COVERING THE SET OF p-ELEMENTS IN FINITE GROUPS BY PROPER SUBGROUPS

ATTILA MARÓTI, JUAN MARTÍNEZ, AND ALEXANDER MORETÓ

Abstract. Let p be a prime and let G be a finite group which is generated by the set G_p of its p-elements. We show that if G is solvable and not a p-group, then the minimal number $\sigma_p(G)$ of proper subgroups of G whose union contains G_p is equal to 1 less than the minimal number of proper subgroups of G whose union is G. For p-solvable groups G, we always have $\sigma_p(G) \geq p+1$. We study the case of alternating and symmetric groups G in detail.

1. Introduction

A celebrated but elementary result of Gustafson [20] asserts that a finite group G is abelian if and only if the probability $P(G)$ that two uniformly and randomly chosen elements in G commute is larger than 5/8. The invariant $P(G)$ is equal to $k(G)/|G|$ where $k(G)$ is the number of conjugacy classes of the finite group G. Throughout the paper let p be a prime. Burness, Guralnick, Moretó, Navarro [6, Theorem A] proved the following deep theorem. A finite group G has an abelian normal Sylow p-subgroup if and only if the probability that two randomly chosen p-elements in G commute is larger than $(p^2+p-1)/p^3$. This result may be viewed as a local analog of Gustafson's theorem.

For a group G, let G_p denote the set of p-elements in G. In this paper G will always denote a finite group, except in the last part of this section where we briefly comment on infinite groups. As usual, $O_p(G)$ denotes the largest normal p-subgroup

Date: July 17, 2024.

2020 *Mathematics Subject Classification.* Primary 20D10, 20D06; Secondary 05E16.

Key words and phrases. covering, p-element, solvable group, alternating group.

Parts of this work were done when the second and third authors were visiting the Alfréd Rényi Institute and when the first author was visiting the University of València. We thank the Alfréd Rényi Institute and the CARGRUPS research team at the University of València for their hospitality. All authors were supported by Ministerio de Ciencia e Innovación (Grant PID2019-103854GB-I00 funded by MCIN/AEI/10.13039/501100011033) and CIAICO/2021/163. The second author was also supported by CIACIF/2021/228. The first author has also received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 741420) and was also supported by the National Research, Development and Innovation Office (NKFIH) Grant No. K138596, No. K132951 and Grant No. K138828.

in G. The previously mentioned theorem [6, Theorem A] is derived from [6, Theorem C] stating that whenever x is an element of $G_p \setminus O_p(G)$ then $|C_G(x)_p|/|G_p| \leq 1/p$.

Let S be a subset of G and let $H_1, \ldots, H_n$ be subgroups of G. We say that $\{H_1, \ldots, H_n\}$ covers S or that $\{H_1, \ldots, H_n\}$ is a covering for S if $S \subseteq \cup_{i=1}^{n} H_i$. A quick consequence of [6, Theorem C] is that if $x_1, \ldots, x_n$ are elements of $G_p \setminus O_p(G)$, for a finite group G and a prime p, such that $\{C_G(x_1), \ldots, C_G(x_n)\}$ covers G_p, then $n \geq p+1$.

The noncommuting graph is the graph whose vertex set is $G \setminus Z(G)$, where G is a finite group and $Z(G)$ is the center of G, with two vertices connected by an undirected edge if and only if they do not commute. See Cameron's article [7] for information about this graph. For example, [7, Corollary 9.8] states that the noncommuting graph is not only connected but has diameter at most 2. Assume that $O_p(G)$ is central in G. Consider the induced graph of the noncommuting graph defined on the elements of $G_p \setminus Z(G)$. We claim that this graph also has diameter at most 2. Indeed, if x and y are elements of $G_p \setminus O_p(G)$ then $\{C_G(x), C_G(y)\}$ is not a cover for G_p by the previous paragraph, so there must be an element z of G_p outside $O_p(G)$ such that z is connected to both x and y.

The noncommuting graph implicitly appeared in a work of Pyber [31]. Let n be the size of a largest complete subgraph of the noncommuting graph Γ of a finite group G. Let $cc(\Gamma)$ be the minimal number of empty subgraphs covering Γ. Clearly, $n \leq cc(\Gamma) \leq |G : Z(G)|$. Answering a question of Neumann and solving a problem of Erdős, Pyber [31] showed that there exists a constant c such that $|G : Z(G)| \leq c^n$. Under the condition that $O_p(G)$ is central in a finite group G for a fixed prime p, [6, Theorem C] implies that if n denotes the size of a largest complete subgraph in the induced subgraph of Γ defined on the vertex set $G_p \setminus Z(G)$ then $n \geq p+1$. In particular, if $\mathcal{A}$ is a set of abelian subgroups of G covering G_p then $|\mathcal{A}| \geq p+1$.

Our aim in this paper is to study the minimal size of a covering of G_p by arbitrary proper subgroups of G. We note that in [28] we study another variation of this problem, namely whether a proper subset of the set of Sylow p-subgroups of a finite group G covers G_p. The results in this paper do not depend on [28] and the techniques used are completely different.

These problems can be seen as a local version of the widely studied problem of covering a finite group by proper subgroups. It is an elementary fact that a group cannot be expressed as the union of two proper subgroups. Let G be a noncyclic finite group. Cohn [8] introduced the invariant $\sigma(G)$ as the minimal size of a covering for G which consists of proper subgroups of G. Tomkinson [34] proved that $\sigma(G)$ is always a prime power plus 1 for any (noncyclic and finite) solvable group G. More precisely, he showed that if G is a finite solvable group and H/K is the smallest chief factor of G having more than one complement in G, then $\sigma(G) = |H/K| + 1$. There is a large literature on σ. The numbers $\sigma(G)$ were computed (or bounds were given) for various classes of nonsolvable groups G; for certain symmetric groups [27], [33], [18], for certain linear groups [5], [4], for sporadic groups [13], for Suzuki groups [25], or for certain wreath products [24]. There are many positive integers n for which there is no group G with $\sigma(G) = n$ (see [34], [22], [23]).

Let G be a finite group generated by its p-elements or equivalently G with $G = O^{p'}(G)$. Let $\sigma_p(G)$ be the minimal size of a cover for G_p consisting of proper subgroups of G. If G is a cyclic p-group, then there is no such cover and we say that $\sigma_p(G) = \infty$. It is well-known, and easy to see, that if G is a non-cyclic p-group then $\sigma(G) = p+1$. It is also clear that in this case $\sigma_p(G) = \sigma(G)$. Perhaps surprisingly, we prove that $\sigma(G)$ and $\sigma_p(G)$ are also closely related when G is any solvable group generated by its p-elements.

Our first result is the following.

Theorem A. *Let G be a solvable group generated by its p-elements. If G is not a p-group, then $\sigma_p(G) = \sigma(G) - 1$.*

We mention that a related invariant to $\sigma_p(G)$ was studied by Fumagalli and Garonzi in [17]. A covering of the set of prime power elements of a finite group G (for all prime divisors of the order of G) is called a primary covering. The minimal size of a primary covering consisting of proper subgroup of G is denoted by $\sigma_0(G)$. This invariant was studied for solvable groups and symmetric groups. Our proof of Theorem A relies on results from this paper.

Theorem A is not true for p-solvable groups. In fact, we show in Example 2.3 that for any $\varepsilon > 0$ there exist p-solvable groups G generated by p-elements such that $\sigma_p(G)/\sigma(G) < \varepsilon$. Therefore, when G is not solvable $\sigma(G)$ and $\sigma_p(G)$ are two completely different invariants.

Our second result is the following.

Theorem B. *If p is a prime and G is a p-solvable group generated by its p-elements, then $\sigma_p(G) \geq p+1$.*

Our proof of Theorem B relies on results of M. Hall [14] and G. Navarro [29] which are just false for non-p-solvable groups. R. Guralnick found a proof of this result for arbitrary finite groups using ideas related to [3]. This will appear elsewhere. This implies our first consequence of [6, Theorem C]. The following result may be viewed as the next step in studying $\sigma_p(G)$ for non-p-solvable groups G.

Theorem C. *Let p be a prime and let G be A_n or S_n. If $G = A_n$ or $(p, G) = (2, S_n)$, then $\sigma_p(G) \to \infty$ as $n \to \infty$ (and p fixed). Otherwise G is not generated by its p-elements and thus $\sigma_p(G)$ is not defined.*

We obtain a more detailed result than Theorem C with good lower bounds and sometimes even identities for $\sigma_p(G)$ when G is an alternating or symmetric group. For example, we establish in Proposition 5.10 that $\sigma_p(A_n) = p+1$, provided that $\max\{5, p+1\} \leq n \leq 2p-1$. Theorem 4.2 states that $\sigma_p(\mathrm{PSL}(2,q)) = q+1$ for any integer power $q \geq 4$ of the prime p. In Corollary 5.14 we investigate the relationship between $\sigma_p(G)$ and $\sigma(G)$ where G is an alternating or symmetric group.

In this paragraph let G be an infinite group. One may consider the coverings of G_p by proper subgroups. A theorem of Neumann [30] states that if G is the union of m proper subgroups where m is finite and small as possible, then the intersection of these subgroups is a subgroup of finite index in G. This is the reason why one may assume that G is finite when computing $\sigma(G)$, the minimal number of proper subgroups needed to cover G. It would be interesting to know if there is a local analogue of Neumann's theorem, namely that in determining $\sigma_p(G)$ (under suitable conditions) could one assume that G is finite.

2. The relationship with $\sigma(G)$

In this section we prove Theorem A. In the following lemma we argue as in the proof of Lemma 1 of [12]. We include the proof for the reader's convenience.

