{-1} \in M$ since $[K, N] \leq K \cap N \leq M$. By calculating this product, we get

$$\begin{aligned}(1,k)(b,l)(1,k)^{-1}(b,l)^{-1} &= (1,k)(b,l)(1,k^{-1})(b^{-1},(l^{-1})^{b^{-1}}) \\ &= (1, k \cdot (lk^{-1}l^{-1})^{b^{-1}})\end{aligned}$$

so $k \cdot (lk^{-1}l^{-1})^{b^{-1}} \in M$, $k^b \cdot (lk^{-1}l^{-1}) \in M^b = M$. In other words, the cosets $Mk^b$ and $Mlkl^{-1}$ coincide. But $Mlkl^{-1} = Mk$ as $k, l \in K$ and $K/M$ is Abelian. Thus $Mk^b = Mk$ and this holds for any$(b, l) \in N$ and $k \in K$. By our assumption the action of $H$ on $K/M$ is faithful, so if there is a $b \in H$ with $Mk^b = Mk$ for all $k \in K$, it can only be $b = 1$. So it follows from the calculations above that for any $(b, l) \in N$, necessarily $b = 1$. In other words, $N \leq K$, thus $N$ is semiregular on the right cosets of $H$ and this is what we wanted to show. □

**Corollary 2.1.** *If $G = H \ltimes K$ is a solvable semiprimitive group on the right cosets of $H$ and $N < K$ is a normal subgroup of $G$, then the quotient group $G/N$ is also semiprimitive on the right cosets of $HN/N$.*

*Proof.* This is simply because every $H$-invariant quotient of $K/N$ is also an $H$-invariant quotient of $K$. □

**Corollary 2.2.** *A finite group $K$ with a characteristic subgroup of index 2 cannot be the kernel of a solvable semiprimitive group. In particular, cyclic groups of even order and groups of order $2m$ with $m$ odd cannot appear as kernels of solvable semiprimitive groups.*

*Proof.* If $G = H \ltimes K$ and $M$ is a characteristic subgroup of index 2 in $K$, then $M$ is normal in $G$ and $H$ can only have the trivial action on $K/M$. □

**Theorem 2.4.** *Suppose that $G$ is a solvable semiprimitive group in which both the kernel $K$ and the point stabilizer $H$ are commutative. Then $H$ is necessarily cyclic and $G$ is a Frobenius group.*

*Proof.* Choose a maximal $H$-invariant subgroup $M$ in $K$. Then $H$ acts irreducibly and faithfully on $K/M$: irreducibly because of the maximality of $M$ and faithfully by Theorem 2.3. A finite Abelian group acting irreducibly and faithfully on another Abelian group is necessarily cyclic (see for example Theorem 3.2.3 in [5]), and here $K/M$ is Abelian, so $H$ is cyclic.

In order to verify that $G$ is a Frobenius group, it is sufficient to show that $C_K(h) = 1$ for any non-trivial element $h$ in $H$. But first let us notice that the order of $H$ must be relatively prime to the order of $K$. For if $p$ is a prime number dividing $|K|$, then $K$ has a characteristic subgroup $L$ with $K/L$ being an elementary Abelian $p$-group (since $K$ is Abelian). Let $M$ be a maximal $H$-invariant subgroup of $K$ containing $L$. Then again, $H$ acts faithfully and irreducibly on the vector space $K/M$, therefore $H$ must be contained in a Singer subgroup of $K/M$. In particular, $|H|$ divides $|K/M|-1$, and so $p$ does not divide $|H|$.

Now choose an element $1 \neq h \in H$. We will show that $C_P(h) = 1$ for all Sylow subgroups $P$ of $K$, this indeed implies $C_K(h) = 1$. So fix a prime divisor $p$ of $|K|$ and let $P$ be the Sylow $p$-subgroup of $K$. Further, let $Q$ be the direct product of all the other Sylow subgroups of $K$, so that $K = P \times Q$. Since $h$ is a $p'$-element and $P$ is an Abelian $p$-group, we have $P = C_P(h) \times [P, \langle h \rangle]$ (see Theorem 5.2.3 in [5]). Here, $[P, \langle h \rangle]$ is an $H$-invariant subgroup of $K$, sice $H$ normalizes both $P$ and $\langle h \rangle$. Moreover, $H$ acts trivially on $P/[P, \langle h \rangle]$ and thus on $K/(Q \times [P, \langle h \rangle])$ as well. Hence $K/(Q \times [P, \langle h \rangle])$ is an $H$-invariant quotient of $K$ on which the action of $H$ is not faithful, therefore $K = Q \times [P, \langle h \rangle]$. This means $P = [P, \langle h \rangle]$ and hence $C_P(h) = 1$ as wanted. □

