Acute sets in Euclidean spaces

Viktor Harangi

April 22, 2011

Abstract

A finite set $\mathcal{H}$ in $\mathbb{R}^d$ is called an acute set if any angle determined by three points of $\mathcal{H}$ is acute. We examine the maximal cardinality $\alpha(d)$ of a d-dimensional acute set. The exact value of $\alpha(d)$ is known only for $d \leq 3$. For each $d \geq 4$ we improve on the best known lower bound for $\alpha(d)$. We present different approaches. On one hand, we give a probabilistic proof that $\alpha(d) > c \cdot 1.2^d$. (This improves a random construction given by Erdős and Füredi.) On the other hand, we give an *almost* exponential constructive example which outdoes the random construction in low dimension ($d \leq 250$). Both approaches use the small dimensional examples that we found partly by hand ($d = 4, 5$), partly by computer ($6 \leq d \leq 10$).

We also investigate the following variant of the above problem: what is the maximal size $\kappa(d)$ of a d-dimensional cubic acute set (that is, an acute set contained in the vertex set of a d-dimensional hypercube). We give an *almost* exponential constructive lower bound, and we improve on the best known upper bound.

2010 Mathematics Subject Classification: 05D40, 51M16.
Keywords: acute sets; probabilistic method; acute angles; strictly antipodal.

Department of Analysis, Eötvös Loránd University,
Pázmány Péter sétány 1/c, H-1117 Budapest, Hungary
Email address: harangi@gmail.com

Acknowledgement: The author was supported by Hungarian Scientific Foundation grant no. 72655.

1 Introduction

Around 1950 Erdős conjectured that given more than 2^d points in $\mathbb{R}^d$ there must be three of them determining an obtuse angle. The vertices of the d-dimensional cube show that 2^d points exist with all angles at most $\pi/2$.

In 1962 Danzer and Grünbaum proved this conjecture [7] (their proof can also be found in [2]). They posed the following question in the same paper: what is the maximal number of points in $\mathbb{R}^d$ such that all angles determined are acute? (In other words, this time we want to exclude right angles as well as obtuse angles.) A set of such points will be called an *acute set* in the sequel.

The exclusion of right angles seemed to decrease the maximal number of points dramatically: they could only give $2d-1$ points, and they conjectured that this is the best possible. However, this was only proved for $d = 2, 3$. (For the non-trivial case $d = 3$, see Croft [6], Schütte [10], Grünbaum [9].)

Then in 1983 Erdős and Füredi disproved the conjecture of Danzer and Grünbaum. They used the probabilistic method to prove the existence of a d-dimensional acute set of cardinality exponential in d. Their idea was to choose random points from the vertex set of the d-dimensional unit cube, that is $\{0,1\}^d$. We denote the maximal size of acute sets in $\mathbb{R}^d$ and in $\{0,1\}^d$ by $\alpha(d)$ and $\kappa(d)$, respectively; clearly $\alpha(d) \geq \kappa(d)$. The random construction of Erdős and Füredi implied the following lower bound for $\kappa(d)$ (thus for $\alpha(d)$ as well)

$$\kappa(d) > \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d > 0.5 \cdot 1.154^d. \tag{1}$$

In their paper they claimed (without proof) that "a bit more complicated random process gives" $\kappa(d) > \left(\sqrt[4]{2} - o(1)\right)^d > 1.189^d$. The best (published) lower bound both for $\alpha(d)$ and for $\kappa(d)$ (for large values of d) is due to Ackerman and Ben-Zwi from 2009 [1]. They improved (1) with a factor $\sqrt{d}$:

$$\alpha(d) \geq \kappa(d) > c\sqrt{d}\left(\frac{2}{\sqrt{3}}\right)^d. \tag{2}$$

In Section 2 we modify the random construction of Erdős and Füredi to get

$$\alpha(d) > c\left(\sqrt[10]{\frac{144}{23}}\right)^d > c \cdot 1.2^d. \tag{3}$$

A different approach can be found in Section 3 where we recursively construct acute sets. These constructions outdo (3) up to dimension 250. In Theorem 3.10 we will show that this constructive lower bound is *almost* exponential in the following sense: given any positive integer r, for infinitely many values of d we have a d-dimensional acute set of cardinality at least

$$\exp(d/\underbrace{\log\log\cdots\log}_{r}(d)).$$

See Table 2 in the Appendix for the best known lower bounds of $\alpha(d)$ ($d \leq 84$). These bounds are new results except for $d \leq 3$.

Both the probabilistic and the constructive approach use small dimensional acute sets as building blocks. So it is crucial for us to construct small dimensional acute sets of large cardinality. In Section 4 we present an acute set of 8 points in $\mathbb{R}^4$ and an acute set of 12 points in $\mathbb{R}^5$ (disproving the conjecture of Danzer and Grünbaum for $d \geq 4$ already). We used computer to find acute sets in dimension $6 \leq d \leq 10$, see Section 4. Table 1 shows our results compared to the construction of Danzer and Grünbaum ($2d-1$) and the examples found by Bevan using computer.

As far as $\kappa(d)$ is concerned, in large dimension (2) is still the best known lower bound. Bevan used computer to determine the exact values of $\kappa(d)$ for $d \leq 9$ [4]. He also gave a recursive construction improving upon the random constructions in low dimension. The

Table 1: Results for $\alpha(d)$ ($d \leq 10$)

dim(d)	*D,G*[7]	*Bevan*[4]	*Our result*
2	$= 3$		
3	$= 5$		
4	≥ 7		≥ 8
5	≥ 9		≥ 12
6	≥ 11		≥ 16
7	≥ 13	≥ 14	≥ 20
8	≥ 15	≥ 16	≥ 23
9	≥ 17	≥ 19	≥ 27
10	≥ 19	≥ 23	≥ 31

constructive approach of Section 3 yields a lower bound not only for $\alpha(d)$ but also for $\kappa(d)$, which further improves the bounds of Bevan in low dimension. Table 3 in the Appendix shows the best known lower bounds for $\kappa(d)$ ($d \leq 82$). These bounds are new results except for $d \leq 12$ and $d = 27$.

The following notion plays an important role in both approaches.

Definition 1.1. A triple A, B, C of three points in $\mathbb{R}^d$ will be called *bad* if for each integer $1 \leq i \leq d$ the i-th coordinate of B equals the i-th coordinate of A or C.

We denote by $\kappa_n(d)$ the maximal size of a set $S \subset \{0, 1, \ldots, n-1\}^d$ that contains no *bad triples*. It is easy to see that $\kappa_2(d) = \kappa(d)$ but our main motivation to investigate $\kappa_n(d)$ is that we can use sets without bad triples to construct acute sets recursively (see Lemma 2.2). We give an upper bound for $\kappa_n(d)$ (Theorem 3.1) and two different lower bounds (Theorem 2.3 and 3.5). In the special case $n = 2$ the upper bound yields $\kappa(d) \leq 3(\sqrt{2})^{d-1}$ which improves the bound $\sqrt{2}\left(\sqrt{3}\right)^d$ given by Erdős and Füredi in [8]. Note that for $\alpha(d)$ the best known upper bound is $2^d - 1$.

Although we can make no contribution to it, we mention that there is an affine variant of this problem. A finite set $\mathcal{H}$ in $\mathbb{R}^d$ is called *strictly antipodal* if for any two distinct points $P, Q \in \mathcal{H}$ there exist two parallel hyperplanes, one through P and the other through Q, such that all other points of $\mathcal{H}$ lie strictly between them. Let $\alpha'(d)$ denote the maximal cardinality of a d-dimensional strictly antipodal set. An acute set is strictly antipodal, thus $\alpha'(d) \geq \alpha(d)$. For $\alpha'(d)$ Talata gave the following constructive lower bound [11]:

$$\alpha'(d) \geq \sqrt[4]{5}^d/4 > 0.25 \cdot 1.495^d.$$

A weaker result (also due to Talata) can be found in [5, Lemma 9.11.2].