Lemma 2.1. *Let G be a finite group and let $N \trianglelefteq G$. Then:*

(i) $\sigma_p(G) \leq \sigma_p(G/N)$.
(ii) *If $N \leq \Phi(G)$, then $\sigma_p(G) = \sigma_p(G/N)$.*
(iii) *If M is a maximal subgroup of G such that $\sigma_p(M) > \sigma_p(G)$, then M belongs to every minimal covering of G_p.*

Proof. Since any covering of $(G/N)_p$ lifts to a covering of G_p, part (i) is clear. Since the minimal size of a covering of G_p by proper subgroups is the minimal size of a covering of G_p by maximal subgroups, it follows that if $N \leq \Phi(G)$ then $\sigma_p(G) = \sigma_p(G/N)$.

Finally, let M be a maximal subgroup of G such that $\sigma_p(M) > \sigma_p(G)$. Let $H_1, \ldots, H_n$ be a covering of G_p, where $n = \sigma_p(G)$. Then $H_1 \cap M, \ldots, H_n \cap M$ covers M_p. Since $\sigma_p(M) > n$, it follows that there exists $i \leq n$ such that $M \leq H_i$, whence $M = H_i$. The result follows. $\square$

The following is Theorem A.

Theorem 2.2. *Let G be a solvable group generated by p-elements which is not a cyclic p-group. Then*

$$\sigma_p(G) \geq \sigma(G) - 1.$$

Moreover, $\sigma_p(G) = \sigma(G) - 1$ if and only if G is not a p-group.

Proof. If G is a p-group, then we clearly have $\sigma_p(G) = \sigma(G)$, so we may assume that G is not nilpotent.

Let $N \trianglelefteq G$ be such that $\sigma_p(G) = \sigma_p(G/N)$ with $|G/N|$ as small as possible. Note that this implies that $\Phi(G/N) = 1$, by Lemma 2.1(ii). Let K/N be a chief factor of G, so that $\sigma_p(G) = \sigma_p(G/N) < \sigma_p(G/K)$, Since K/N is a non-Frattini chief factor, it has a complement M/N, for some M maximal subgroup of G. Therefore,

$M/N \cong G/K$, so $\sigma_p(G/N) < \sigma_p(M/N)$. By Lemma 2.1(iii) this implies that M/N belongs to every minimal covering of $(G/N)_p$.

Let b be the number of complements of K/N in G/N, so that $b \leq \sigma_p(G)$. We claim that there exists K/N with $b > 1$. If $b = 1$, then K/N has a unique complement in G/N and hence $G/N = K/N \times M/N$. Since K/N is a chief factor, it is abelian. Therefore, K/N is central in G/N and has prime order.

If the claim is false, then we have that any minimal normal subgroup of G/N is central of prime order. Since $\Phi(G/N) = 1$, $F(G/N)$ is the direct product of the minimal normal subgroups of G/N. It follows that $F(G/N)$ is central in G/N, so $G/N = F(G/N)$ is a direct product of cyclic groups of prime order. Since G/N is generated by p-elements, G/N is a non-cyclic elementary abelian p-group and hence K/N possesses more than one complement, which is impossible.

Therefore, we have that $b > 1$ for some chief factor K/N. By [12, Lemma 2], this implies that $b \geq |K/N|$. On the other hand, by [34, Theorem 2.2],

$$\sigma_p(G) \leq \sigma(G) \leq 1 + |K/N| \leq 1 + b \leq 1 + \sigma_p(G),$$

and the inequality follows. It only remains to prove the statement on the equality. By the above inequalities, $\sigma_p(G) \in \{\sigma(G), \sigma(G) - 1\}$ and $\sigma_p(G) = \sigma(G) - 1$ if and only if we have equality above except in the first inequality. This happens if and only if $\sigma_p(G) = b$ and $\sigma(G) = b + 1$. We claim that $\sigma_p(G) = b$ and $\sigma(G) = b + 1$ if and only if K/N is a p'-group.

Let $\{M_i/N\}_{i=1}^b$ be all the complements of K/N. We know that all M_i/N appear in every minimal covering of G_p. As a consequence, we have that $\sigma_p(G) = b$ if and only if $\{M_i/N\}_{i=1}^b$ covers $(G/N)_p$, or equivalently, if and only if K/N is a p'-group. Thus, the "if" part of the claim follows. It only remains to prove the "only if" part.

Assume that K/N is a p'-group. We know that $\sigma_p(G) = b$ and hence we only have to prove that $\sigma(G) > b$. Suppose that $\sigma(G) = b$. Then there exists a collection $\{H_i/N\}_{i=1}^b$ of maximal groups which covers G/N. In particular, $\{H_i/N\}_{i=1}^b$ is a minimal covering of $(G/N)_p$ and, by the conditions on the M_i/N, we have that $\{H_i/N\}_{i=1}^b = \{M_i/N\}_{i=1}^b$. Thus, we have that

$$G/N = \bigcup_{i=1}^{b} H_i/N = \bigcup_{i=1}^{b} M_i/N \subseteq G/N \setminus (K/N \setminus \{1\}) \neq G/N,$$

which is a contradiction. Now, the result on the equality follows from the claim. □

Note that this result, together with [34, Theorem 2.2], allows us to give another proof of Theorem B for solvable groups. We conclude this section showing that Theorem A cannot be extended to p-solvable groups.

Example 2.3. *Let $p \notin \{2, 3, 5, 11\}$ be a prime. Let $k \geq 1$. Let $G_k = \mathrm{M}_{11} \wr C_{p^k}$, so that G_k is p-solvable. We claim that $\sigma_p(G_k)/\sigma(G_k) \to 0$ when $k \to \infty$.*

By Corollary 1.2 of [24], we know that $\sigma(G_k) = 1 + 11^{p^k} + 12^{p^k}$. On the other hand, let $H = \mathrm{M}_{10} \wr C_{p^k}$, so that $|G : H| = 11^{p^k}$. Since Sylow p-subgroups of G_k are G_k-conjugate and H contains a Sylow p-subgroup of G, the 11^{p^k} G_k-conjugates of H cover $(G_k)_p$. Thus $\sigma_p(G_k) \le 11^{p^k}$. Therefore,

$$\frac{\sigma_p(G_k)}{\sigma(G_k)} \le \left(\frac{11}{12}\right)^{p^k} \to 0,$$

when $k \to \infty$, as wanted.

3. On the minimal size of a covering of G_p for p-solvable groups G

In this section, we prove Theorem B. We use some results on the number of Sylow p-subgroups that are just false outside p-solvable groups. The first one is a theorem of M. Hall. If p is a prime and G a finite group, we write $\nu_p(G)$ to denote the number of Sylow p-subgroups of G.

Lemma 3.1. *Let G be a p-solvable group. Let $\nu_p(G) = p_1^{a_1} \dots p_t^{a_t}$ be the factorization of $\nu_p(G)$ as a product of prime powers. Then $p_i^{a_i} \equiv 1 \pmod p$.*

Proof. This was proved by P. Hall [15] for solvable groups. The result for p-solvable groups follows from the proof of Theorem 2.2 of [14]. □

The following result may have independent interest.

Theorem 3.2. *Let p be a prime and G a p-solvable group. If H is a proper subgroup of G such that $|H|_p = |G|_p$, then $\nu_p(H) < \nu_p(G)/p$.*

Proof. By way of contradiction, suppose that $\nu_p(H) \ge \nu_p(G)/p$. Since the number of Sylow p-subgroups is 1 modulo p, we have that $\nu_p(H) > \nu_p(G)/p$. We claim that $\nu_p(H) = \nu_p(G)/p$. By By [29], we know that $\nu_p(H)$ divides $\nu_p(G)$, so we may write $\nu_p(H) = p_1^{a_1} \dots p_t^{a_t}$ and $\nu_p(G) = p_1^{b_1} \dots p_t^{b_t}$, with $a_j \le b_j$ for every j. Without loss of generality, we may assume that $a_1 < b_1$. Since $\nu_p(G)/\nu_p(H) < p$, it follows that $p_1^{b_1-a_1} < p$. By Lemma 3.1, $p_1^{b_1} - p_1^{a_1} = p_1^{a_1}(p_1^{b_1-a_1} - 1)$ is a multiple of p. Hence, $p < p_1^{b_1-a_1}$, a contradiction. This proves the claim.

Therefore, H contains all the p-elements of G. Since G is generated by p-elements, we conclude that $H = G$, a contradiction. □

This is false when G is not p-solvable, as $G = A_5$ and $H = A_4$ show for $p = 3$. We will obtain a version of this result for non-p-solvable groups elsewhere.

Now, we complete the proof of Theorem B for p-solvable groups.

Theorem 3.3. *If p is a prime and G is a finite p-solvable group generated by its p-elements, then $\sigma_p(G) \ge p + 1$.*

Proof. Arguing by contradiction, suppose that there exist $H_1, \ldots, H_p < G$ such that $G_p \subseteq \bigcup_{i=1}^{p} H_i$. Let P be a Sylow p-subgroup of G. Hence

$$P = \bigcup_{i=1}^{p} H_i \cap P.$$

Since a p-group cannot be covered by less than $p+1$ proper subgroups, we deduce that there exists i such that $P \subseteq H_i$. Thus, every Sylow p-subgroup is contained in some of the subgroups H_i. By the pigeonhole principle, we may assume that $H = H_1$ contains a proportion of at least $1/p$ of the Sylow p-subgroups of G. This contradicts Theorem 3.2. □

4. Some projective special linear groups

In this section let $q = p^n$ be a prime power for a prime p and a positive integer n. Consider the projective special linear group $G = \mathrm{PSL}(2, p^n)$. We will show that $\sigma_p(G) = q + 1$, which is the number of Sylow p-subgroups of G.

Lemma 4.1. *Let $q \geq 4$. If $S < G$, then $|S_p| \leq |G|_p = q$.*

Proof. It suffices to prove that $|S_p| \leq q$ for all maximal subgroups S of G. We may also assume that p divides the order of S.

Assume first that $p > 5$. By Dickson's classification of subgroups of $\mathrm{PSL}(2, q)$ (see [16, p. 213-214] or [9, p. 285]), we have that S must be isomorphic to one of the following.