**Corollary 2.3.** *If a point stabilizer of a solvable semiprimitive group is commutative, then it is necessarily cyclic.*

*Proof.* Suppose that the point-stabilizer $H$ of a solvable semiprimitive group $G$ is Abelian. Denote the kernel of $G$ by $K$. Then $G/[K,K]$ is semiprimitive on the set of orbits of $[K,K]$ with a commutative kernel and a commutative point-stabilizer. Hence by Theorem 2.4, $H$ is cyclic and $G/[K,K]$ is a Frobenius group. □

**Lemma 2.5.** *If $G$ is a solvable semiprimitive group and its kernel $K$ is a cyclic group of odd order, then $G$ is necessarily a Frobenius group.*

*Proof.* The point stabilizer of $G$ is Abelian and hence cyclic. So $G$ is indeed a Frobenius group by the results above. □

**Theorem 2.5.** *Let $G$ be a semiprimitive group with solvable kernel $K$, let $H$ be a point-stabilizer, and suppose that $D$ is a subgroup of $H$ so that $D \cdot C_H(D) = H$. Then $DK$ is also semiprimitive.*

*Proof.* By way of contradiction, suppose that $DK$ is not semiprimitive, and then the solvability of $K$ implies that there is a $D$-invariant proper normal subgroup $L$ in $K$ so that the action of $D$ on $K/L$ is not faithful. Denote the kernel of this action by $B$. Then for all $b \in B$ and $k \in K$ we have that $k^{-1}k^b \in L$.

Now let $b$ and $c$ be any elements in $B$ and $C_H(D)$, respectively. Then for any $k \in K$,

$$(k^c)^{-1}(k^c)^b = (k^{-1})^c(k^b)^c = (k^{-1}k^b)^c \in L^c$$

and since $k^c$ can be any element in $K$, it follows that we have $k^{-1}k^b \in L^c$ for any $k \in K$. Thus if we take $M = \bigcap_{c \in C_H(D)} L^c$, then

$$k^{-1}k^b \in M$$

for any $k \in K$ and $b \in B$.

However, $M$ is an $H$-invariant normal subgroup of $K$. For if we take any element $h$ in $H$, by our assumption we can write $h = dx$ for some $d \in D$ and $x \in C_H(D)$ and then

$$M^h = \bigcap_{c \in C_H(D)} L^{ch} = \bigcap_{c \in C_H(D)} L^{cdx} = \bigcap_{c \in C_H(D)} L^{dcx} = \bigcap_{c \in C_H(D)} L^{cx} = M\,.$$

(As $c$ centralizes $D$, $L$ is $D$-invariant, and $cx$ runs through the elements of $C_H(D)$ as does $c$.) Moreover, $M$ is not only an $H$-invariant normal subgroup of $K$, but the action of $H$ on $K/M$ is not faithful: we have shown above that $B$ is contained in the kernel of this action. Therefore the group $G = HK$ is not semiprimitive (by Theorem 2.3), and this contradiction implies that $DK$ must be semiprimitive. □

Two immediate corollaries are the following.

**Theorem 2.6.** *Let $G$ be a semiprimitive group with solvable kernel $K$, let $H$ be a point-stabilizer, and assume that $C$ is a non-trivial central subgroup of $H$. Then $CK$ is also semiprimitive. In particular, the center of $H$ must be cyclic.*

*Proof.* Put $D = C$ and apply Theorem 2.5. □

**Theorem 2.7.** *Let $G$ be a semiprimitive group with solvable kernel $K$, let $H$ be a point-stabilizer, and suppose that $H$ has a direct decomposition $H = A \times B$. Then $AK$ and $BK$ are also semiprimitive.*

*Proof.* Put $D = A$ and apply Theorem 2.5. □

**Theorem 2.8.** *Let $G$ be a solvable transitive group with a regular normal subgroup $K$. Then $G$ is semiprimitive if and only if $G/\Phi(K)$ is faithful and semiprimitive on the set of orbits of the Frattini subgroup $\Phi(K)$ of $K$.*