2 The probabilistic approach

As we mentioned in the introduction, in 1983 Erdős and Füredi proved the existence of acute sets of exponential cardinality [8]. Since then their proof has become a well-known example to demonstrate the probabilistic method. In this section we use similar arguments to prove a better lower bound for $\alpha(d)$. The following problem plays a key role in our approach.

Question 2.1. What is the maximal cardinality $\kappa_n(d)$ of a set $S \subset \{0, 1, \ldots, n-1\}^d$ that contains no bad triples? (Recall Definition 1.1.)

In the case $n = 2$, given three distinct points $A, B, C \in \{0, 1\}^d$, $\angle ABC = \pi/2$ holds if and only if A, B, C is a bad triple, otherwise $\angle ABC$ is acute. So a set $S \subset \{0, 1\}^d$ contains no bad triples if and only if S is an acute set, thus $\kappa_2(d) = \kappa(d)$.

If $n > 2$, then a triple being bad still implies that the angle determined by the triple is $\pi/2$ but we can get right angles from good triples as well, moreover, we can even get obtuse angles. So for $n > 2$ the above problem is not directly related to acute sets. However, the following simple lemma shows how one can use sets without bad triples to construct acute sets recursively.

Lemma 2.2. *Suppose that $\mathcal{H} = \{h_0, h_1, \ldots, h_{n-1}\} \subset \mathbb{R}^m$ is an acute set of cardinality n. If $S \subset \{0, 1, \ldots, n-1\}^d$ contains no bad triples, then*

$$\mathcal{H}^S \stackrel{\text{def}}{=} \{(h_{i_1}, h_{i_2}, \ldots, h_{i_d}) : (i_1, i_2, \ldots, i_d) \in S\} \subset \underbrace{\mathcal{H} \times \mathcal{H} \times \ldots \times \mathcal{H}}_{d} \subset \mathbb{R}^{md}$$

is also an acute set. Consequently,

$$\alpha(md) \geq \kappa_{\alpha(m)}(d) \quad \textit{and} \quad \kappa(md) \geq \kappa_{\kappa(m)}(d). \tag{4}$$

Proof. Take three distinct points of S:

$$i = (i_1, i_2, \ldots, i_d); \quad j = (j_1, j_2, \ldots, j_d); \quad k = (k_1, k_2, \ldots, k_d),$$

and the corresponding points in $\mathcal{H}^S$:

$$h_i = (h_{i_1}, h_{i_2}, \ldots, h_{i_d}); \quad h_j = (h_{j_1}, h_{j_2}, \ldots, h_{j_d}); \quad h_k = (h_{k_1}, h_{k_2}, \ldots, h_{k_d}).$$

We show that $\angle h_i h_j h_k$ is acute by proving that the scalar product

$$\langle h_i - h_j, h_k - h_j \rangle = \sum_{r=1}^{d} \langle h_{i_r} - h_{j_r}, h_{k_r} - h_{j_r} \rangle$$

is positive. Since $\mathcal{H}$ is an acute set, the summands on the right-hand side are positive with the exception of those where j_r equals i_r or k_r, in which case the r-th summand is 0. This cannot happen for each r though, else i, j, k would be a bad triple in S.

To prove (4) we set $|\mathcal{H}| = n = \alpha(m)$ and $|S| = \kappa_n(d) = \kappa_{\alpha(m)}(d)$. Then $\alpha(md) \geq |\mathcal{H}^S| = |S| = \kappa_{\alpha(m)}(d)$. A similar argument works for $\kappa(md)$. (Note that if $\mathcal{H} \subset \{0, 1\}^m$, then $\mathcal{H}^S \subset \{0, 1\}^{md}$.) □

In view of the above lemma, it would be useful to construct large sets without bad triples. One possibility is using the probabilistic method. The next theorem is a generalization of the original random construction of Erdős and Füredi.

Theorem 2.3.

$$\kappa_n(d) > \frac{1}{2}\left(\frac{n^2}{2n-1}\right)^{\frac{d}{2}} > \frac{1}{2}\left(\frac{n}{2}\right)^{\frac{d}{2}} = \left(\frac{1}{2}\right)^{\frac{d+2}{2}} n^{\frac{d}{2}}.$$

Proof. For a positive integer m, we take $2m$ (independent and uniformly distributed) random points in $\{0, 1, \ldots, n-1\}^d$: $A_1, A_2, \ldots, A_{2m}$. What is the probability that the triple A_1, A_2, A_3 is bad? For a fixed i, the probability that the i-th coordinate of A_2 is equal to the i-th coordinate of A_1 or A_3 is clearly $(2n-1)/n^2$. These events are independent so the probability that this holds for every i (that is to say A_1, A_2, A_3 is a bad triple) is

$$p = \left(\frac{2n-1}{n^2}\right)^d.$$

We get the same probability for all triples, thus the expected value of the number of bad triples is

$$3\binom{2m}{3}p = \frac{2m(2m-1)(2m-2)}{2}p < 4m^3 p \le m, \text{ where we set } m = \left\lfloor \frac{1}{2\sqrt{p}} \right\rfloor.$$

Consequently, the $2m$ random points determine less than m bad triples with positive probability. Now we take out one point from each bad triple. Then the remaining at least $m+1$ points obviously contain no bad triples. So we have proved that there exist

$$m + 1 > \frac{1}{2\sqrt{p}} = \frac{1}{2}\left(\sqrt{\frac{n^2}{2n-1}}\right)^d$$

points in $\{0, 1\}^d$ without a bad triple. (Note that the original $2m$ random points might contain duplicated points. However, a triple of the form A, A, B is always bad, thus the final (at least) $m+1$ points contain no duplicated points.) □

Combining Lemma 2.2 and Theorem 2.3 we readily get the following.

Corollary 2.4. *Suppose that we have an m-dimensional acute set of size n. Then for any positive integer t*

$$\alpha(mt) > \frac{1}{2}\left(\sqrt{\frac{n^2}{2n-1}}\right)^t,$$

which yields the following lower bound in general dimension:

$$\alpha(d) \ge \alpha\left(m\left\lfloor\frac{d}{m}\right\rfloor\right) > \frac{1}{2}\left(\sqrt[2m]{\frac{n^2}{2n-1}}\right)^{m\left\lfloor\frac{d}{m}\right\rfloor} \ge c\left(\sqrt[2m]{\frac{n^2}{2n-1}}\right)^d.$$

Using this corollary with $m = 5$ and $n = 12$ (see Example 4.2 for a 5-dimensional acute set with 12 points) we obtain the following.

Theorem 2.5.

$$\alpha(d) > c\left(\sqrt[10]{\frac{144}{23}}\right)^d > c \cdot 1.2^d,$$

that is, there exist at least $c \cdot 1.2^d$ points in $\mathbb{R}^d$ such that any angle determined by three of these points is acute. (If d is divisible by 5, then c can be chosen to be $1/2$, for general d we need to use a somewhat smaller c.)

Remark 2.6. We remark that one can improve the above result with a factor $\sqrt{d}$ by using the method suggested by Ackerman and Ben-Zwi in [1].