(i) $C_{(q-1)/d} \rtimes C_p^n$, where $d = \gcd(2, q-1)$.
(ii) $\mathrm{PSL}(2, p^f)$ with $f < n$ dividing n.
(iii) $\mathrm{PGL}(2, p^f)$ with $2f$ dividing n.

In Case (i) we have that $|S_p| = |S|_p = q$ and in Cases (ii) and (iii) we have that $|\mathrm{PGL}(2, p^f)_p| = |\mathrm{PSL}(2, p^f)_p| = p^{2f} \leq q$.

Let $p \in \{2, 3, 5\}$. In this case S is either one of the groups mentioned above or (under some conditions on q) $S \in \{A_4, S_4, A_5\}$. The lemma follows trivially for $q \geq 9$ and, for the case $q \leq 8$, the result can be checked by a closer look at Dickson's list. Now, if $q = 5^n$ with $n \geq 2$, then S can also be isomorphic to A_5 and hence $|S_5| = 25 \leq q$. Similarly, if $p \in \{2, 3\}$, then S can also be isomorphic to a member of $\{A_4, S_4, A_5\}$ and simple calculations give the result. □

We borrow the following definition from [27, p. 99]. Let G be a finite group and Π a subset of G. A set $\mathcal{H} = \{H_1, \ldots, H_m\}$ of m proper subgroups in G is said to be definitely unbeatable on Π if $\Pi \subseteq \cup_{i=1}^{m} H_i$, if $\Pi \cap H_i \cap H_j = \emptyset$ for all $i \neq j$, and if $|S \cap \Pi| \leq |H_i \cap \Pi|$ holds whenever $1 \leq i \leq m$ and when $S \notin \mathcal{H}$ is a proper subgroup of G. If $\mathcal{H}$ is definitely unbeatable on Π, then $|\mathcal{H}| = \sigma(\Pi) \leq \sigma(G)$ where for a subset Σ of G the minimal number of proper subgroups of G needed to cover Σ is denoted by $\sigma(\Sigma)$ (in particular, $\sigma_p(G)$ is just $\sigma(\Sigma)$ when Σ is the set of all p-elements in G).

Theorem 4.2. *If $q = p^n \geq 4$ where p is a prime and n is an integer, then*

$$\sigma_p(\mathrm{PSL}(2,q)) = q + 1.$$

Proof. Let $G = \mathrm{PSL}(2,q)$. Let $\mathcal{F}$ be the family of maximal subgroups of G of the form $C_{(q-1)/d} \rtimes C_p^n$, where $d = \gcd(2, q-1)$. We know that for every Sylow p-subgroup P of G there exists a unique $M_P \in \mathcal{F}$ containing P and that $\mathcal{F} = \{M_P | P \in \mathrm{Syl}_p(G)\}$. Thus, $|\mathcal{F}| = q + 1$ and hence it suffices to prove that $\mathcal{F}$ is definitely unbeatable on $\Pi := G_p \setminus \{1\}$. Clearly, $\Pi \subseteq \bigcup_{P \in \mathrm{Syl}_p(G)} P \subseteq \bigcup_{P \in \mathrm{Syl}_p(G)} M_P$. Let P and Q be two different Sylow p-subgroups of G. Then $P \cap Q = 1$ and so $\Pi \cap P \cap Q \subseteq (P \cap Q) \setminus \{1\} = \varnothing$. Finally, Lemma 4.1 implies that if $M \in \mathcal{F}$ and $S < G$, then $|S \cap \Pi| \leq q - 1 = |M \cap \Pi|$. □

5. Symmetric and alternating groups

In this section we study symmetric and alternating groups. In particular, we prove Theorem C.

Let p be a prime and let n be an integer at least $\max\{5, p\}$. Let S_n and A_n be the symmetric and alternating group of degree n respectively. Let G be A_n or S_n, latter only if $p = 2$. There is a unique way to write n in the form $\sum_{i=0}^{k} a_i p^i$ where k and the a_i are non-negative integers such that $a_k \neq 0$ and $a_i \leq p - 1$ for every i with $0 \leq i \leq k$. Let j be $\min_{0 \leq i \leq k}\{i \mid a_i \neq 0\}$.

For a positive integer m, let P_m be a Sylow p-subgroup of S_m. For a non-negative integer a, let $(P_m)^a$ denote the direct product of a copies of the group P_m. It is well-known (see [11]) that P_n is conjugate in S_n to the group $\prod_{i=1}^{k} (P_{p^i})^{a_i}$.

We state a useful lemma.

Lemma 5.1. *Let X be a finite group and p a prime. Let x be a p-element in X with $C_X(x) = \langle x \rangle$. Let H be a maximal and non-normal subgroup of X such that $x \in H$, $x^H = x^X \cap H$, and $|X : H|$ is not divisible by p. Moreover, assume that whenever S is a proper subgroup of X then $|x^X \cap S| \leq |x^H|$. Then $\sigma_p(X) = |X : H|$.*

Proof. Let $\mathcal{H}$ be the set of all conjugates of H in X. Since $N_X(H) = H$, we have $|\mathcal{H}| = |X : H|$. Since $|X : H|$ is not divisible by p, the set $\mathcal{H}$ is a covering for G_p by Sylow's theorem. This gives $\sigma_p(X) \leq |X : H|$.

We claim that $\mathcal{H}$ is definitely unbeatable on x^X. Clearly, $\mathcal{H}$ covers x^X. Since $x^H = x^X \cap H$ and $C_X(x) = \langle x \rangle$, every conjugate of H contains $|H|/|\langle x \rangle|$ elements from x^X. Since $|\mathcal{H}| = |X : H|$ and $|x^X| = |X|/|\langle x \rangle|$, we find that $x^X \cap H_1 \cap H_2 = \emptyset$ for any pair of distinct subgroups H_1 and H_2 from $\mathcal{H}$. The third condition of the definition is also satisfied since $|x^X \cap S| \leq |x^H|$ for any proper subgroup S of X. This gives $\sigma_p(X) \geq |\mathcal{H}|$, which proves the lemma. □

We will need the following theorem of Jones [21, Theorem 3].

Theorem 5.2 (Jones [21])**.** *A primitive permutation group H of finite degree n has a cyclic regular subgroup if and only if one of the following holds.*

(i) $H = S_n$ *for some* $n \geq 2$ *or* $H = A_n$ *for some odd* $n \geq 3$.
(ii) $C_p \leq H \leq \mathrm{AGL}(1, p)$ *where* $n = p$ *is prime.*
(iii) $\mathrm{PGL}(d, q) \leq H \leq \mathrm{P\Gamma L}(d, q)$ *where* $n = (q^d - 1)/(q - 1)$ *for some* $d \geq 2$.
(iv) $H = \mathrm{PSL}(2, 11)$, M_{11} *or* M_{23} *where* $n = 11$, 11 *or* 23 *respectively.*

Consider the case when $n = p$. Since $n \geq 5$, we also have $p \geq 5$ and thus $G = A_n$.

Assume first that $n = p \notin \{11, 23\}$ and n is not of the form $(q^d - 1)/(q - 1)$ where q is a prime power and $d \geq 2$ is an integer. Let H be a maximal subgroup of G containing a p-cycle. According to Theorem 5.2, the group H is $\mathrm{AGL}(1, p) \cap G$. We have $\sigma_p(G) = (n - 2)!$ by Lemma 5.1.

Assume that $n = p \in \{11, 23\}$. Observe that neither 11 nor 23 has the form $(q^d - 1)/(q - 1)$ where q is a prime power and $d \geq 2$ is an integer. Let H be a maximal subgroup of G containing a p-cycle x. Then $H = \mathrm{M}_{11}$ if $n = 11$ or $H = \mathrm{M}_{23}$ if $n = 23$ by Theorem 5.2 and [1]. We have $x^H = x^G \cap H$ by [1]. It is clear that $|G : H|$ is not divisible by p and that $C_G(x) = \langle x \rangle$. Thus $\sigma_p(G) = |G|/|H|$ by Lemma 5.1.

Let $n = p$ be of the form $(q^d - 1)/(q - 1)$ where q is a prime power and $d \geq 2$ is an integer. Since there are $(n - 1)!/2$ conjugates of a given n-cycle x in G and every proper subgroup of G contains at most $|\mathrm{P\Gamma L}(d, q)|/2$ conjugates of x for a certain integer $d \geq 2$ and a prime power q with $n = (q^d - 1)/(q - 1)$ by Theorem 5.2, we have $\sigma_p(G) \geq (n - 1)!/|\mathrm{P\Gamma L}(d, q)|$. Now $|\mathrm{P\Gamma L}(d, q)| < q^{d^2+1}$. Since $q^{d-1} < (q^d - 1)/(q - 1) = n$, we have $q^{(d-1)^2} < n^{\log n}$ where the base of the logarithms is 2. It follows that $|\mathrm{P\Gamma L}(d, q)| < q^{d^2+1} < n^{\log n + 4}$.

We will need the following estimates obtained from Stirling's formula.

Lemma 5.3. *For any positive integer m, we have* $(m/e)^m < m! \leq em(m/e)^m$.

We have $\sigma_p(G) > n^{n - \log n - 5}/e^n$ by Lemma 5.3.

We are left with the possibility that $n \in \{5, 7, 13\}$. Let $n = 5$. The only maximal subgroup of $G = A_5$ containing an element of order $p = 5$ is D_{10}. The set of all conjugates of D_{10} in G is definitely unbeatable on $G_p \setminus \{1\}$. We get $\sigma_p(G) = p + 1$ in this case. Let $n = 7$. There are two conjugacy classes of maximal subgroups of $G = A_7$ containing elements of order $p = 7$. Both consist of subgroups $\mathrm{PSL}(2, 7)$. The group $\mathrm{PSL}(2, 7)$ contains two conjugacy classes of elements of order 7. We get $\sigma_p(G) = 15$ by Lemma 5.1. Let $n = 13$. According to [1], there are three conjugacy classes of maximal subgroups of $G = A_{13}$ containing elements of order $p = 13$. These are two classes consisting of groups $\mathrm{PSL}(3, 3)$ and one class consisting of groups $13 : 6$. In any case, there are $12!/2$ conjugates of a given p-cycle x in G and any proper subgroup of G contains at most $|\mathrm{PSL}(3, 3)|/2$ conjugates of x. This gives the bound $\sigma_p(G) \geq 12!/|\mathrm{PSL}(3, 3)| > 14$.