*Proof.* Since $K$ is regular, $\Phi(K)$ is semiregular. Hence one direction of the theorem follows from Lemma 2.3. Suppose that $G/\Phi(K)$ is faithful and semiprimitive on the set of orbits of $\Phi(K)$. Let $N$ be a normal subgroup of $G$. We must show that $N$ is transitive or semiregular. The group $N\Phi(K)/\Phi(K)$ is normal in the solvable semiprimitive group $G/\Phi(K)$. Hence, by Theorem 2.1, the group $N\Phi(K)$ contains $K$ or it is contained in $K$. In the latter case we conclude that $N$ is semiregular. So suppose that $K \leq N\Phi(K)$. Since $\Phi(K) \leq K$, this implies that $K \leq (N \cap K)\Phi(K)$. We claim that $K \leq N$. This would mean that $N$ is transitive. By way of contradiction, suppose that there exist a maximal subgroup $L$ of $K$ containing $K \cap N$. Then $K \leq (K \cap N)\Phi(K) \leq L$ is a contradiction. Hence $N$ is indeed transitive. This completes the proof of the theorem. □

Let $p$ be an odd prime and let $E$ be a $p$-group with $|E| = p^{1+2d}$ $(d \geq 1)$, $Z(E) = [E, E] = \Phi(E)$ has order $p$, and $E$ has exponent $p$. Then $S = Sp_{2d}(p)$ acts on $E$ as a group of automorphisms. Indeed, let $S_0 = CSp_{2d}(p)$, the group which preserves up to scalar multiples the alternating form preserved by $S$. So $S_0/S$ is cyclic of order $p - 1$, $S_0$ acts as a group of automorphisms on $E$ and $S$ is the normal subgroup which centralizes $Z(E)$. A corollary to Theorem 2.8 is the following.

**Corollary 2.4.** *Let $E$ and $S_0$ be as above. Let $G$ be a finite transitive group with a point-stabilizer $H$ and a regular normal subgroup isomorphic to $E$. Then $G$ is a semiprimitive group with kernel $E$, if and only if, $H \leq S_0$ and $HE/\Phi(E)$ is faithful and semiprimitive on the set of orbits of the Frattini subgroup $\Phi(E)$ of $E$.*

**Theorem 2.9.** *Let $G$ be a solvable transitive group with a regular normal subgroup $K$. Suppose that $K = K_1 \times \ldots \times K_t$ where $|K_i|$ and $|K_j|$ are relatively prime integers for all $1 \leq i < j \leq t$. Then $G$ is semiprimitive, if and only if, for all $1 \leq i \leq t$ the factor group $G/\Phi(K_i)K_i'$ is faithful and semiprimitive on the sets of orbits of $\Phi(K_i)K_i'$ where $\Phi(K_i)$ denotes the Frattini subgroup of $K_i$ and $K_i'$ denotes the direct product of all the $K_j$'s where $j$ is different from $i$.*

*Proof.* Let us use the notations of the theorem. Since $K$ is regular, $\Phi(K_i)K_i'$ is semiregular for all $i$, and hence one direction of the theorem follows from Lemma 2.3. For the other direction, suppose that for all $1 \leq i \leq t$, the factor group $G/\Phi(K_i)K_i'$ is faithful and semiprimitive on the set of orbits of $\Phi(K_i)K_i'$. By the previous theorem and by our hypotheses on the orders of the $K_i$'s, this assumption is equivalent to the assumption that for all $1 \leq i \leq t$, the factor group $G/K_i'$ is faithful and semiprimitive on the set of orbits of $K_i'$. Let $N$ be a normal subgroup in $G$. We must show that $N$ is semiregular or transitive. If for some $i$ the normal subgroup $NK_i'/K_i'$ in $G/K_i'$ is semiregular with respect to its relevant action, then $NK_i'$ and hence $N$ is also semiregular on the original set of points on which $G$ acts transitively. Hence we may (and do) assume that for all $i$ the factor group $NK_i'/K_i'$ acts faithfully and transitively on the orbits of $K_i'$. By Theorem 2.1, this implies that $K \leq NK_i'$ for all $i$. Since $K_i' \leq K$ for all $i$, this is equivalent to $K \leq (N \cap K)K_i'$ for all $i$. By our hypotheses on $K$ and the orders of the $K_i$'s, we see that $K_i \leq N \cap K$ and hence $K_i \leq N$ for all $i$. This implies that $K \leq N$ which means that $N$ is transitive. Since $N$ was an arbitrary normal subgroup of $G$, we conclude that $G$ is semiprimitive. □