Remark 2.7. We could have applied Corollary 2.4 with any specific acute set. The larger the value $\sqrt[2m]{n^2/(2n-1)}$ is, the better the lower bound we obtain. For $m = 1, 2, 3$ the largest values of n are known.

$$\left.\begin{matrix} m=1 \\ n=2 \end{matrix}\right\} \sqrt[2]{\frac{4}{3}} \approx 1.154 \qquad \left.\begin{matrix} m=2 \\ n=3 \end{matrix}\right\} \sqrt[4]{\frac{9}{5}} \approx 1.158 \qquad \left.\begin{matrix} m=3 \\ n=5 \end{matrix}\right\} \sqrt[6]{\frac{25}{9}} \approx 1.185$$

We will construct small dimensional acute sets in Section 4 (see Table 1 for the results). For $m = 4, 5, 6$ these constructions yield the following values for $\sqrt[2m]{n^2/(2n-1)}$.

$$\left.\begin{matrix} m=4 \\ n=8 \end{matrix}\right\} \sqrt[8]{\frac{64}{15}} \approx 1.198 \qquad \left.\begin{matrix} m=5 \\ n=12 \end{matrix}\right\} \sqrt[10]{\frac{144}{23}} \approx \mathbf{1.201} \qquad \left.\begin{matrix} m=6 \\ n=16 \end{matrix}\right\} \sqrt[12]{\frac{256}{31}} \approx 1.192$$

However, we do not know whether these acute sets are optimal or not. If we found an acute set of 9 points in $\mathbb{R}^4$, 13 points in $\mathbb{R}^5$ or 18 points in $\mathbb{R}^6$, we could immediately improve Theorem 2.5.

3 The constructive approach

3.1 On the maximal cardinality of sets without bad triples

In this subsection we investigate Question 2.1 more closely. Recall that $\kappa_n(d)$ denotes the maximal cardinality of a set in $\{0, 1, \dots, n-1\}^d$ containing no bad triples. We have already seen a probabilistic lower bound for $\kappa_n(d)$ (Theorem 2.3). Here we first give an upper bound. As we will see, this upper bound is essentially sharp if n is large enough (compared to d).

Theorem 3.1. *For even d*

$$\kappa_n(d) \le 2n^{d/2},$$

and for odd d

$$\kappa_n(d) \le n^{(d+1)/2} + n^{(d-1)/2}.$$

Proof. Suppose that $S \subset \{0, 1, \dots, n-1\}^d$ contains no bad triples. Let $0 < r < d$ be an integer, and consider the following two projections:

$$\pi_1((x_1, \dots, x_d)) = (x_1, \dots, x_r)\ ;\ \pi_2((x_1, \dots, x_d)) = (x_{r+1}, \dots, x_d).$$

Now we take the set

$$S_0 \stackrel{\text{def}}{=} \{x \in S : \exists y \in (S \setminus \{x\})\ \pi_1(x) = \pi_1(y)\}.$$

By definition π_1 is injective on $S \setminus S_0$, thus $|S \setminus S_0| \le n^r$. We claim that π_2 is injective on S_0, so $|S_0| \le n^{d-r}$. Otherwise there would exist $x, y \in S_0$ such that $\pi_2(x) = \pi_2(y)$. Since $y \in S_0$, there exists $z \in S$ such that $\pi_1(y) = \pi_1(z)$. It follows that the triple x, y, z is bad, contradiction.

Consequently, $|S| \le n^r + n^{d-r}$. Setting $r = \left\lfloor \frac{d}{2} \right\rfloor$ we get the desired upper bound. $\square$

Setting $n = 2$ and using that $\kappa_2(d) = \kappa(d)$ the next corollary readily follows.

Corollary 3.2. *For even d*

$$\kappa(d) \leq 2^{(d+2)/2} = 2\left(\sqrt{2}\right)^d,$$

and for odd d

$$\kappa(d) \leq 2^{(d+1)/2} + 2^{(d-1)/2} = \frac{3}{\sqrt{2}}\left(\sqrt{2}\right)^d.$$

This corollary improves the upper bound $\sqrt{2}(\sqrt{3})^d$ given by Erdős and Füredi in [8]. (We note though that they proved not only that a subset of $\{0,1\}^d$ of size larger than $\sqrt{2}(\sqrt{3})^d$ must contain three points determining a right angle but they also showed that such a set cannot be strictly antipodal which is a stronger assertion.)

If n is a prime power greater than d, then the following constructive method gives better lower bound than the random construction of the previous section. We will need matrices over finite fields with the property that every square submatrix of theirs is invertible. In coding theory the so-called *Cauchy matrices* are used for that purpose.

Definition 3.3. Let $\mathbb{F}_q$ denote the finite field of order q. A $k \times l$ matrix A over $\mathbb{F}_q$ is called a *Cauchy matrix* if it can be written in the form

$$A_{i,j} \stackrel{\text{def}}{=} (x_i - y_j)^{-1} \quad (i = 1, \ldots, k; j = 1, \ldots, l), \tag{5}$$

where $x_1, \ldots, x_k, y_1, \ldots, y_l \in \mathbb{F}_q$ and $x_i \neq y_j$ for any pair of indices i, j.

In the case $k = l = r$, the determinant of a Cauchy matrix A is given by

$$det(A) = \frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{1\leq i,j\leq r}(x_i - y_j)}.$$

This well-known fact can be easily proved by induction. It follows that A is invertible provided that the elements $x_1, \ldots, x_r, y_1, \ldots, y_r$ are pairwise distinct.

Lemma 3.4. *Let q be a prime power and k, l be positive integers. Suppose that $q \geq k + l$. Then there exists a $k \times l$ matrix over $\mathbb{F}_q$ any square submatrix of which is invertible.*

Proof. Let $x_1, \ldots, x_k, y_1, \ldots, y_l$ be pairwise distinct elements of $\mathbb{F}_q$, and take the $k \times l$ Cauchy matrix A as in (5). Clearly, every submatrix of A is also a Cauchy matrix thus the determinant of every square submatrix of A is invertible. □

Now let $k + l = d \geq 2$ and n be a prime power greater than or equal to d. Due to the lemma, there exists a $k \times l$ matrix A over the field $\mathbb{F}_n$ such that each square submatrix of A is invertible. Let us think of $\{0, 1, \ldots, n-1\}^d$ as the d-dimensional vector space $\mathbb{F}_n^d$. We define an $\mathbb{F}_n$-linear subspace of $\mathbb{F}_n^d$: take all points $(x, Ax) \in \mathbb{F}_n^d$ as x runs through $\mathbb{F}_n^l$ (thus $Ax \in \mathbb{F}_n^k$). This is an l-dimensional subspace consisting n^l points. We claim that each of its points has at least $k + 1$ nonzero coordinates. We prove this by contradiction. Assume that there is a point (x, Ax) which has at most k nonzero coordinates. Let the number of nonzero coordinates of x be r. It follows that the number of nonzero coordinates of Ax is at most $k-r$, in other words, Ax has at least r zero coordinates. Consequently, A has an $r \times r$ submatrix which takes a vector with nonzero elements to the null vector. This contradicts the assumption that every square submatrix is invertible.

Setting $k = \lfloor \frac{d}{2} \rfloor$ and $l = \lceil \frac{d}{2} \rceil$ we get a subspace of dimension $\lceil \frac{d}{2} \rceil$, every point of which has at least $\lfloor \frac{d}{2} \rfloor + 1 > \frac{d}{2}$ nonzero coordinates. We claim that this subspace does not contain bad triples. Indeed, taking distinct points $x_1, x_2, x_3 \in \mathbb{R}^l$, the points $(x_1 - x_2, A(x_1 - x_2))$ and $(x_3 - x_2, A(x_3 - x_2))$ are elements of the subspace, thus both have more than $\frac{d}{2}$ nonzero coordinates which means that there is a coordinate where both of them take nonzero value. We have proved the following theorem.