Let $n = p^k > p$.

We continue with the following proposition.

Proposition 5.4. *Let $n = p^k > p$. If p is odd or $(p, G) = (2, S_n)$, then*

$$\sigma_p(G) = \frac{n!}{(n/p)!^p p!}.$$

Proof. Let $H = (S_{n/p} \wr S_p) \cap G$ be a maximal imprimitive subgroup of G. This contains a Sylow p-subgroup of G. Let $\mathcal{H}$ be the set of all conjugates of H in G. This is a cover for G_p. Thus $\sigma_p(G) \leq |G : H| = n!/(n/p)!^p p!$.

All n-cycles in S_n are conjugate and there are two conjugacy classes of n-cycles in A_n. It is easy to see that all n-cycles in $S_a \wr S_b$ are conjugate in $S_a \wr S_b$ for any integers $a > 1$ and $b > 1$ with $n = ab$. There are two conjugacy classes of n-cycles in $(S_a \wr S_b) \cap A_n$.

Let x be an n-cycle in G. All the conditions of Lemma 5.1 are satisfied (with $X = G$) with the possible exception of the last. In order to prove the proposition, it is therefore sufficient to show that whenever S is a proper subgroup of G then $|x^G \cap S| \leq |x^H|$.

For the claim, we may assume that S is transitive. If S is imprimitive, then the bound follows from the proof of [26, Lemma 2.1]. Let S be primitive. Then $S = \mathrm{P\Gamma L}(d, q) \cap G$ where $n = (q^d - 1)/(q - 1)$ for some $d \geq 2$ and prime power q, by Theorem 5.2. We have $|S| \leq n^{1+\log n}$ by [26, Theorem 1.1]. Thus $|x^G \cap S| \leq (2d|S|)/n \leq n^{1+\log n}$ by [21, Corollary 2]. On the other hand, Lemma 5.3 and the fact that $k \geq 2$ give $(\sqrt{n}/e)^n < |H|$. If $n \geq 120$, then $n^{2+\log n} < (\sqrt{n}/e)^n$ and so $|x^G \cap S| < |x^H|$. We may assume that $n \leq 119$. Since $n = p^k \geq 5$ and $k \geq 2$, the possibilities for n are 8, 16, 32, 64, 9, 27, 81, 25, and 49. None of these integers have the form $(q^d - 1)/(q - 1)$ (where $d \geq 2$ and q is a prime power), except 8 and 32. Let $n \in \{8, 32\}$. We have $|x^G \cap S| \leq (d|S|)/n$ by [21, Corollary 2]. The inequalities $(d|S|)/n \leq |S_{n/2} \wr S_2|/n = |H|/n = |x^H|$ can be verified using Gap [19]. □

We continue to assume that p is odd (and $G = A_n$), or $p = 2$ and $G = S_n$. We have $\sigma_p(G) = n!/((n/p)!^p p!)$ by Proposition 5.4. If $p = 2$, then this clearly tends to infinity as $n \to \infty$. Let $p \geq 3$. An easy computation gives $n!/((n/p)!^p p!) \geq p^{n-\sqrt{n}}/e^n$, showing that $\sigma_p(G) \to \infty$ as $n \to \infty$. Chebyshev's theorem (or Bertrand's postulate) implies that there is a prime divisor r of $n!/((n/p)!^p p!)$ such that $r \geq n/2$. We conclude that $\sigma_p(G) \geq n/2 = p^k/2$.

We continue to have $n = p^k > p$. In order to prove Theorem C, we may assume that $p = 2$ and $G = A_n$.

Let x be a permutation in G whose disjoint cycle decomposition consists of two cycles each of length $n/2$. Since n is a power of 2, the subgroup $C_G(x)$ is a 2-group. Let P be a Sylow 2-subgroup containing $C_G(x)$. The group P is contained in a conjugate of $(S_{n/2} \wr S_2) \cap G$ in G which we denote by H. This is a maximal subgroup in G. All hypotheses of Lemma 5.1 are satisfied with the possible exception of the last. Let S be a proper subgroup of G containing x. We claim that $|x^G \cap S| \leq |H|/n = |x^H|$, provided that $k \geq 4$. The group S is transitive. If S is imprimitive, then $|x^G \cap S| = |S|/n \leq |H|/n$ by the proof of [26, Lemma 2.1]. Let S be primitive.

Assume that S is a subgroup of $S_m \wr S_r$, with $m \geq 5$ and $r \geq 1$, containing $(A_m)^r$, where the action of S_m is on ℓ-element subsets of $\{1, \ldots, m\}$ and the wreath product has the product action of degree $n = \binom{m}{\ell}^r$. Since n is a power of 2, the integer ℓ must be 1 by Sylvester's theorem (which is a generalization of Bertrand's postulate), stating that if $m \geq 2\ell$ then at least one of the numbers $m, m-1, \ldots, m-\ell+1$ has a prime divisor larger than ℓ, (and thus m is a power of 2). In any case, the order of every 2-element in S must have order at most mr. On the other hand, x has order $m^r/2$. This forces $r = 1$, contradicting the fact that S is a proper subgroup of G. It follows that $|S| < n^{1+\log n} = 2^{k(k+1)}$ by [26, Theorem 1.1]. This is at most $2^{2^k-2} = 2^{n-2} \leq ((n/2)!)^2/n = |H|/n$ for $k \geq 5$. If $k = 4$, then the inequality $|S| \leq |H|/n$ may be checked directly. If $k = 3$, then $\sigma_p(G) = 15$ by [1] and a version of Lemma 5.1. If $k = 2$, then $\sigma_p(G)$ is not defined.

The previous argument together with Lemma 5.1 give the following.

Proposition 5.5. *Let $n = 2^k > 4$ and $p = 2$. If $k \geq 4$, then $\sigma_p(G) = \binom{n}{n/2}/2$. Otherwise, $\sigma_p(G) = 15$.*

Clearly, $\sigma_2(G) \geq 3$ and $\sigma_2(G) \to \infty$, under the conditions of Proposition 5.5.

We proved (i) of Theorem C and (v) of Theorem C in the special case when n is a power of p.

We now turn to the case when $n \neq p^k$. In this case $p = 2$ (and $G \in \{A_n, S_n\}$) or $G = A_n$. Recall that p^j is a smallest member in the p-adic expansion of n and p^k is a largest.

We start with an easy upper bound.

Lemma 5.6. *If $n \neq p^k$, then $\sigma_p(G) \leq \binom{n}{p^j}$.*

Proof. A Sylow p-subgroup P_n of S_n is intransitive and its smallest orbit has size p^j by the definition of j. Thus P_n lies inside a maximal subgroup isomorphic to $S_{n-p^j} \times S_{p^j}$. The set of all maximal subgroups of G conjugate to $(S_{n-p^j} \times S_{p^j}) \cap G$ contains all Sylow p-subgroups of G. The result follows. □

For p odd, let x be an element in G with the property that its disjoint cycle decomposition has a_i cycles of length p^i for every i with $0 \leq i \leq k$. For $(p, G) = (2, S_n)$ and $\sum_{i=1} a_i$ odd, let x be again an element in G with the property that its disjoint cycle decomposition has a_i cycles of length 2^i for every i with $0 \leq i \leq k$. For $(p, G) = (2, S_n)$ and $\sum_{i=1} a_i$ even, let x be an element of G whose disjoint cycle decomposition has a_i cycles of length 2^i for every i with $0 \leq i \leq k-2$, and $a_{k-1} + 2$ cycles of length 2^{k-1}. For $(p, G) = (2, A_n)$ and $\sum_{i=1} a_i$ odd, let x be again an element of G whose disjoint cycle decomposition has a_i cycles of length 2^i for every i with $0 \leq i \leq k-2$, and $a_{k-1}+2$ cycles of length 2^{k-1}. For $(p,G)=(2,A_n)$ and $\sum_{i=1} a_i$ even, let x be an element in G with the property that its disjoint cycle decomposition has a_i cycles of length 2^i for every i with $0 \leq i \leq k$. Let b_i denote the number of cycles of length p^i in the disjoint cycle decomposition of x for every i with $0 \leq i \leq k$. Let Π be the set of all conjugates of x in G.

Lemma 5.7. *Let G act on a set Ω of size n. Let S be a subgroup of G which leaves a nonempty proper subset Δ of Ω invariant. Assume that S is maximal subject to this condition. Let $x \in S$. For each nonnegative integer r, let the disjoint cycle decomposition of x on Δ have c_r cycles of lengths p^r. Then $\Pi \cap S$ is a conjugacy class in S and $|C_G(x)| \leq \prod_{r=0}^{k} \binom{b_r}{c_r} |C_S(x)|$. Moreover, if each c_r is 0 except c_j which is 1, then $|C_G(x)| = b_j |C_S(x)|$.*

Proof. Let y be an element of Π contained in S. This induces a permutation y_1 on Δ and a permutation y_2 on $\Omega \setminus \Delta$ such that $y = y_1 y_2 = y_2 y_1$, the set of points moved by y_1 is Δ, and the set of points moved by y_2 is $\Omega \setminus \Delta$.