**Corollary 2.5.** *Let $G$ be a semiprimitive group with a nilpotent kernel $K$ and a nilpotent point-stabilizer $H$. Then $|K|$ and $|H|$ are relatively prime.*

*Proof.* Let $K = K_1 \times \ldots \times K_t$ where $K_i$ is a $p_i$-group for distinct prime numbers $p_i$ for $1 \leq i \leq t$. Similarly, let $H = H_1 \times \ldots \times H_u$ where $H_j$ is a $q_j$-group for distinct prime numbers $q_j$ for $1 \leq j \leq u$. By repeated use of Theorem 2.7, the groups $H_jK$ are semiprimitive for all $1 \leq j \leq u$. Similarly, by Theorem 2.9 (and the idea of its proof), the group $H_jK_i$ is semiprimitive on the set of orbits of the complement of $K_i$ in $K$ for all $1 \leq j \leq u$ and all $1 \leq i \leq t$. By Lemma 2.2, the groups $H_j K_i$ cannot be nilpotent, and so $q_j \neq p_i$ for all $1 \leq j \leq u$ and all $1 \leq i \leq t$. Hence $|H|$ and $|K|$ are indeed relatively prime integers. □

Some of the observations we had before may suggest that semi-primitive groups regarding their structures may be close to Frobenius groups. But there is a significant difference as well. It is known that the kernel of a solvable (quasi)primitive group is elementary Abelian and (by Thompson's theorem) the kernel of a Frobenius group is always nilpotent. However, the kernel of a solvable semiprimitive group does not need to be nilpotent. For example, take the symmetric group $S_4$ with its action on the cosets of a subgroup $H$ generated by a single transposition. This action is semiprimitive and the kernel is the alternating group $A_4$, which is not nilpotent.

This example inspired Pyber to point on the following fact.

**Theorem 2.10.** *A transitive group $G$ with a simple one-point stabilizer is semiprimitive if and only if the set-theoretic union of all point stabilizers generate $G$.*

*Proof.* Let $G$ be such a group, with a simple point stabilizer $H$ and suppose that $N$ is a normal subgroup in $G$, which is not semiregular. It means that for some $g \in G$, the intersection $H^g \cap N$ is non-trivial. But this intersection is normal in $H^g$, so it must be equal to $H^g$ and now it follows that $N$ contains all the point-stabilizers, therefore $N = G$. In the other direction, if the point stabilizers do not generate $G$, then they generate a normal subgroup that is not semiregular and not transitive either (for any transitive group containing a point stabilizer would be $G$ itself). □

For our last theorem, which is a consequence of Theorem 2.10, let us modify Definition 2.1 to include infinite permutation groups as well. A transitive permutation group $G$ is semiprimitive if it is not regular and if every normal subgroup of $G$ is transitive or semiregular.

**Theorem 2.11.** *Let $(W, S)$ be a Coxeter system such that any two involutions of the generating set $S$ are conjugate within the Coxeter group $W$. Then any quotient group of $W$ may be viewed as a semiprimitive group with a point stabilizer conjugate to a cyclic group generated by some element of $S$.*

Note that the hypothesis saying that $S$ is a subset of a conjugacy class of $W$ can not be omitted from the theorem. Indeed, a dihedral 2-group is a Coxeter group with not all involutions conjugate, but it is never semiprimitive.

For example, the group $D_4 = 2^3 : S_4$ can be viewed as a semiprimitive permutation group on 96 points. This group is solvable, and its kernel is not nilpotent.

# 3 Classifications for certain degrees

Now we can classify the solvable semiprimitive groups of certain degrees. From Corollary 2.2 it follows that there is no solvable semiprimitive group of degree $2m$ with $m$ odd. If the degree is a prime number, then any transitive group is also primitive, so semiprimitive or regular. If the degree is a square of a prime

$p$, then by Theorem 2.1, the kernel $K$ of a solvable semiprimitive group has order $p^2$. Given that the action of the point-stabilizer $H$ is faithful on every non-trivial $H$-invariant quotient of $K$ (see Theorem 2.3), one of the following three cases holds.

- $p$ is odd, $K$ is cyclic and $G$ is a Frobenius group;
- $K \cong C_p \times C_p$ and $H$ acts irreducibly on $K$, in which case $G$ is primitive;
- $K \cong C_p \times C_p$ and $H$ is reducible on $K$, in which case $H$ is cyclic, say $H = \langle h \rangle$, $|H|$ divides $p-1$ and there are elements $x, y$ in $K$ such that $h^{-1}xh = x^a$ and $h^{-1}yh = y^b$ for some integers $a$ and $b$, having the same multiplicative order modulo $p$. Such an example is always a Frobenius group.