Theorem 3.5. *If $d \geq 2$ is an integer and $n \geq d$ is a prime power, then*

$$\kappa_n(d) \geq n^{\lceil \frac{d}{2} \rceil}.$$

If n is not a prime power, then there exists no finite field of order n. We can still consider matrices over the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. If we could find a $\lfloor \frac{d}{2} \rfloor \times \lceil \frac{d}{2} \rceil$ matrix with all of its square submatrices invertible, it would imply the existence of a set without bad triples and of cardinality $n^{\lceil \frac{d}{2} \rceil}$. For example, in the case $d = 3$ the matrix $(1 \;\; 1)$ over $\mathbb{Z}_n$ is clearly good for any n so the next theorem follows.

Theorem 3.6. *For arbitrary positive integer n it holds that $\kappa_n(3) \geq n^2$.*

Proof. We can prove this directly by taking all points in the form $(i, j, i+j)$ where i, j run through $\mathbb{Z}_n$ (addition is meant modulo n). Clearly, there are no bad triples among these n^2 points. □

Finally we show that the upper bound given in Theorem 3.1 is sharp apart from a constant factor provided that n is sufficiently large compared to d.

Theorem 3.7. *We have*

$$\kappa_n(d) > (1 - \varepsilon(d, n)) \cdot n^{\lceil \frac{d}{2} \rceil},$$

where for any fixed d the error term $\varepsilon(d, n)$ converges to 0 as $n \to \infty$. In fact, for any $\delta > 0$ there exists C_δ such that $\varepsilon(d, n) < \delta$ provided that $n \geq C_\delta \cdot d^{2.11}$.

Proof. It was proved in [3] that for any sufficiently large n there is a prime number q in the interval $[n - n^{0.525}, n]$. If $q \geq d$, then by Theorem 3.5 we can find a set $S \subset \{0, 1, \ldots, q-1\}^d \subset \{0, 1, \ldots, n-1\}^d$ such that S contains no bad triples and

$$|S| \geq q^{\lceil \frac{d}{2} \rceil} \geq (n - n^{0.525})^{\lceil \frac{d}{2} \rceil} = \left(1 - n^{-0.475}\right)^{\lceil \frac{d}{2} \rceil} n^{\lceil \frac{d}{2} \rceil}.$$

For fixed d the coefficient of $n^{\lceil \frac{d}{2} \rceil}$ clearly converges to 1 as $n \to \infty$. To obtain the stronger claim we use the well-known fact that $(1 - 1/x)^{x-1} > 1/e$ for any $x > 1$. Consequently, if the exponent $\lceil \frac{d}{2} \rceil$ is at most $(n^{0.475} - 1)\delta$, then we have

$$|S| > (1/e)^\delta n^{\lceil \frac{d}{2} \rceil} > (1 - \delta) n^{\lceil \frac{d}{2} \rceil},$$

whence $\varepsilon(d, n) < \delta$. A simple calculation completes the proof. □

3.2 Constructive lower bounds for $\alpha(d)$ and $\kappa(d)$

Random constructions of acute sets (as the original one of Erdős and Füredi or the one given in Section 2) give exponential lower bound for $\alpha(d)$. However, these only prove existence without telling us exactly how to find such large acute sets. Also, one can give better (constructive) lower bound if the dimension is small.

The first (non-linear) constructive lower bound is due to Bevan[4]:

$$\alpha(d) \geq \kappa(d) > \exp\left(cd^{\mu}\right), \text{ where } \mu = \frac{\log 2}{\log 3} = 0.631... \tag{6}$$

For small d this is a better bound than the probabilistic ones.

Our goal in this section is to obtain even better constructive bounds. The key will be the next theorem which follows readily from Lemma 2.2, Theorem 3.5 and Theorem 3.6 setting $d = 2s - 1$. (In fact, the special case $s = 2$ was already proved by Bevan, see [4, Theorem 4.2]. He obtained (6) by the repeated application of this special case.)

Theorem 3.8. *Let $s \geq 2$ be an integer, and suppose that $n \geq 2s-1$ is a prime power. (In the case $s = 2$ the theorem holds for arbitrary positive integer n.) If $\mathcal{H} \subset \mathbb{R}^m$ is an acute set of cardinality n, then we can choose n^s points of the set*

$$\underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{2s-1} \subset \mathbb{R}^{(2s-1)m}$$

that form an acute set.

Remark 3.9. If $\mathcal{H}$ is cubic (that is, $\mathcal{H} \subset \{0,1\}^m$), then the obtained acute set is also cubic (that is, it is in $\{0,1\}^{(2s-1)m}$).

Now we start with an acute set $\mathcal{H}$ of prime power cardinality and we apply the previous theorem with the largest possible s. Then we do the same for the obtained larger acute set (the cardinality of which is also a prime power). How large acute sets do we get if we keep doing this? For the sake of simplicity, let us start with the $d_0 = 4$ dimensional acute set of size $n_0 = 8$ that we will construct in Section 4. Let us denote the dimension and the size of the acute set we obtain in the k-th step by d_k and n_k, respectively. Clearly n_k is a power of 2, thus at step $(k+1)$ we can apply Theorem 3.8 with $s_k = n_k/2$. Setting $u_k = \log_2 n_k$ we get the following:

$$d_{k+1} = d_k(2s_k - 1) < d_k n_k; \quad n_{k+1} = n_k^{s_k} = n_k^{n_k/2};$$
$$u_{k+1} = u_k(n_k/2) = u_k 2^{u_k - 1} \geq 2 \cdot 2^{u_k-1} = 2^{u_k}.$$

It follows that $d_{k+1}/u_{k+1} \leq 2d_k/u_k$ so

$$d_k \leq \frac{d_0}{u_0} u_k 2^k = \frac{4}{3} 2^k u_k.$$

It yields that in dimension d_k we get an acute set of size

$$n_k = 2^{u_k} \geq 2^{(3/4)2^{-k} d_k}.$$

Due to the factor 2^{-k} in the exponent, n_k is not exponential in d_k. However, the inequality $u_{k+1} \geq 2^{u_k}$ implies that u_k grows extremely fast (and so does n_k and d_k) which means that n_k is *almost* exponential. For instance, we can easily obtain that for any positive integer r there exists k_0 such that for $k \geq k_0$ it holds that

$$n_k > \exp(d_k / \underbrace{\log\log\cdots\log}_{r}(d_k)).$$

We have given a constructive proof of the following theorem.

Theorem 3.10. *For any positive integer r we have infinitely many values of d such that*

$$\alpha(d) > \exp(d / \underbrace{\log\log\cdots\log}_{r} d).$$

We can also get a constructive lower bound for $\kappa(d)$. We do the same iterated process but this time we start with an acute set in $\{0,1\}^{d_0}$. (For instance, we can set $d_0 = 3$ and $n_0 = 4$.) Then the acute set obtained in step k will be in $\{0,1\}^{d_k}$. This way we get an almost exponential lower bound for $\kappa(d)$ as well.

However, Theorem 3.8 gives acute sets only in certain dimensions. In the remainder of this section we consider the problems investigated so far in a slightly more general setting to get large acute sets in any dimension. (The proofs of these more general results are essentially the same as the original ones. Thus we could have considered this general setting in the first place, but for the sake of better understanding we opted not to.)

Let $n_1, n_2, \ldots, n_d \geq 2$ be positive integers and consider the $n_1 \times \cdots \times n_d$ lattice, that is the set $\{0,1,\ldots,n_1-1\} \times \cdots \times \{0,1,\ldots,n_d-1\}$.

Question 3.11. What is the maximal cardinality of a subset S of the $n_1 \times \cdots \times n_d$ lattice containing no bad triples?