Let p be odd. Since the p-adic decomposition of any positive integer is unique, the sets Δ and $\Omega \setminus \Delta$ determine the sets of cycle lengths of the permutations y_1 and y_2. This conclusion is also true when $p = 2$ by the construction of x (with the fact that x has at most three cycles of length 2^{k-1}). It follows that all elements in $\Pi \cap (\mathrm{Sym}(\Delta) \times \mathrm{Sym}(\Omega \setminus \Delta))$ are conjugate in $\mathrm{Sym}(\Delta) \times \mathrm{Sym}(\Omega \setminus \Delta)$ independent from p being odd or even. We find that all elements in $\Pi \cap S$ are conjugate in S unless possibly if $p = 2$ and $|\Delta| \in \{2, n-2\}$. This claim can also be checked when $p = 2$ and $|\Delta| \in \{2, n-2\}$ using the assumption that $n \geq 5$.

The group $C_G(x)$ has a normal subgroup C which is a direct product of cyclic groups such that C and $\langle x \rangle$ have the same orbits on Ω, and $C_G(x)/C$ is embedded in a direct product of symmetric groups. This subgroup C lies inside S and therefore in $C_S(x)$. For each nonnegative integer r with $b_r \geq 1$, the group $C_G(x)$ acts as S_{b_r} on the set of b_r cycles of the disjoint cycle decomposition of x of lengths p^r (using the assumption that $n \geq 5$). It follows that $C_S(x)$ has index at most $\prod_{r=0}^{k} \binom{b_r}{c_r}$ in $C_G(x)$. If all c_r are 0 except c_j which is 1, then the index of $C_S(x)$ in $C_G(x)$ is equal to b_j. $\square$

Let H be an intransitive maximal subgroup of G conjugate to $(S_{n-p^j} \times S_{p^j}) \cap G$.

Lemma 5.8. *If S is as in Lemma 5.7, then $|\Pi \cap S| \leq |\Pi \cap H|$.*

Proof. Let S and Δ be as in Lemma 5.7. Let the size of Δ be denoted by a.

Assume first that x acts as a cycle of length $a = p^r$ on Δ for some $r > j$. The group S is conjugate to $(S_{n-p^r} \times S_{p^r}) \cap G$. We have $|C_G(x)| \leq b_r |C_S(x)|$ by Lemma 5.7. It also follows by Lemma 5.7 that $|\Pi \cap S| = |S|/|C_S(x)| \leq b_r |S|/|C_G(x)|$ and $|\Pi \cap H| = |H|/|C_H(x)| = b_j |H|/|C_G(x)|$. We claim that $|\Pi \cap S| \leq |\Pi \cap H|$, that is,

$b_r|S| \leq b_j|H|$. This latter inequality is equivalent to the inequality

$$b_r((p^j+1)\cdots p^r) \leq b_j((n-p^r+1)\cdots(n-p^j)), \tag{5.1}$$

which is true since both sides of (5.1) have the same number of factors, $p^j + i \leq n - p^r + i$ for every i with $1 \leq i \leq p^r - p^j$, and $b_r p^r \leq n - p^j$.

We may assume that the disjoint cycle decomposition of x on Δ consists of at least two cycles. Let a longest cycle have length p^m.

Assume that $m \leq k-1$. In this case $a = \sum_{r=0}^{k} c_r p^r$ where the c_r are all non-negative and at most $p-1$, we have $c_r = 0$ for each r at least $m+1$, and $c_m \geq 1$. In order to prove the lemma in this case, it is sufficient to establish the following statement by Lemma 5.7 and the argument in the previous claim. For this, put $a_1 = a$ and $a_2 = a_1 - p^m$. Let S_1 and S_2 be subgroups conjugate to $(S_{a_1} \times S_{n-a_1}) \cap G$ and $(S_{a_2} \times S_{n-a_2}) \cap G$ in G respectively. After rearranging, it would be sufficient to prove the inequality

$$\binom{b_m}{c_m}(n-a_1)!a_1! \leq \binom{b_m}{c_m-1}(n-a_2)!a_2!.$$

This inequality is equivalent to

$$\binom{b_m}{c_m}(a_2+1)\cdots a_1 \leq \binom{b_m}{c_m-1}(n-a_1+1)\cdots(n-a_2). \tag{5.2}$$

There are the same number of factors on both sides of the inequality (5.2). From the second to the last pairs of factors these are increasing as in the proof above. Take the last pair of factors together with the first ones. It would be sufficient to establish the inequality $\binom{b_m}{c_m}a_1 \leq \binom{b_m}{c_m-1}(n-a_2)$, that is, the inequality

$$(b_m - c_m + 1)a_1 \leq c_m(n-a_2).$$

This in turn is equivalent to $(b_m+1)a_1 \leq c_m(n+p^m)$. Since $n \geq p^k + b_m p^m$ and $a_1 \leq (c_m+1)p^m$, it would be sufficient to establish the inequality

$$(b_m+1)(c_m+1)p^m \leq c_m(p^k + (b_m+1)p^m).$$

But $(b_m+1)p^m \leq p^{m+1} \leq c_m p^k$, since $m \leq k-1$.

We may thus assume that $m = k$, that is, the longest cycle in the disjoint cycle decomposition of x on Δ has length p^k. In particular, we may assume that $b_k \geq 2$ (otherwise we may replace Δ by $\Omega \setminus \Delta$ if necessary). After rearranging (5.2), we get

$$(b_k - c_k + 1)(a_2+1)\cdots a_1 \leq c_k(n-a_1+1)\cdots(n-a_2), \tag{5.3}$$

where a_1 and a_2 are defined as before. After substituting $a_2 = a_1 - p^k$ in (5.3) and using the inequalities $b_k - c_k + 1 < p$ and $c_k \geq 1$, we see that in order to prove the inequality (5.3), it is sufficient to establish the inequality

$$p - 1 \leq \prod_{i=1}^{p^k} \frac{n - a_1 + i}{a_1 - p^k + i}. \tag{5.4}$$

The inequality (5.4) would follow from the statement

$$(p-1)^{1/p^k}(a_1 - p^k + i) \leq n - a_1 + i$$

for every i such that $1 \leq i \leq p^k$. Since $i \leq p^k$ and $(p-1)^{1/p^k} - 1 \geq 0$, this statement would follow from the inequality

$$(1 + (p-1)^{1/p^k}) \cdot a_1 \leq n + p^k. \tag{5.5}$$

We may assume that $a_1 \leq n/2$. After replacing a_1 by $n/2$ in (5.5) and rearranging, we get

$$((p-1)^{1/p^k} - 1) \cdot \frac{n}{2} \leq p^k. \tag{5.6}$$

Since $n \leq p^{k+1}$, inequality (5.6) would follow from the inequality $p - 1 \leq (1 + \frac{2}{p})^{p^k}$. But this is true for every integer $k \geq 2$ and prime p.

We may finally assume that $m = k = 1$. We have $n = b_1 p + b_0$ and $a = c_1 p + c_0$. Assume first that $b_0 \geq 1$. From the beginning of this proof and Lemma 5.7, it is sufficient to establish the inequality

$$\binom{b_0}{c_0}\binom{b_1}{c_1} a!(n-a)! \leq b_0(n-1)!,$$

which, after a rearrangement, is

$$\binom{b_0}{c_0}\binom{b_1}{c_1} \cdot 2 \cdot 3 \cdot \ldots \cdot a \leq b_0 \cdot (n-a+1) \cdot (n-a+2) \cdot \ldots \cdot (n-1). \tag{5.7}$$

Since both b_0 and b_1 are at most $p-1$, we have $\binom{b_0}{c_0}\binom{b_1}{c_1} \leq 2^{2p-2}$. In order to establish (5.7), it would be sufficient to see

$$2^{2p-2} \leq b_0 \cdot 2^{a-2} \cdot \frac{n-1}{a},$$

assuming that $a \leq n/2$. This inequality is true if $c_1 \geq 2$. We may thus assume that $c_1 = 1$ and $a \leq n/2$. From the first half of the proof of this lemma we may also assume that $c_0 \geq 1$. In this case $a = p + c_0$ and $n = b_1 p + b_0$. Assume first that $b_0 \geq 2$. In order to establish (5.7), it would be sufficient to see

$$2^{b_0} \cdot b_1 \leq b_0 \cdot 2^{a-2} \cdot \Big(\frac{n-1}{a}\Big).$$

This follows from noting that $2^{b_0} \leq 2^{a-2}$ and that $b_1 a \leq b_0(n-1)$. Let $b_0 = 1$. In this case (5.7) follows from $b_1 \leq p - 1 \leq 2^{p-1} \leq b_0 \cdot 2^{a-2} \cdot ((n-1)/a)$.

We are left with the case when $m = k = 1$ and $b_0 = 0$, that is, when $n = b_1 p$ and $a = c_1 p$. Let us assume that $2 \leq c_1 \leq b_1/2$, that is, $2p \leq a \leq n/2$. We would like to establish the inequality

$$\binom{b_1}{c_1} a!(n-a)! \leq b_1 p!(n-p)!,$$

which becomes

$$\binom{b_1}{c_1} \cdot (p+1) \cdot (p+2) \cdot \ldots \cdot (c_1 p) \leq b_1 \cdot ((b_1 - c_1)p + 1) \cdot ((b_1 - c_1)p + 2) \cdot \ldots \cdot ((b_1 - 1)p). \tag{5.8}$$

There are $(c_1 - 1)p + 1$ factors on both sides of (5.8). If $c_1 \geq 3$, then

$$\binom{b_1}{c_1} \leq 2^{b_1} \leq b_1 \cdot (3/2)^{(c_1 - 1)p}$$

and so (5.8) follows. We may now assume that $c_1 = 2$. In this case (5.8) becomes

$$(b_1 - 1) \cdot (p+1) \cdot (p+2) \cdot \ldots \cdot (2p) \leq 2 \cdot ((b_1 - 2)p + 1) \cdot ((b_1 - 2)p + 2) \cdot \ldots \cdot ((b_1 - 1)p).$$

But this follows from the inequality $b_1 - 1 \leq p - 2 \leq 2 \cdot (3/2)^p$. □

We continue to assume that H is an intransitive maximal subgroup of G conjugate to $(S_{n-p^j} \times S_{p^j}) \cap G$.