Next we will analyze the situation when the degree is a product of two different odd prime numbers.

**Lemma 3.1.** *Let $G$ be a solvable semiprimitive group of degree $pq$ where $p$ and $q$ are different prime numbers. Then the degree is odd, the kernel $K$ of $G$ is a cyclic group of order $pq$ and $G$ is a Frobenius group.*

*Proof.* As we mentioned earlier, there is no solvable semiprimitive group of degree $2m$ with $m$ odd. So we may assume that $p$ and $q$ are both odd. If $K$ is cyclic, then the statement follows from Lemma 2.5. Let us assume that $K \cong C_p \ltimes C_q$, in particular $p$ divides $q-1$. Let $M$ be the cyclic normal subgroup of order $q$ in $K$, then $M$ is also characteristic in $K$, so $M$ is normal in $G$. It follows that the centralizer of $M$ in $G$, $C_G(M)$ is also normal in $G$, but it cannot contain $K$, as $K$ does not centralize $M$. Therefore $C_G(M) < K$, which means that $C_G(M) = M$. Now $G/M = N_G(M)/C_G(M)$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$ which is a cyclic group of order $q-1$, so $G/M$ is cyclic. On the other hand, by Lemma 2.3, $G/M$ is a semiprimitive group on the orbits of $M$, but this is now a contradiction, since we know from Lemma 2.2 that a cyclic group cannot be semiprimitive. This completes the proof. □

It was proved by Suprunenko [10] that there is no Frobenius group with kernel isomorphic to a non-Abelian group of order $p^3$ and of exponent $p^2$ (for any prime number $p$). Almost the same holds for solvable semiprimitive groups as well, but here there are two exceptions.

**Theorem 3.1.** *Let $G$ be a solvable semiprimitive group with kernel $K$ isomorphic to non-Abelian group of order $p^3$ and exponent $p^2$. Then $K \cong Q$ (the quaternion group) and $G \cong GL_2(3)$ or $SL_2(3)$.*

*Proof.* Suppose $G = H \ltimes K$ is a solvable semiprimitive group with $H$ being a point-stabilizer and the kernel $K$ is a non-Abelian group of order $p^3$ and exponent $p^2$ for some prime number $p$. It means that either $K$ is isomorphic to the quaternion group of order 8 or

$$K \cong C_p \ltimes C_{p^2} \cong \langle x, y \mid x^p = y^{p^2} = 1\,,\, x^{-1}yx = y^{p+1} \rangle$$

for some prime number $p$. By way of contradiction assume the latter case.

First we notice that $H$ is isomorphic to a subgroup of $C_{p-1}$. This is true because $K$ has a characteristic subgroup $L$ of order $p^2$ (if $p = 2$ then let $L$ be the cyclic subgroup of order 4, and if $p$ is odd then let $L = \Omega_1(K)$, the subgroup of the elements of order $p$ together with the unit element), and thus $L$ is normal in $G$, so $G/L$ acts faithfully as a semiprimitive group on the $p$ orbits of $L$. This implies that the point-stabilizer of this action, which is isomorphic to $(G/L)/(K/L) \cong G/K \cong H$, is a subgroup in $C_{p-1}$. Now if $p = 2$ then $H$ has to be trivial and $G = K$, contrary to our definition, that a regular permutation group is not semiprimitive.

Assume that $p$ is odd, $K \cong C_p \ltimes C_{p^2}$, and next we will show that there is a decomposition $K = T \ltimes S$ with $T \cong C_p$, $S \cong C_{p^2}$ such that $H$ normalizes $S$ and centralizes $T$. To see this, first notice that since $\gcd(|H|, |L|) = 1$ and $L$ is a vector space of dimension 2 over the field of $p$ elements, $H$ must have a semisimple action on $L$. Therefore there is a subgroup $T$ of order $p$ in $L$ which gives an $H$-module decomposition $L = T \times C(K)$. (Here $C(K)$ denotes the center of $K$.) Furthermore, the action of $H$ on $K/C(K) \cong C_p \times C_p$ is also semisimple, so there is also an $H$-module decomposition $K/C(K) = L/C(K) \times S/C(K)$ for some group $S$ of order $p^2$ containing the center of $K$. Since $L$ contains all the elements of order $p$ in $K$, $S$ must be a cyclic group of order $p^2$. Now $S$ does not contain $T$, since the only subgroup of order $p$ in $S$ is $C(K)$. So we have a decomposition $K = T \ltimes S$ such that $H$ normalizes both $T$ and $S$. But we also want to show that $H$ actually centralizes $T$. In order to do this, choose elements $x$ and $y$ from $T$ and $S$, respectively such that $T = \langle x \rangle$, $S = \langle y \rangle$ and $x^{-1}yx = y^{p+1}$. This is possible as $T$ acts non-trivially on $S$. Further, let $H = \langle h \rangle$. Now $h^{-1}xh = x^m$ and $h^{-1}yh = y^n$ with some positive integers $m$ and $n$ that are not divisible by $p$. Then