We claim that if $n \geq \max\{n_1,\ldots,n_d\}$ and the set $S_0 \subset \{0,1,\ldots,n-1\}^d$ contains no bad triples, then we can get a set in the $n_1 \times \cdots \times n_d$ lattice without bad triples and of cardinality at least

$$\frac{n_1}{n} \cdots \frac{n_d}{n} |S_0|.$$

Indeed, starting with the $n \times \ldots \times n$ lattice, we replace the n's one-by-one with the n_i's; in each step we keep those n_i *sections* that contain the biggest part of S_0. Combining this argument with Theorem 3.5 and 3.6 we get the following for the odd case $d = 2s-1$.

Theorem 3.12. *Let $s \geq 2$, and suppose that $n \geq 2s-1$ is a prime power (in the case $s = 2$ the theorem holds for arbitrary positive integer n). For positive integers $n_1, \ldots, n_{2s-1} \leq n$ in the $n_1 \times n_2 \times \cdots \times n_{2s-1}$ lattice at least $\lceil n_1 n_2 \cdots n_{2s-1}/n^{s-1} \rceil$ points can be chosen without any bad triple.*

Also, one can get a more general version of Lemma 2.2 with the same proof.

Lemma 3.13. *Suppose that the set $\mathcal{H}_t = \{h_0^t, h_1^t, \ldots, h_{n_t-1}^t\} \subset \mathbb{R}^{m_t}$ is acute for each $1 \leq t \leq d$. If $S \subset \{0,1,\ldots,n_1-1\} \times \cdots \times \{0,1,\ldots,n_d-1\}$ contains no bad triples, then the set*

$$\left\{ \left(h_{i_1}^1, h_{i_2}^2, \ldots, h_{i_d}^d\right) : (i_1, i_2, \ldots, i_d) \in S \right\} \subset \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_d \subset \mathbb{R}^{m_1+\cdots+m_d}$$

is an $(m_1 + \cdots + m_d)$-dimensional acute set.

Putting these results together we obtain a more general form of Theorem 3.8.

Theorem 3.14. *Let $s \geq 2$, and suppose that $n \geq 2s-1$ is a prime power (in the case $s = 2$ the theorem holds for arbitrary positive integer n). Assume that for each $t = 1, \dots, 2s-1$ we have an acute set of $n_t \leq n$ points in $\mathbb{R}^{m_t}$. Then in $\mathbb{R}^{m_1+\dots+m_{2s-1}}$ there exists an acute set of cardinality at least*

$$\left\lceil n_1 n_2 \cdots n_{2s-1}/n^{s-1} \right\rceil .$$

The obtained acute set is cubic provided that all acute sets used are cubic.

Remark 3.15. We also note that in the case $s = 3$ the theorem can be applied for $n = 4$ as well. Consider the 4-element field $\mathbb{F}_4 = \{0, 1, a, b\}$. Then the 2×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \end{pmatrix}$$

has no singular square submatrix which implies that Theorem 3.5 holds for $d = 5; n = 4$, thus Theorem 3.12 and Theorem 3.14 hold for $s = 3; n = 4$.

Now we can use the small dimensional acute sets of Section 4 as building blocks to build higher dimensional acute sets by Theorem 3.14. Table 2 in the Appendix shows the lower bounds we get this way for $d \leq 84$. (We could keep doing that for larger values of d and up to dimension 250 we would get better bound than the probabilistic one given in Section 2.) These bounds are all new results except for $d \leq 3$.

We can do the same for $\kappa(d)$, see Table 3 in the Appendix for $d \leq 82$. This method outdoes the random construction up to dimension 200. (We need small dimensional cubic acute sets as building blocks. We use the ones found by Bevan who used computer to determine the exact values of $\kappa(d)$ for $d \leq 9$. He also used a recursive construction to obtain bounds for larger d's. His method is similar but less effective: our results are better for $d \geq 13; d \neq 27$. In dimension $d = 63$ we get a cubic acute set of size 65536. This is almost ten times bigger than the one Bevan obtained which contains 6561 points.)

Tables 4 and 5 in the Appendix compare the probabilistic and constructive lower bounds for $\alpha(d)$ and $\kappa(d)$.

Finally we prove the simple fact that $\alpha(d)$ is strictly monotone increasing. We will need this fact in Table 2.

Lemma 3.16. *$\alpha(d+1) > \alpha(d)$ holds for any positive integer d.*

Proof. Assume that we have an acute set $\mathcal{H} = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$. Let P be the convex hull of $\mathcal{H}$ and y be any point in $P \setminus \mathcal{H}$. We claim that $\angle y x_i x_j < \pi/2$ for any $i \neq j$. Let $H_{i,j}$ be the hyperplane that is perpendicular to the segment $x_i x_j$ and goes through x_i. Let $S_{i,j}$ be the open half-space bounded by $H_{i,j}$ that contains x_j. For a point $z \in \mathbb{R}^d$ the angle $\angle z x_i x_j$ is acute if and only if $z \in S_{i,j}$. It follows that $\mathcal{H} \setminus \{x_i\} \subset S_{i,j}$ while x_i lies on the boundary of $S_{i,j}$. Thus $y \in P \setminus \{x_i\} \subset S_{i,j}$ which implies that $\angle y x_i x_j < \pi/2$.

Now let us consider the usual embedding of $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ and let $\mathbf{v}$ denote the unit vector $(0, \dots, 0, 1)$. Consider the point $y^t = y + t\mathbf{v}$ for sufficiently large t. It is easy to see that $\angle y^t x_i x_j < \pi/2$ still holds, but now even the angles $\angle x_i y^t x_j$ are acute. It follows that $\mathcal{H} \cup \{y^t\} \subset \mathbb{R}^{d+1}$ is an acute set. □

Remark 3.17. For $\kappa(d)$ it is only known that $\kappa(d+2) > \kappa(d)$ [4, Theorem 4.1]. In Table 3 we will refer to this result as *almost strict monotonicity*.

4 Small dimensional acute sets

In this section we construct acute sets in dimension $m = 4, 5$ and use computer to find such sets for $6 \le m \le 10$. These small dimensional examples are important because the random construction of Section 2 and the recursive construction of Section 3 use them to find higher dimensional acute sets of large cardinality.

Danzer and Grünbaum presented an acute set of $2m - 1$ points in $\mathbb{R}^m$ [7]. It is also known that for $m = 2, 3$ this is the best possible [6, 10, 9]. Bevan used computer to find small dimensional acute sets by generating random points on the unit sphere. For $m \ge 7$ he found more than $2m - 1$ points [4].

Our approach starts similarly as the construction of Danzer and Grünbaum. We consider the following $2m - 2$ points in $\mathbb{R}^m$:

$$P_i^{\pm 1} = (0, \dots, 0, \underbrace{\pm 1}_{i\text{-th}}, 0, \dots, 0); \qquad i = 1, 2, \dots, m-1.$$

What angles do these points determine? Clearly, $\angle P_i^{-1} P_j^{\pm 1} P_i^{+1} = \pi/2$ for $i \ne j$ and all other angles are acute. We can get rid of the right angles by slightly perturbing the points in the following manner:

$$\widetilde{P}_i^{\pm 1} = (0, \dots, 0, \underbrace{\pm 1}_{i\text{-th}}, 0, \dots, 0, \varepsilon_i); \qquad i = 1, 2, \dots, m-1, \tag{7}$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}$ are pairwise distinct real numbers.

Our goal is to complement the points $\widetilde{P}_i^{\pm 1}$ with some additional points such that they still form an acute set. In fact, we will complement the points $P_i^{\pm 1}$ such that all *new* angles are acute. (Then changing the points $P_i^{\pm 1}$ to $\widetilde{P}_i^{\pm 1}$ we get an acute set provided that the ε_i's are small enough.)