Proposition 5.9. *If $|\Pi \cap S| \leq |\Pi \cap H|$ for any proper transitive subgroup S of G and $n \neq p^k$, then*

$$\frac{1}{b_j}\binom{n}{p^j} \leq \sigma_p(G) \leq \binom{n}{p^j}.$$

Proof. Let $n \neq p^k$. The upper bound follows from Lemma 5.6. We have $|\Pi \cap S| \leq |\Pi \cap H|$ for any intransitive subgroup S of G by Lemma 5.8. If $|\Pi \cap S| \leq |\Pi \cap H|$ for any proper subgroup S of G, then $\sigma_p(G) \geq |\Pi|/|\Pi \cap H|$. On the other hand,

$$\frac{|\Pi|}{|\Pi \cap H|} = \frac{|C_H(x)|}{|C_G(x)|}\frac{|G|}{|H|} = \frac{1}{b_j}\binom{n}{p^j}$$

by Lemma 5.7 and its proof. □

Let S be a maximal imprimitive subgroup of G. This has the form $S = (S_a \wr S_b) \cap G$ where a and b are integers at least 2 such that $n = ab$. Assume that $b < p$. In this case $\Pi \cap S = \Pi \cap (S_a)^b \cap G$. Since $(S_a)^b \cap G$ is intransitive, we see that $|\Pi \cap S| \leq |\Pi \cap H|$ from Lemma 5.8.

Let $n \leq 2p - 1$. Since $n \geq 5$, the prime p must be odd and $G = A_n$.

We will prove the following.

Proposition 5.10. *If $p + 1 \leq n \leq 2p - 1$ (and $n \geq 5$), then $\sigma_p(G) = p + 1$.*

Proof. Note that $G = A_n$. Clearly, G_p is a union of $p+1$ point-stabilizers H in G. Thus $\sigma_p(G) \leq p+1$. The statement now follows from Guralnick's proof for a general version of Theorem B (holding for all finite groups). Since this argument will appear elsewhere, we proceed to prove the inequality $\sigma_p(G) \geq p+1$ in a different way.

As before, let H be an intransitive maximal subgroup of G conjugate to $(S_{n-1} \times S_1) \cap G$, and let Π be the conjugacy class in G of x. Observe that if S is an intransitive subgroup of G intersecting Π non-trivially, then there is a conjugate H^g of H in G with $g \in G$ such that $\Pi \cap S \subseteq \Pi \cap H$. This means that in finding a covering $\mathcal{H}$ of G_p, consisting of proper subgroup of G, with $|\mathcal{H}|$ minimal, we may replace any intransitive subgroup in $\mathcal{H}$ by a conjugate of H.

Let $\mathcal{H}$ be a cover for G_p, with smallest possible size, consisting of proper subgroups of G with the property that if $S \in \mathcal{H}$ is intransitive then S is conjugate in G to H. We may also assume that $\mathcal{H}$ consists of maximal subgroup in G.

If $S = (S_a \wr S_b) \cap G$ is a maximal imprimitive subgroup of G, then $b < p$ and $a < p$ and so $\Pi \cap S = \emptyset$. It follows that $\mathcal{H}$ does not contain imprimitive subgroups of G.

Let S be a primitive proper subgroup of G such that S intersects Π non-trivially. The group S does not contain A_n. A theorem of Jordan [10, Theorem 3.3E] implies that $n = p+1$ or $n = p+2$. In the latter case S is 3-transitive by another theorem of Jordan [10, Theorem 7.4A (ii)].

If $n \geq p+3$, then $\mathcal{H}$ consists only of conjugates of H and it is easy to see that $|\mathcal{H}| > p$.

Let $n = p+2$. Assume that $\mathcal{H}$ contains $p+1-r$ conjugates of H for some positive integer r. Let $H_1, \ldots, H_{p+1-r}$ be the list of conjugates of H which are contained in $\mathcal{H}$. There are $(p-1)!\binom{r+1}{2}$ elements of Π not contained in $\bigcup_{i=1}^{p+1-r} H_i$. By the above, there must be at most r proper primitive subgroups of G each contained in $\mathcal{H}$ such that there union contains these $(p-1)!\binom{r+1}{2}$ elements of Π. In particular, $(p-1)!\binom{r+1}{2} \leq rm$ where m is the maximal size of a proper primitive subgroup of G. This gives $2((n-3)/e)^{n-3} < 2(n-3)! = (p-1)!(r+1) \leq 2m \leq 2n^{n/2}$ by Lemma 5.3 and a result of Bochert [2]. This forces $n \in \{5, 7, 9, 13, 15\}$.

Recall that S is 3-transitive (and S does not contain A_n). Since $n \in \{5, 7, 9, 13, 15\}$, we must have $n = 9$, may assume that $S = \mathrm{P\Gamma L}(2,8)$, and $|\Pi \cap S| = 216$, by Gap [19]. This gives $6!(r(r+1)/2) = (p-1)!\binom{r+1}{2} \leq 216r$, which is a contradiction.

Finally, let $n = p+1$. It is sufficient to see that all the conditions of Lemma 5.1 are satisfied with $X = G$. Only the last condition is to be checked. Let S be a proper subgroup of G. We would like to show that $|x^G \cap S| \leq (p-1)!/2 = |x^H|$. This is clear if S is conjugate to H. If S is not primitive and not conjugate to H, then $x^G \cap S = \emptyset$. Assume that S is primitive. Let K be a point-stabilizer. There are the following possibilities for K according to Theorem 5.2. The group K could be a subgroup of $\mathrm{AGL}(1,p)$. In this case $|x^G \cap S| = (p+1)(p-1)/2 \leq (p-1)!/2$. The prime p could be 11 and $K \leq \mathrm{M}_{11}$ (since $\mathrm{PSL}(2,11)$ is contained in M_{11} by [1]), or the prime p could be 23 and $K = \mathrm{M}_{23}$. We have seen that in this case $|x^G \cap S| = (p+1)|K|/p$, which is smaller than $(p-1)!/2$. Finally, the group K could be a subgroup of $\mathrm{P\Gamma L}(d,q)$ where $p = (q^d-1)/(q-1)$ for some integer $d \geq 2$ and some prime power q. First let $p \geq 17$. In the paragraph after Lemma 5.3 we saw that $(p-1)!/|\mathrm{P\Gamma L}(d,q)| \geq p+1$. From this we deduce that

$$|x^G \cap S| \leq (p+1)|K|/2 \leq (p+1)|\mathrm{P\Gamma L}(d,q)|/2 \leq (p-1)!/2.$$

Finally, let $p \in \{5,7,13\}$. In this case the inequality $|x^G \cap S| \leq (p-1)!/2$ may be checked by Gap [19]. □

This proves (ii) of Theorem C.

Let $n < p^2$. Since $n \geq 5$, the prime p is odd and $G = A_n$. We will prove the following.

Lemma 5.11. *If $2p \leq n < p^2$ and $n > 3^{16}$, then $|\Pi \cap S| \leq |\Pi \cap H|$ for any proper transitive subgroup S of G.*

Proof. Write n in the form $b_1p + b_0$ where $2 \leq b_1 \leq p-1$ and $0 \leq b_0 \leq p-1$. The disjoint cycle decomposition of x consists of b_1 cycles of length p and b_0 fixed points.

If $b_0 \geq 1$, then

$$|\Pi \cap H| = \frac{(n-1)!}{p^{b_1} b_1! (b_0-1)!}, \tag{5.9}$$

while if $b_0 = 0$, then

$$|\Pi \cap H| = \frac{p!(n-p)!}{p^{b_1}(b_1-1)!}. \tag{5.10}$$

If $b_0 \geq 1$, then (5.3) gives

$$\frac{(n-1)!}{p^{b_1} b_1! (b_0-1)!} > e^{-n} \frac{n^{n-1}}{(pb_1)^{b_1} \cdot {b_0}^{b_0-1}} > e^{-n} \frac{n^{n-1}}{n^{b_1} \cdot n^{b_0-1}} = e^{-n} n^{n-b_1-b_0}$$

and so (5.9) provides

$$|\Pi \cap H| > e^{-(b_1+1)p} n^{(b_1-1)p}. \tag{5.11}$$

If $b_0 = 0$, then (5.3) gives

$$\frac{p!(n-p)!}{p^{b_1}(b_1-1)!} > \frac{(n-p)!}{p^{b_1}} = \frac{((1-1/b_1)n)!}{p^{b_1}} > \Big(\frac{(1-1/b_1)n}{e}\Big)^{(1-1/b_1)b_1p} \cdot \frac{1}{p^{b_1}}$$

and so (5.10) provides

$$|\Pi \cap H| > \Big(\frac{(1-1/b_1)n}{e}\Big)^{(1-1/b_1)b_1p} \cdot \frac{1}{p^{b_1}}. \tag{5.12}$$

We may assume that S is a maximal subgroup in G. Assume first that S is conjugate to $(S_a \wr S_b) \cap G$ where a and b are integers at least 2 with $ab = n$. If $b < p$, then $|\Pi \cap S| \leq |\Pi \cap H|$ by the argument before Proposition 5.10. Assume that $b \geq p$. It follows that $a < p$. Let Σ be the system of imprimitivity for S. The action of any elements in $\Pi \cap S$ on Σ has c fixed points, for some integer c, and moves $b - c$ blocks in cycles of lengths p. It follows that $b_0 = ac$ and $b_1 p = a(b-c)$. Since $b_1 \geq 2$, all elements conjugate to x in S_n belong to Π. All elements in $\Pi \cap (S_a \wr S_b) = \Pi \cap S$ are conjugate in $S_a \wr S_b$. Since there is always an odd element in $S_a \wr S_b$ centralizing x, we find that $\Pi \cap S$ is a single conjugacy class in S. We find that

$$|\Pi \cap S| = \frac{a!^b \cdot b!}{a!^c \cdot c! \cdot a!^{(b-c)/p} \cdot ((b-c)/p)! \cdot p^{(b-c)/p}} \leq \left(a!^{(1-(1/p))} \cdot b\right)^{(b-c)}. \tag{5.13}$$

The right-hand side of (5.13) is less than $(a^a \cdot b)^{(b-c)} = \left(a \cdot b^{1/a}\right)^{a(b-c)} < \left(a \cdot n^{1/a}\right)^{b_1 p}$. The derivative of the function $f(x) = x \cdot n^{1/x}$ is $f'(x) = n^{1/x}(x - \ln n)/x$. It follows that the maximum of $f(x)$ in the interval $[2, \sqrt{n}]$ is $\max\{2\sqrt{n}, n^{1/\sqrt{n}}\sqrt{n}\} < 3\sqrt{n}$. These give

$$|\Pi \cap S| \leq 3^{b_1 p} \cdot n^{b_1 p/2}. \tag{5.14}$$

Let $b_0 \geq 1$. If $b_1 \geq 3$, then, comparing the bounds in (5.11) and (5.14), we see that $|\Pi \cap S| \leq |\Pi \cap H|$, provided that $n > 3^{14}$. Let $b_1 = 2$. In this case (5.9) and (5.14) give

$$|\Pi \cap H| = \frac{(n-1)!}{2p^2(b_0-1)!} \geq \frac{(2p)!}{2p^2} \geq 3^{2p} \cdot n^p \geq |\Pi \cap S|,$$

provided that $n \geq 1500$.