$$h^{-1}(yx)h = y^n x^m = x^m y^{n(p+1)^m} = x^m y^{n(mp+1)} ,$$

but also, since $yx = xy^{p+1}$ we have

$$h^{-1}(yx)h = h^{-1}(xy^{p+1})h = x^m y^{n(p+1)} .$$

It follows that $n(mp + 1) \equiv n(p + 1)$ modulo $p^2$, and since $n$ is not divisible by $p$, this implies $m \equiv 1$ modulo $p$. In other words, $h$ centralizes $x$, so $H$ also centralizes $T$.

Now $HS$ is a subgroup of $G$, because $H$ normalizes $S$. Furthermore, $T$ normalizes $HS$ because $S$ is normal in $K$, $T < K$, and as we just saw, $T$ centralizes $H$. Hence $SH$ is a normal subgroup of $G$. However $SH$ is not contained in $K$ and does not contain $K$ either, a contradiction.

From now on, we only need to deal with the case when $K$ is isomorphic to the quaternion group of order 8. Since $C(K)$ is normal in $G$, $H$ must act faithfully on the quotient $K/C(K) \cong C_2 \times C_2$, that is, $H$ is isomorphic to a subgroup of $\mathrm{GL}_2(2) \cong S_3$. $H$ cannot be a group of order 2 (in that case $G$ would be nilpotent, and nilpotent groups cannot be semiprimitive), so either $H \cong S_3$ or $H \cong C_3$. One can easily check that up to isomorphism, there is only one possible action of $S_3$ (resp. $C_3$) on the quaternion group, and the semidirect product of this action appears in $\mathrm{GL}_2(3)$ (resp. $\mathrm{SL}_2(3)$), hence $G = H \ltimes K$ is isomorphic to $\mathrm{GL}_2(3)$ or $\mathrm{SL}_2(3)$. □

It is known that there are Frobenius groups with their kernel a non-Abelian group of order $p^3$ and exponent $p$. As we know, these are also semiprimitive.

However, beside these there are more semiprimitive groups with this property. For example, let $p$ be any odd prime, let

$$K = \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = 1\,,\ [x, y] = z \rangle$$

and let $H$ be the cyclic group generated by the single automorphism $h$ of $K$ that takes $x$, $y$ and $z$ to $x^a$, $y^b$ and $z^{ab}$, respectively, where $a$ and $b$ are positive integers, less than $p$. Such an automorphism exists, because $x^a$, $y^b$ and $z^{ab}$ satisfy the same relations as $x$, $y$, and $z$. Moreover, the order of such an automorphism divides $p - 1$. Now if we choose $a$ and $b$ so that they are multiplicative roots modulo $p$, but their product is not (for instance, if $b$ is the mod $p$ inverse of $a$), then a certain non-trivial power of $h$ centralizes $C(K) = \langle z \rangle$. So $G = H \ltimes K$ is not a Frobenius group in this case, but it is semiprimitive on the cosets of $H$. For if $1 \neq L \neq K$ is an $H$-invariant normal subgroup of $K$ then $L$ contains $C(K)$ and $K/L$ can be considered as a vector space of characteristic $p$, on which $H$ acts as the whole group of scalar multiplications. Hence $H$ acts faithfully on each non-trivial $H$-invariant quotient of $K$. This shows that there are more semiprimitive groups then Frobenius groups of degree $p^3$ for any odd prime $p$.

**Theorem 3.2.** *Let $G$ be a solvable semiprimitive group with an elementary Abelian kernel $K$ of order $p^3$. If $p \neq 3$, then $G$ is a primitive permutation group or $G$ is a Frobenius group.*

*Proof.* Let $G$ and $K$ be as in the statement of the lemma. Let $H$ be a point stabilizer of $G$.