Under what condition can a point $\mathbf{x} = (x_1, \dots, x_m)$ be added in the above sense? Simple calculation shows that the exact condition is

$$\|\mathbf{x}\| > 1 \text{ and } |x_i| + |x_j| < 1 \text{ for } 1 \le i, j \le m-1; i \ne j. \tag{8}$$

For example, the point $A = (0, \dots, 0, a)$ can be added for $a > 1$. This way we get an acute set of size $2m-1$. Basically, this was the construction of Danzer and Grünbaum. We know that this is the best possible for $m = 2, 3$. However, we can do better if $m \ge 4$.

Suppose that we have two points $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ both satisfying (8) (that is, they can be separately added). Both points can be added (at the same time) if and only if

$$|x_i + y_i| < 1 + \langle \mathbf{x}, \mathbf{y} \rangle \text{ and } |x_i - y_i| < \min\left(\|\mathbf{x}\|^2, \|\mathbf{y}\|^2\right) - \langle \mathbf{x}, \mathbf{y} \rangle \text{ for } 1 \le i \le m-1. \tag{9}$$

We can find two such points in the following simple form: $A_1 = (a_1, a_1, \dots, a_1, a_2)$ and $A_2 = (-a_1, -a_1, \dots, -a_1, a_2)$. Then points A_1 and A_2 can be added if and only if

$$\frac{1}{m-1} < a_1 < \frac{1}{2} \text{ and } a_2^2 > \left|1 - (m-1)a_1^2\right|. \tag{10}$$

Such a_1 and a_2 clearly exist if $m \ge 4$.

Example 4.1. For sufficiently small and pairwise distinct ε_i's the 8 points below form an acute set in $\mathbb{R}^4$.

$$
\begin{array}{cccccc}
(& 1 & 0 & 0 & \varepsilon_1 &) \\
(& -1 & 0 & 0 & \varepsilon_1 &) \\
(& 0 & 1 & 0 & \varepsilon_2 &) \\
(& 0 & -1 & 0 & \varepsilon_2 &) \\
(& 0 & 0 & 1 & \varepsilon_3 &) \\
(& 0 & 0 & -1 & \varepsilon_3 &) \\
(& 0.4 & 0.4 & 0.4 & 1 &) \\
(& -0.4 & -0.4 & -0.4 & 1 &)
\end{array}
$$

For $m = 5$, we can even add four points of the following form:

$$A_1 = (a_1, a_1, a_1, a_1, a_2); A_2 = (-a_1, -a_1, -a_1, -a_1, a_2);$$
$$B_1 = (b_1, b_1, -b_1, -b_1, -b_2); B_2 = (-b_1, -b_1, b_1, b_1, -b_2).$$

We have seen that $1/4 < a_1, b_1 < 1/2$ must hold so we set $a_1 = 1/4 + \delta$ and $b_1 = 1/2 - \delta$. Then we set $a_2 = \sqrt{3}/2$ and $b_2 = 2\sqrt{\delta}$ so that $\|A_i\|$ and $\|B_i\|$ are slightly bigger than 1.

Example 4.2. Let us fix a positive real number $\delta < 1/48$ and consider the points below.

$$
\begin{array}{rcccccc}
A_1 = (& 1/4+\delta & 1/4+\delta & 1/4+\delta & 1/4+\delta & \sqrt{3}/2 &) \\
A_2 = (& -1/4-\delta & -1/4-\delta & -1/4-\delta & -1/4-\delta & \sqrt{3}/2 &) \\
B_1 = (& 1/2-\delta & 1/2-\delta & -1/2+\delta & -1/2+\delta & -2\sqrt{\delta} &) \\
B_2 = (& -1/2+\delta & -1/2+\delta & 1/2-\delta & 1/2-\delta & -2\sqrt{\delta} &)
\end{array}
$$

Then the set $\{\widetilde{P}_i^{\pm 1} : i = 1, 2, 3, 4\} \cup \{A_1, A_2, B_1, B_2\}$ is an acute set of 12 points in $\mathbb{R}^5$ assuming that ε_i's are sufficiently small and pairwise distinct.
(This specific example is important because the random method presented in Section 2 gives the best result starting from this example.)

Proof. We need to prove that A_1, A_2, B_1, B_2 can be added to $P_i^{\pm 1}$'s in such a way that all *new* angles are acute. First we prove that any pair of these 4 points can be added. Since each of them satisfies (8), we only have to check that each pair satisfies (9). For the pair A_1, A_2 we are done since they satisfy (10). It goes similarly for the pair B_1, B_2. For the pairs A_i, B_j (9) yields the condition $3/4 < 1 - \sqrt{3\delta} \Leftrightarrow \delta < 1/48$.

Now we have checked all new angles except those that are determined by three new points. The squares of the distances between the 4 new points are:

$$d(A_1, A_2)^2 = 1 + 8\delta + 16\delta^2; d(B_1, B_2)^2 = 4 - 16\delta + 16\delta^2; d(A_i, B_j)^2 = 2 + 2\sqrt{3\delta} + 2\delta + 8\delta^2.$$

Now for any triangle in $\{A_1, A_2, B_1, B_2\}$ the square of each side is less than the sum of the squares of the two other sides which means that the triangle is acute-angled. □

For $m \geq 6$ we used computer to add further points to the system (7). We generated random points on the sphere with radius $1 + \delta$ and we added the point whenever it was possible. Table 1 shows the cardinality of acute sets we found this way compared to previous results. The reader can also find examples for $m = 6, 7, 8$ in the Appendix. For $m \geq 11$, the recursive construction presented in Section 3 gives better result than the computer search (see Table 2 in the Appendix for the best known lower bounds of $\alpha(d)$ for $d \leq 84$).

Appendix A Small dimensional acute sets

We have seen in Section 4 that the following $2m-2$ points form an acute set in $\mathbb{R}^m$:

$$\widetilde{P}_i^{\pm 1} = (0,\ldots,0,\underbrace{\pm 1}_{i\text{-th}},0,\ldots,0,\varepsilon_i); \qquad i=1,2,\ldots,m-1,$$

where $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{m-1}$ denote sufficiently small, pairwise distinct real numbers. We used computer to add points to this system in such a way that they still form an acute set. Below the reader can find the obtained additional points in dimension $m=6,7,8$. In order to get integer coordinates, we start with the enlarged system $999\cdot\widetilde{P}_i^{\pm 1}$; $i=1,2,\ldots,m-1$.

With the following additional points we have an acute set of 16 points in $\mathbb{R}^6$.

$$\begin{array}{lrrrrrrr}
(& -88 & 2 & -244 & -35 & 124 & -957 &) \\
(& 1 & -448 & -458 & -482 & 485 & 349 &) \\
(& -537 & 364 & -358 & -227 & -426 & 466 &) \\
(& -386 & 473 & 494 & -420 & 455 & -18 &) \\
(& 455 & 467 & -47 & 490 & 296 & 494 &) \\
(& 435 & 411 & -431 & -533 & -39 & -413 &)
\end{array}$$

With the following additional points we have an acute set of 20 points in $\mathbb{R}^7$.

$$\begin{array}{lrrrrrrrr}
(& -398 & -425 & -271 & 548 & 316 & -191 & -389 &) \\
(& -29 & 174 & -320 & 278 & 322 & 250 & 789 &) \\
(& -413 & -261 & -498 & -295 & -263 & -288 & 524 &) \\
(& 453 & -273 & -380 & -241 & -493 & 438 & -288 &) \\
(& -224 & 473 & -260 & -410 & 73 & 319 & -619 &) \\
(& -398 & 28 & 348 & 475 & -511 & 479 & 60 &) \\
(& -117 & -420 & 377 & -422 & 548 & 386 & 199 &) \\
(& 506 & -444 & 490 & 292 & -233 & -409 & -20 &)
\end{array}$$

With the following additional points we have an acute set of 23 points in $\mathbb{R}^8$.