Let $b_0 = 0$. If $b_1 \geq 3$, then, comparing the bounds in (5.12) and (5.14), we see that $|\Pi \cap S| \leq |\Pi \cap H|$, provided that $n > 3^{16}$. Let $b_1 = 2$. In this case (5.10), (5.3), and (5.14) give

$$|\Pi \cap H| = \frac{p!^2}{p^2} > \left(\frac{p}{e}\right)^{2p} \frac{1}{p^2} \geq 3^{2p} \cdot (2p)^p \geq |\Pi \cap S|,$$

provided that $n = 2p > 3^7$.

From now on we may assume that S is a proper primitive subgroup of G. We have

$$|\Pi \cap S| < |S| \leq 50 \cdot n^{\sqrt{n}} < 50 \cdot n^p \tag{5.15}$$

by [26, Corollary 1.1].

Let $b_0 \geq 1$. If $b_1 \geq 3$, then (5.15) and (5.11) give

$$|\Pi \cap S| < 50 \cdot n^p < e^{-(b_1+1)p} n^{(b_1-1)p} < |\Pi \cap H|,$$

provided that $n \geq 3^8$. If $b_1 = 2$, then (5.15) and (5.9) give

$$|\Pi \cap S| < 50 \cdot n^p < \frac{(2p)!}{2p^2} < |\Pi \cap H|,$$

provided that $n \geq 3^7$.

Finally, let $b_0 = 0$. If $b_1 \geq 3$, then (5.15) and (5.12) give

$$|\Pi \cap S| < 50 \cdot n^{\sqrt{n}} < \Big(\frac{(2/3)n}{e}\Big)^{(2/3)n} \cdot \frac{1}{p^{n/p}} < |\Pi \cap H|,$$

provided that $n \geq 3^6$. If $b_1 = 2$, then (5.15) and (5.10) give

$$|\Pi \cap S| < 50 \cdot (2p)^{\sqrt{2p}} < \frac{(p!)^2}{p^2} < |\Pi \cap H|,$$

provided that $n \geq 3^5$. □

Let $n \geq p^2$. In this case $G \in \{A_n, S_n\}$.

Lemma 5.12. *If $n \geq \max\{p^2, 3^{11}\}$ and $n \neq p^k$, then $|\Pi \cap S| \leq |\Pi \cap H|$ for every proper transitive subgroup S of G.*

Proof. We first present an upper bound for $|C_G(x)|$. Recall that $n = \sum_{i=j}^{k} b_i p^i$ where $k \geq 2$ and where each b_i satisfies $0 \leq b_i \leq p-1$ unless possibly if $p = 2$ when $0 \leq b_{k-1} \leq 3$. We have

$$|C_{S_n}(x)| \leq \Big(\prod_{i=j}^{k} b_i!\Big)\Big(\prod_{i=j}^{k} p^{ib_i}\Big) \leq \frac{3}{2} \cdot \Big(\prod_{i=j}^{k} p^p\Big)\Big(\prod_{i=j}^{k} p^{ib_i}\Big) \leq \frac{3}{2} \cdot p^{k(k+3)p/2}.$$

It follows that

$$|\Pi \cap H| \geq \frac{(n-p^j)!(p^j)!}{3 \cdot p^{k(k+3)p/2}} \geq \frac{n!}{3 \cdot 2^n \cdot p^{k(k+3)p/2}} \geq \frac{n!}{3 \cdot 2^n \cdot n^{(5/2)\sqrt{n}}}. \tag{5.16}$$

If S is primitive, then

$$|\Pi \cap S| < |S| \leq 50 \cdot n^{\sqrt{n}} \leq \frac{n!}{3 \cdot 2^n \cdot n^{(5/2)\sqrt{n}}} \leq |\Pi \cap H|,$$

by [26, Corollary 1.1] and (5.16), provided that $n \geq 3^{11}$.

Let S be an imprimitive subgroup of G. We may assume that it is maximal, that is, it is the group $(S_a \wr S_b) \cap G$ where a and b are integers at least 2 with $ab = n$. Thus

$$|\Pi \cap S| < |S| \leq a!^b \cdot b! \leq e^{b+1}\Big(\frac{a}{e}\Big)^n\Big(\frac{b}{e}\Big)^b \leq e^{1-n} \cdot a^n \cdot b^b = e^{1-n} \cdot n^n \cdot b^{b-n},$$

by Lemma 5.3. This, (5.16), and Lemma 5.3 give

$$|\Pi \cap S| < e^{1-n} \cdot n^n \cdot b^{b-n} \leq \frac{n^n}{3 \cdot 2^n \cdot e^n \cdot n^{(5/2)\sqrt{n}}} < \frac{n!}{3 \cdot 2^n \cdot n^{(5/2)\sqrt{n}}} \leq |\Pi \cap H|,$$

provided that $b \geq 3$ and $n \geq 3^8$.

Let $b = 2$. If p is odd, then we argue as in the lines preceding Proposition 5.10. Let $p = 2$. We have

$$|C_{S_n}(x)| \leq 6 \cdot \left(\prod_{i=j}^{k} 2^{ib_i} \right) = 6 \cdot 2^{\sum_{i=j}^{k} ib_i} \leq \frac{3}{2} \cdot 2^{k(k+3)/2} \leq \frac{3}{2} \cdot n^{(\log_2 n + 3)/2},$$

since $n > 2^k > 4$. It follows that

$$|\Pi \cap H| \geq \frac{(n-2^j)!(2^j)!}{3 \cdot n^{(\log_2 n+3)/2}}. \tag{5.17}$$

On the other hand, $|\Pi \cap S| < |S| \leq 2(n/2)!^2$. We have $|\Pi \cap S| \leq |\Pi \cap H|$ if the inequality

$$6 \cdot (n/2)!^2 \cdot n^{(\log_2 n+3)/2} \leq (n-2^j)!(2^j)! \tag{5.18}$$

holds by (5.17). After rearranging (5.18), we get

$$6 \cdot n^{(\log_2 n+3)/2} \cdot (2^j + 1) \cdots (n/2) \leq ((n/2) + 1) \cdots (n - 2^j). \tag{5.19}$$

Observe that

$$\frac{(n/2) + i}{2^j + i} \geq \frac{4}{3}$$

for every positive integer i at most $(n/2) - 2^j$. Observe also that $(n/2) - 2^j \geq n/6$. From these it follows that (5.19) is satisfied provided that

$$6 \cdot n^{(\log_2 n+3)/2} \leq \left(\frac{4}{3}\right)^{n/6}. \tag{5.20}$$

It is easy to see that (5.20) holds if $n \geq 2^{10}$. □

We summarize the last results in this section in the following proposition.

Proposition 5.13. *If $n \geq 2p$, $n \neq p^k$, and $n > 3^{16}$, then*

$$\frac{1}{b_j}\binom{n}{p^j} \leq \sigma_p(G) \leq \binom{n}{p^j}.$$

Proof. This follows from Proposition 5.9 and Lemmas 5.11 and 5.12. □

We are now in the position to prove Theorem C.

Proof of Theorem C. This follows from the discussion above in the case when $n = p^k$. We have $\sigma_p(G) = p + 1$ when $p + 1 \leq n < 2p$ by Proposition 5.10. Let $n > 3^{16}$. In the remaining cases

$$\sigma_p(G) \geq \frac{1}{b_j}\binom{n}{p^j} \tag{5.21}$$

by Proposition 5.13. Since $1 \leq p^j \leq n/3$ and $b_j \leq \max\{3, p-1\}$, the right-hand side of (5.21) goes to infinity as $n \to \infty$ (for every fixed p). □

To conclude this section we investigate the relationship between $\sigma_p(G)$ and $\sigma(G)$.

Corollary 5.14. *Let p be a prime. The following are true.*

(i) *If n runs over the set of positive integers not of the form p^k, then both the sequence $\sigma_p(A_n)/\sigma(A_n)$ and the sequence $\sigma_2(S_n)/\sigma(S_n)$ tend to 0.*
(ii) *The sequence $\sigma_2(S_{2^k})/\sigma(S_{2^k})$ tends to 1 as k grows.*
(iii) *The sequence $\sigma_2(A_{2^k})/\sigma(A_{2^k})$ tends to 0 as k grows.*
(iv) *If $p > 2$, then the sequence $\sigma_p(A_{p^k})/\sigma(A_{p^k})$ tends to 1 as k grows.*

Proof. Let n be different from p^k, that is, different from a power of p. In this case $p^j < p^k$. We have $\sigma_p(S_n) \leq \binom{n}{p^j}$ and $\sigma_p(A_n) \leq \binom{n}{p^j}$ by Lemma 5.6. In the former case $p^j \leq n/3$. Every binomial coefficient of n is at most $c \cdot 2^n/\sqrt{n}$ for some universal constant c, by Stirling's approximation. If $n > 9$, then $\sigma(A_n) \geq 2^{n-2}$ by [27, Theorem 1.1]. This proves the first statement of (i). For the proof of the second statement of (i) we assume that $p = 2$. If $n > 9$ is odd, then $\sigma(S_n) = 2^{n-1}$ by [27, Theorem 1.1], while if n is even and tends to infinity then $\sigma(S_n) \sim \binom{n}{n/2}/2$ by [27, Theorem 3.2]. The observation $p^j \leq n/3$ together with the sentence after [27, Theorem 3.2] now imply (i).