If $H$ is Abelian, then $G$ is a Frobenius group by Theorem 2.4. If there is no $H$-invariant normal subgroup in $K$, then $G$ is a primitive permutation group. Let $L$ be a proper $H$-invariant normal subgroup in $K$. If $|L| = p^2$, then $G/L$ is a solvable semiprimitive group with kernel of order $p$, the group $H$ is cyclic, and so $G$ is a Frobenius group by Theorem 2.4.

We may (and do) assume that every proper $H$-invariant normal subgroup $K$ has order $p$. We can (and will) also assume that there is a unique $H$-invariant normal subgroup $L$ in $K$ of order $p$. (Otherwise $G$ would be Frobenius.) Then $G/L$ is semiprimitive with kernel $K/L \cong C_p \times C_p$ and point-stabilizer $H$ considered as an irreducible subgroup of $GL(2, p)$.

If the order of $H$ is relatively prime to $p$, then, by Maschke's theorem, $H$ is completely reducible on $K$, and hence there is an $H$-invariant normal subgroup of order $p^2$ in $K$ and this contradicts our assumptions. So we may (and do) assume that $H$ contains a subgroup of order $p$. The stabilizer $H$ cannot contain exactly one subgroup of order $p$, since in this case $H$ would be reducible on $K/L$ from Dickson's list of subgroups of $GL(2, p)$. Hence we may (and do) assume that $H$ is a solvable irreducible subgroup of $GL(2, p)$ containing at least two subgroups of order $p$. By Dickson's list, this can only happen when $p = 2$ and $H = S_3$ or if $p = 3$ and $H \cong A_4$, $SL(2, 3)$, or $GL(2, 3)$.

Suppose that $p = 2$ and $H = S_3$. Then $H'$ is completely reducible on $K$ since $|H'| = 3$, and hence there exists an $H'$-invariant normal subgroup $M$ in $K$ such that $|M| = p^2$ and $L \cap M = \{1\}$. The group $H$ acts by conjugation on the set of all subgroups of order $p^2$ in $K$. The orbit containing $M$ has two elements: $M$ and say $N$. But then $M \cap N$ is an $H$-invariant subgroup of $K$ of order $p$ and different from $L$. This is a contradiction. □

**Theorem 3.3.** *Let $G$ be a solvable semiprimitive group with kernel $K$. If $K$ is Abelian of order a product of at most 3 primes and $K$ is not an elementary Abelian group of order 27, then $G$ is primitive or $G$ is a Frobenius group.*

*Proof.* All cases were considered in this section except when $|K| = p^2q$ or $|K| = pqr$ where $p$, $q$, $r$ are distinct primes. In both remaining cases there exists a characteristic subgroup $L$ of $G$ contained in $K$ so that $(K : L) = q$. By Lemma 2.3, this implies that the point-stabilizer of $G$ must be cyclic. Hence, by Theorem 2.4, $G$ is a Frobenius group. $\square$

# 4 Appendix on collapsing monoids

This section contains the self-contained definition of a collapsing permutation group and some background material from universal algebra. We refer to the Appendix only in the middle of the Introduction and it is not used in the proofs. This section was written only for the interested reader who is not familiar with the basic concepts of universal algebra related to this work.

Let $A$ be a set. For a non-negative integer $n$ let $\mathcal{O}_A^{(n)}$ denote the set of all $n$-ary operations on $A$. If $n \geq 1$ and $1 \leq i \leq n$, then the $n$-ary operation $p_i : A^n \longrightarrow A$ such that $(a_1, \ldots, a_n) \longmapsto a_i$ is called the $i$-th projection. If $n = 1$, then this projection is denoted by $id_A$. Put $\mathcal{O}_A = \cup_{n\geq 0}\mathcal{O}_A^{(n)}$. A subset $\mathcal{C}$ of $\mathcal{O}_A$ is said to be closed under superposition if for all non-negative integers $n$ and $k$, whenever $f \in \mathcal{C} \cap \mathcal{O}_A^{(n)}$ and $f_1, \ldots, f_n \in \mathcal{C} \cap \mathcal{O}_A^{(k)}$, then $f(f_1, \ldots, f_n) \in \mathcal{C}$. A subset $\mathcal{C}$ of $\mathcal{O}_A$ is called a clone if it contains all projections and is closed under superposition. The unary operations in a clone $\mathcal{C}$ form a transformation monoid. This monoid is called the unary part of the clone and it is denoted by $\mathcal{C}^{(1)}$. Let $T(A)$ be the symmetric monoid on the set $A$, and let $M$ be an arbitrary submonoid of $T(A)$. We call $[M] = \cap_{M\subseteq\mathcal{C}}\mathcal{C}$ the clone generated by the monoid $M$. The stabilizer $Stab(M)$ of $M$ is defined to be the clone consisting of all $n$-ary operations $f(x_1, \ldots, x_n)$ (for all $n \geq 0$) such that for all elements $m_1, \ldots, m_n$ of $M$ the operation $f(m_1(x), \ldots, m_n(x))$ is in $M$. Clearly, $[M]^{(1)} = Stab(M)^{(1)} = M$. In general, if $\mathcal{C}$ is a clone with unary part $M$, then $[M] \subseteq \mathcal{C} \subseteq Stab(M)$. In other words, clones with unary part $M$ form an interval $I(M)$ in the lattice of all clones on $A$. Every clone on $A$ is a member of $I(M)$ for some monoid $M$.