$$\begin{array}{lrrrrrrrrr}
(& -403 & 160 & 381 & 120 & -438 & 470 & 435 & -226 &) \\
(& -3 & 470 & -158 & -424 & -375 & 423 & 233 & 447 &) \\
(& -456 & 349 & 387 & -135 & -32 & -538 & -438 & 145 &) \\
(& 166 & -170 & -16 & 286 & -35 & -314 & 188 & 853 &) \\
(& 239 & -281 & 451 & -297 & -521 & 255 & -454 & 173 &) \\
(& 271 & 273 & 438 & -543 & 204 & 446 & 148 & -321 &) \\
(& 384 & 149 & -408 & 476 & -499 & 116 & -195 & -370 &) \\
(& -239 & -414 & -499 & -151 & -230 & -273 & 99 & -603 &) \\
(& 563 & 410 & 93 & 219 & -399 & -415 & 354 & 26 &)
\end{array}$$

Appendix B Best known bounds in low dimension

The following tables show the best known lower bounds for $\alpha(d)$ and $\kappa(d)$. Beside the dimension and the bound itself, we stated the value of s, n and the product $n_1 \cdots n_{2s-1}/n^{s-1}$ with which Theorem 3.14 is applied. From the n_i's the reader can easily obtain the m_i's. *Str. mon.* and *a. str. mon.* stand for *strict monotonicity* (cf. Lemma 3.16) and *almost strict monotonicity* (cf. Remark 3.17).

For example, in dimension 39 in Table 2 we see that $s = 5$ and $n = 9$. (Note that n is indeed a prime power and $n \geq 2s - 1$ holds.) The expression $8^6 \cdot 9^3/9^4$ means that we need to apply Theorem 3.14 with $n_1 = n_2 = \ldots = n_6 = 8$ and $n_7 = n_8 = n_9 = 9$. (Note that they are all indeed at most n.) Then for each i we take the smallest dimension m_i in which we have an acute set containing at least n_i points. In our case the corresponding dimensions are $m_1 = m_2 = \ldots = m_6 = 4$ and $m_7 = m_8 = m_9 = 5$. Consequently, the total dimension is $6 \cdot 4 + 3 \cdot 5 = 39$. We obtain that in $\mathbb{R}^{39}$ there exists an acute set of cardinality at least $\lceil 8^6 \cdot 9^3/9^4 \rceil = 29128$.

Recall that in the case $s = 2$ we can take arbitrary n (it does not need to be a prime power). See dimension 14 and 15 in Table 2.

Also, according to Remark 3.15, in the case $s = 3$ we can have $n = 4$ (even though $n \geq 2s - 1$ does not hold). See dimension 13 and 15 in Table 3.

Table 2: Best known lower bound for $\alpha(d)$ $(1 \le d \le 84)$

dim	l. b.	s	n		dim	l. b.	s	n	
1	2				43	85184	5	11	$8^2 \cdot 11^7/11^4$
2	3				44	120439	5	13	$8^1 \cdot 12^8/13^4$
3	5				45	180659	5	13	$12^9/13^4$
4	8			construction	46	195714	5	13	$12^8 \cdot 13^1/13^4$
5	12			construction	47	212023	5	13	$12^7 \cdot 13^2/13^4$
6	16			computer	48	229692	5	13	$12^6 \cdot 13^3/13^4$
7	20			computer	49	262144	6	11	$8^6 \cdot 11^5/11^5$
8	23			computer	50	360448	6	11	$8^5 \cdot 11^6/11^5$
9	27			computer	51	495616	6	11	$8^4 \cdot 11^7/11^5$
10	31			computer	52	681472	6	11	$8^3 \cdot 11^8/11^5$
11	40	2	8	$5^1 \cdot 8^2/8^1$	53	937024	6	11	$8^2 \cdot 11^9/11^5$
12	64	2	8	$8^3/8^1$	54	1334092	6	13	$8^1 \cdot 12^{10}/13^5$
13	65			str. mon.	55	2001138	6	13	$12^{11}/13^5$
14	96	2	12	$8^1 \cdot 12^2/12^1$	56	2167900	6	13	$12^{10} \cdot 13^1/13^5$
15	144	2	12	$12^3/12^1$	57	2348558	6	13	$12^9 \cdot 13^2/13^5$
16	145			str. mon.	58	2544271	6	13	$12^8 \cdot 13^3/13^5$
17	192	2	16	$12^1 \cdot 16^2/16^1$	59	2756293	6	13	$12^7 \cdot 13^4/13^5$
18	256	2	16	$16^3/16^1$	60	2985984	6	16	$12^6 \cdot 16^5/16^5$
19	320	3	8	$5^1 \cdot 8^4/8^2$	61	4378558	7	13	$8^4 \cdot 12^9/13^6$
20	512	3	8	$8^5/8^2$	62	6567837	7	13	$8^3 \cdot 12^{10}/13^6$
21	513			str. mon.	63	9851755	7	13	$8^2 \cdot 12^{11}/13^6$
22	514			str. mon.	64	14777632	7	13	$8^1 \cdot 12^{12}/13^6$
23	704	3	11	$8^2 \cdot 11^3/11^2$	65	22166447	7	13	$12^{13}/13^6$
24	982	3	13	$8^1 \cdot 12^4/13^2$	66	24013651	7	13	$12^{12} \cdot 13^1/13^6$
25	1473	3	13	$12^5/13^2$	67	26014789	7	13	$12^{11} \cdot 13^2/13^6$
26	1600	4	8	$5^2 \cdot 8^5/8^3$	68	28182688	7	13	$12^{10} \cdot 13^3/13^6$
27	2560	4	8	$5^1 \cdot 8^6/8^3$	69	30531245	7	13	$12^9 \cdot 13^4/13^6$
28	4096	4	8	$8^7/8^3$	70	33075516	7	13	$12^8 \cdot 13^5/13^6$
29	4097			str. mon.	71	35831808	7	16	$12^7 \cdot 16^6/16^6$
30	4098			str. mon.	72	47775744	7	16	$12^6 \cdot 16^7/16^6$
31	4099			str. mon.	73	63700992	7	16	$12^5 \cdot 16^8/16^6$
32	5632	4	11	$8^3 \cdot 11^4/11^3$	74	84934656	7	16	$12^4 \cdot 16^9/16^6$
33	7744	4	11	$8^2 \cdot 11^5/11^3$	75	113246208	7	16	$12^3 \cdot 16^{10}/16^6$
34	10873	4	13	$8^1 \cdot 12^6/13^3$	76	150994944	7	16	$12^2 \cdot 16^{11}/16^6$
35	16310	4	13	$12^7/13^3$	77	201326592	7	16	$12^1 \cdot 16^{12}/16^6$
36	20457	5	9	$8^9/9^4$	78	268435456	7	16	$16^{13}/16^6$
37	23015	5	9	$8^8 \cdot 9^1/9^4$	79	268435457			str. mon.
38	25891	5	9	$8^7 \cdot 9^2/9^4$	80	268435458			str. mon.
39	29128	5	9	$8^6 \cdot 9^3/9^4$	81	322486272	8	16	$12^9 \cdot 16^6/16^7$
40	36864	4	16	$12^2 \cdot 16^5/16^3$	82	429981696	8	16	$12^8 \cdot 16^7/16^7$
41	49152	4	16	$12^1 \cdot 16^6/16^3$	83	573308928	8	16	$12^7 \cdot 16^8/16^7$
42	65536	4	16	$16^7/16^3$	84	764411904	8	16	$12^6 \cdot 16^9/16^7$