Let $p = 2$ and $n = 2^k > 16$. Both $\sigma_2(S_n)$ and $\sigma_2(A_n)$ are equal to $\binom{n}{n/2}/2$ by Proposition 5.5. Statement (ii) follows from the previously mentioned fact that $\sigma(S_n) \sim \binom{n}{n/2}/2$. Statement (iii) follows from the facts that $\binom{n}{n/2}/2 \leq c \cdot 2^n/\sqrt{n}$ and $\sigma(A_n) \geq 2^{n-2}$.

We turn to the proof of (iv). Let us assume that $n = p^k > p$ for $p > 2$. We have

$$\sigma_p(A_n) = \frac{n!}{p!((n/p)!)^p}$$

by Proposition 5.4. Let us denote this formula by $h(n)$. It follows from [27, Theorem 4.2] that $h(n) < \sigma(A_n) < h(n) + g(n)$ where $g(n)$ is a function with the property that $g(n)/h(n)$ tends to 0. Thus

$$\frac{h(n)}{h(n) + g(n)} < \frac{\sigma_p(A_n)}{\sigma(A_n)} < 1$$

and the left-hand side of this inequality tends to 1. □

6. Concluding remarks

The following theorem is a consequence of Theorems C and D of [6] and a form of this was mentioned in the Introduction.

Theorem 6.1. *Let G be a group generated by its p-elements. If $x_1, \ldots, x_p \in G_p - Z(O_p(G))$, then $G_p \not\subseteq C_G(x_1) \cup \cdots \cup C_G(x_p)$.*

As mentioned in the Introduction, our results have some consequences on the noncommuting graph.

Corollary 6.2. *Let G be a p-solvable group. Let $x_1, \dots, x_p \in G - Z(G)$. Then there exists $y \in G_p$ such that y is joined to $x_1, \dots, x_p$. In particular, the induced subgraph by the noncommuting graph of G on $G_p - Z(G)$ has diameter at most 2.*

Proof. There exists $y \in G_p - (C_G(x_1) \cup \cdots \cup C_G(x_p))$ by Theorem B. The result follows. □

Corollary 6.3. *Let G be a group generated by its p-elements. Let Δ be the induced subgraph of the noncommuting graph of G on $G_p - Z(O_p(G))$. Then for any $x_1, \dots, x_p \in G_p - Z(O_p(G))$ there exists $y \in G_p - Z(O_p(G))$ such that y is joined to x_i for every i. In particular, Δ is connected with diameter ≤ 2.*

Proof. There exists $y \in G_p - (C_G(x_1) \cup \cdots \cup C_G(x_p))$ by Theorem 6.1. The result follows. □

It is not clear if we really need to remove the elements from $Z(O_p(G))$ both in Theorem 6.1 and in Corollary 6.3.

Our work in this paper suggests that perhaps the following question has an affirmative answer.

Question 6.4. *Let G be a group generated by its p-elements and let p be a prime. Let $\sigma_p(G) = n < \infty$. Does there exist $\{H_1, \dots H_n\}$ cover of G_p such that every H_i contains a Sylow p-subgroup of G?*

We do not even know an example of a group G with a covering $\{H_1, \dots, H_n\}$ of G_p of size $\sigma_p(G) = n$ such that $|H_i| < |G|_p$ for some i.

Another problem that seems interesting is the following. Sambale and Tărnăuceanu proved in [32] that there exists $c = c(n) > 0$ such that if a finite group G is not covered by $\{H_1, \dots, H_n\}$ then the proportion of elements of G in $G \setminus (H_1 \cup \cdots \cup H_n)$ is at least c. (Actually, they proved stronger and more precise results.) The following is the p-version of this.

Question 6.5. *Does there exist $c = c(n) > 0$ (possibly depending on p too) such that if G is a finite group and G_p is not covered by $H_1, \dots, H_n < G$ then $|G_p \setminus (H_1 \cup \cdots \cup H_n)|/|G_p| \geq c$?*

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, ATLAS of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985.

[2] A. Bochert, Über die Transitivitätsgrenze der Substitutionengruppen, welche die Alternierende ihres Grades nicht einhalten. *Math. Ann.* **33**, (1889), 572–583.

[3] T. Breuer, R. M. Guralnick, Finite groups can be generated by a π-subgroup and a π'-subgroup. To appear in *Israel J. Math.*

[4] J. R. Britnell, A. Evseev, R. M. Guralnick, P. E. Holmes, A. Maróti, Sets of elements that pairwise generate a linear group. *J. Combin. Theory Ser. A* **115** (2008), no. 3, 442–465.

[5] R. A. Bryce, V. Fedri, L. Serena, Subgroup coverings of some linear groups. *Bull. Austral. Math. Soc.* **60** (1999), no. 2, 227–238.

[6] T. C. Burness, R. M. Guralnick, A. Moretó, and G. Navarro, On the commuting probability of p-elements in a finite group. *Algebra & Number Theory* **17** (2023), no. 6, 1209–1229.

[7] P. Cameron, Graphs defined on groups. *Int. J. Group Theory* **11** (2022), 53–107.

[8] J. H. E. Cohn, On n-sum groups. *Math. Scand.* **75** (1994), no. 1, 44–58.

[9] L. E. Dickson, Linear groups: With an exposition of the Galois field theory. Dover Publications, Inc., New York 1958.

[10] J. D. Dixon, B. Mortimer, Permutation groups. *Springer-Verlag, New York* 1996.

[11] W. Findlay, The Sylow subgroups of the symmetric group. *Trans. Amer. Math. Soc.* **24** (1904), 263–278.

[12] F. Fumagalli, M. Garonzi, On the primary coverings of finite solvable and symmetric groups. *J. Group Theory* **24** (2021), 1189–1121.

[13] P. E. Holmes, A. Maróti, Pairwise generating and covering sporadic simple groups. *J. Algebra* **324** (2010), no. 1, 25–35.

[14] M. Hall, On the number of Sylow subgroups in a finite group. *J. Algebra* **7** (1967), 363–371.

[15] P. Hall, A note on soluble groups. *J. London Math. Soc.* **3** (1928),98–105.

[16] B. Huppert, Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften, Band 134 Springer-Verlag, Berlin-New York 1967.

[17] F. Fumagalli, M. Garonzi, On the primary coverings of finite solvable and symmetric groups. *J. Group Theory* **24** (2021), no. 6, 1189–1211.

[18] F. Fumagalli, M. Garonzi, A. Maróti, On the maximal number of elements pairwise generating the symmetric group of even degree. *Discrete Math.* **345** (2022), no. 4, Paper No. 112776.

[19] The GAP Group, GAP Groups, algorithms, and programming, version 4.12.1, 2022, http://www.gap-system.org.

[20] W. H. Gustafson, What is the probability that two elements commute?, *Amer. Math. Monthly* **80** (1973), 1031–1034.

[21] G. A. Jones, Cyclic regular subgroups of primitive permutation groups. *J. Group Theory* **5** (2002), no. 4, 403–407.

[22] M. Garonzi, Finite groups that are the union of at most 25 proper subgroups. *J. Algebra Appl.* **12** (2013), no. 4, 1350002.

[23] M. Garonzi, L-C. Kappe, E. Swartz, On integers that are covering numbers of groups. *Exp. Math.* **31** (2022), no. 2, 425–443.

[24] M. Garonzi, A. Maróti, Covering certain wreath products with proper subgroups. *J. Group Theory* **14** (2011), no. 1, 103–125.

[25] M. S. Lucido, On the covers of finite groups. Groups St. Andrews 2001 in Oxford. Vol. II, 395–399, London Math. Soc. Lecture Note Ser., 305, Cambridge Univ. Press, Cambridge, 2003.

[26] A. Maróti, On the orders of primitive groups. *J. Algebra* **258** (2002), no. 2, 631–640.

[27] A. Maróti, Covering the symmetric groups with proper subgroups. *J. Combin. Theory Ser A.* **110** (2005), 97–111.

[28] A. Maróti, J. Martínez, A. Moretó, Covering the set of p-elements in finite groups by Sylow p-subgroups. Submitted for publication.

[29] G. Navarro, Number of Sylow subgroups in p-solvable groups. *Proc. Amer. Math. Soc.* **131** (2003), 3019–3020.

[30] B. H. Neumann, Groups covered by finitely many cosets. *Publ. Math. Debrecen* **3** (1954), 227–242.

[31] L. Pyber, The number of pairwise noncommuting elements and the index of the centre in a finite group. *J. London Math. Soc.* (2) **35** (1987), no. 2, 287–295.

[32] B. Sambale, M. Tărnăuceanu, On the size of coset unions. *J. Algebraic Combin.* **55** (2022), no. 3, 979–987.

[33] E. Swartz, On the covering number of symmetric groups having degree divisible by six. *Discrete Math.* **339** (2016), no. 11, 2593–2604.

[34] M. J. Tomkinson, Groups as the union of proper subgroups. *Math. Scand.* **81** (1997), no. 2, 191–198.

Alfréd Rényi Institute, Reáltanoda Utca 13-15, H-1053, Budapest, Hungary
Email address: maroti@renyi.hu

Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain
Email address: Juan.Martinez-Madrid@uv.es

Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain
Email address: alexander.moreto@uv.es