Let $A$ be a finite set with at least three elements. In this case there are uncountably many clones on $A$ and only finitely many submonoids of $T(A)$. Hence $\{I(M)\}_{M\subseteq T(A)}$ partitions the lattice of clones into finitely many disjoint intervals. The problem of classifying those transformation monoids $M$ for which $I(M)$ is finite was posed by Szendrei in [11]. The full transformation monoid $M = T(A)$ (see [2]) and the symmetric group $M = S_A$ are examples where $I(M)$ is finite: $|I(T(A))| = |A| + 1$ and $|I(S_A)| = 1$. A large family of monoids $M$ with $I(M)$ finite was provided by Pálfy in [7]: if $M$ consists of all constants and some permutations, then $|I(M)| \leq 2$; moreover, $|I(M)| = 1$ unless $M$ coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field.

The above results motivate the following definition (see [8]). If the interval $I(M)$ has only one element, the clone $[M]$, then the transformation monoid $M$ is called collapsing, and in the special case when $M$ is a permutation group, then $M$ is called a collapsing permutation group.

Dormán [3] gives examples of collapsing monoids. These monoids form large intervals in the submonoid lattice of the full transformation semigroups. Some of these intervals have cardinalities at least $2^{2^{cn}}$ where $n = |A|$. (If $A$ is a finite set with $n \geq 2$ elements, then the full transformation semigroup $T(A)$ has at least $2^{2^{cn}}$ and at most $2^{n^n}$ subsemigroups for some positive constant $c$ (see [3]).)

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# References

[1] Bamberg, J.; Praeger, C. E. Finite permutation groups with a transitive minimal normal subgroup. *Proc. London Math. Soc.* (3) **89** (2004), no. 1, 71–103.

[2] Burle, G. A. Classes of $k$-valued logic which contain all functions of a single variable. (Russian) *Diskret. Analiz* No. **10** 1967 3–7.

[3] Dormán, M. Intervals of collapsing monoids. *Acta Sci. Math.* (Szeged) **68** (2002), no. 3-4, 561–569.

[4] Feit, W. Characters of finite groups. N. Y., W. A. Benjamin, 1967.

[5] Gorenstein, D. Finite groups. Second edition. Chelsea Publishing Co., New York, 1980.

[6] Kearnes, K. A; Szendrei, Á. Collapsing permutation groups. *Algebra Universalis* **45**, (2001), no. 1, 35–51.

[7] Pálfy, P. P. Unary polynomials in algebras. I. *Algebra Universalis* **18** (1984), no. 3, 262–273.

[8] Pálfy, P. P; Szendrei, Á. Unary polynomials in algebras. II. *Contributions to general algebra* **2**, (Klagenfurt, 1982), 273–290, Hlder-Pichler-Tempsky, Vienna, 1983.

[9] Praeger, C. E. An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs. *J. London Math. Soc. (2)* **47**, (1993), no. 2, 227–239.

[10] Suprunenko, D. A. On the Frobenius group. *Ukrain. Mat. Zh.* **43**, (1991), no. 7-8, 1021–1030.

[11] Szendrei, Á. Clones in universal algebra. *Séminaire de Mathématiques Supérieures* [Seminar on Higher Mathematics], **99**. Presses de l'Universit de Montral, Montreal, QC, 1986.

*Áron Bereczky, Department of Mathematics and Statistics, Sultan Qaboos University, Al Khodh, Sultanate of Oman*

*E-mail address: aron@squ.edu.om*

*Attila Maróti, Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA*

*E-mail address: maroti@usc.edu*