Table 3: Best known lower bound for $\kappa(d)$ $(1 \le d \le 82)$

dim	l. b.	s	n		dim	l. b.	s	n	
1	2				42	4096	4	8	$8^7/8^3$
2	2				43	4096			
3	4				44	4097			a. str. mon.
4	5			Bevan	45	4097			
5	6			Bevan	46	4608	4	9	$8^3 \cdot 9^4/9^3$
6	8			Bevan	47	5184	4	9	$8^2 \cdot 9^5/9^3$
7	9			Bevan	48	5832	4	9	$8^1 \cdot 9^6/9^3$
8	10			Bevan	49	6561	4	9	$9^7/9^3$
9	16	2	4	$4^3/4^1$	50	7991	5	9	$5^2 \cdot 8^7/9^4$
10	16				51	10229	5	9	$4^1 \cdot 8^8/9^4$
11	20	2	5	$4^1 \cdot 5^2/5^1$	52	12786	5	9	$5^1 \cdot 8^8/9^4$
12	25	2	5	$5^3/5^1$	53	15343	5	9	$6^1 \cdot 8^8/9^4$
13	32	3	4	$2^1 \cdot 4^4/4^2$	54	20457	5	9	$8^9/9^4$
14	32				55	23015	5	9	$8^8 \cdot 9^1/9^4$
15	64	3	4	$4^5/4^2$	56	25891	5	9	$8^7 \cdot 9^2/9^4$
16	64				57	29128	5	9	$8^6 \cdot 9^3/9^4$
17	65			a. str. mon.	58	32768	5	9	$8^5 \cdot 9^4/9^4$
18	80	3	5	$4^2 \cdot 5^3/5^2$	59	36864	5	9	$8^4 \cdot 9^5/9^4$
19	100	3	5	$4^1 \cdot 5^4/5^2$	60	41472	5	9	$8^3 \cdot 9^6/9^4$
20	125	3	5	$5^5/5^2$	61	46656	5	9	$8^2 \cdot 9^7/9^4$
21	125				62	52488	5	9	$8^1 \cdot 9^8/9^4$
22	126			a. str. mon.	63	65536	4	16	$16^7/16^3$
23	126				64	65536			
24	133	3	7	$5^1 \cdot 6^4/7^2$	65	65537			a. str. mon.
25	160	3	8	$4^1 \cdot 5^1 \cdot 8^3/8^2$	66	65537			
26	200	3	8	$5^2 \cdot 8^3/8^2$	67	65538			a. str. mon.
27	256	2	16	$16^3/16^1$	68	67505	6	11	$8^9 \cdot 9^2/11^5$
28	320	3	8	$5^1 \cdot 8^4/8^2$	69	75943	6	11	$8^8 \cdot 9^3/11^5$
29	384	3	8	$6^1 \cdot 8^4/8^2$	70	85436	6	11	$8^7 \cdot 9^4/11^5$
30	512	3	8	$8^5/8^2$	71	102400	5	16	$5^2 \cdot 16^7/16^4$
31	512				72	131072	5	16	$4^1 \cdot 8^1 \cdot 16^7/16^4$
32	513			a. str. mon.	73	163840	5	16	$5^1 \cdot 8^1 \cdot 16^7/16^4$
33	576	3	9	$8^2 \cdot 9^3/9^2$	74	196608	5	16	$6^1 \cdot 8^1 \cdot 16^7/16^4$
34	681	4	7	$5^1 \cdot 6^6/7^3$	75	262144	5	16	$4^1 \cdot 16^8/16^4$
35	817	4	7	$6^7/7^3$	76	327680	5	16	$5^1 \cdot 16^8/16^4$
36	1024	4	8	$4^2 \cdot 8^5/8^3$	77	393216	5	16	$6^1 \cdot 16^8/16^4$
37	1280	4	8	$4^1 \cdot 5^1 \cdot 8^5/8^3$	78	524288	5	16	$8^1 \cdot 16^8/16^4$
38	1600	4	8	$5^2 \cdot 8^5/8^3$	79	589824	5	16	$9^1 \cdot 16^8/16^4$
39	2048	4	8	$4^1 \cdot 8^6/8^3$	80	655360	5	16	$10^1 \cdot 16^8/16^4$
40	2560	4	8	$5^1 \cdot 8^6/8^3$	81	1048576	5	16	$16^9/16^4$
41	3072	4	8	$6^1 \cdot 8^6/8^3$	82	1048576			

Appendix C Comparing the two approaches

To compare the lower bounds given by the probabilistic and the constructive approach we do the following. For a small value of m we take an m-dimensional acute set of prime power cardinality n, then we apply Theorem 3.8 with the largest possible s to get an acute set of size n^s in dimension $d = (2s-1)n$. Then we compare this to the probabilistic bound $\alpha(d) > (1/2)(144/23)^{d/10}$ (in fact, we obtained this result only for d divisible by 5; for general d it only holds with a somewhat smaller constant factor). For the sake of simplicity we consider the base-10 logarithm of the bounds. (See Table 2 for values of n used here.)

Table 4: Comparing constructive and probabilistic lower bound of $\alpha(d)$

m	n	s	dimension $d = (2s-1)m$	constructive l.b. $s \lg n$	probabilistic l.b. $\lg \frac{1}{2} + \frac{d}{10} \lg \frac{144}{23}$
4	8	4	28	**3.61**	1.92
5	11	6	55	**6.24**	4.08
6	16	8	90	**9.63**	6.86
7	19	10	133	**12.78**	10.29
8	23	12	184	**16.34**	14.35
9	27	14	243	**20.03**	19.05
10	31	16	310	23.86	**24.39**
11	37	19	407	29.79	**32.12**
12	64	32	756	57.79	**59.92**

We can do the same for $\kappa(d)$. We apply Theorem 3.8 for small dimensional acute sets in $\{0,1\}^d$ with the largest possible s and compare what we get to the bound $\kappa(d) > (1/2)(4/3)^{d/2}$ given by Erdős and Füredi. (See Table 3 for values of n used here.)

Table 5: Comparing constructive and probabilistic lower bound of $\kappa(d)$

m	n	s	dimension $d = (2s-1)m$	constructive l.b. $s \lg n$	probabilistic l.b. $\lg \frac{1}{2} + \frac{d}{2} \lg \frac{4}{3}$
4	5	3	20	**2.09**	0.94
6	8	4	42	**3.61**	2.32
9	16	8	135	**9.63**	8.13
11	19	10	209	**12.78**	12.75
12	25	13	300	18.17	**18.43**
13	32	16	403	24.08	**24.87**
15	64	32	945	57.79	**58.73**

References

[1] E. Ackerman, O. Ben-Zwi, On sets of points that determine only acute angles, *European J. Combin.* **30** (2009) no. 4, 908–910.

[2] M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, 3rd ed. Springer-Verlag (2003), 79–83.

[3] R.C. Baker, G. Harman, J. Pintz, The difference between consecutive primes, II., *Proc. London Math. Soc.* **83** (2001) no. 3, 532–562.

[4] D. Bevan, Sets of points determining only acute angles and some related colouring problems, *Electron. J. Combin.* **13** (2006).

[5] K. Böröczky Jr., *Finite packing and covering, Cambridge Tracts in Math., vol. 154*, Cambridge Univ. Press (2004), 269–270.

[6] H.T. Croft, On 6-point configurations in 3-space, *J. London Math. Soc.* **36** (1961), 289–306.

[7] L. Danzer, B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V.L. Klee, *Math. Zeitschrift* **79** (1962), 95–99.

[8] P. Erdős, Z. Füredi, The greatest angle among n points in the d-dimensional Euclidean space, *Ann. Discrete Math.* **17** (1983), 275–283.

[9] B. Grünbaum, Strictly antipodal sets, *Israel J. Math.* **1** (1963), 5–10.

[10] K. Schütte, Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand, *Math. Ann.* **150** (1963), 91–98.

[11] I. Talata, Personal communication.