Reducts of countable homogeneous structures

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Homogeneous structures

A structure (model) is $\langle M, R_i \rangle_{i \in I}$, where

- for today: the universe M is countable.
- R_i's are relations on M; finite language unless otherwise stated.
- M is a relational structure: no functions on M in the language.

Definition

M is homogeneous if every isomorphism between finite substructures extends to an automorphism.

Generic examples:

- $\langle \mathbb{Q}, < \rangle$ countable linear dense order without endpoints; extend finite order-preserving maps using a back-and-forth argument. Direct limit of finite total orders.
- Disjoint unions of complete graphs all of the same size.
- Random graph: Set of vertices $= \omega$ and for each pair $\{n, m\}$ toss a coin and draw an edge or non-edge accordingly. Or: consider the countable graph that satisfies for all distinct $v_0, \ldots, v_n$ and $u_0, \ldots, u_k$ there is a vertex x such that

$$\bigwedge_{i<n} E(v_i, x) \wedge \bigwedge_{i<k} \neg E(u_i, x).$$

Construction of homogeneous structures

The age of a structure M is a collection of finite structures which are isomorphic to a substructure of M. Age(M) has the following properties:

(i) Closed under isomorphism and substructure,

(ii) Has countably many members up to isomorphism,

(iii) Has the joint embedding property (JEP): if $U, V \in \text{Age}(M)$ then there is $W \in \text{Age}(M)$ such that both U and V embed in W.

Conversely, if $\mathcal{C}$ is a class of finite structures (of the same language) satisfying (i), (ii) and (iii), then there is a countable M with $\text{Age}(M) = \mathcal{C}$. (build M as a union of a chain of finite structures in $\mathcal{C}$, repeatedly using (iii)).

(iv) Amalgamation property (AP) whenever A, B_1, $B_2 \in \mathcal{C}$ and $f_i : A \to B_i$ are embeddings, there is $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Theorem (Fraïssé)

- *If M is homogeneous, then* $\mathrm{Age}(M)$ *has the amalgamation property.*
- *If $\mathcal{C}$ is a non-empty class of finite structures satisfying (i)–(iv), then there is a homogeneous structure M with* $\mathrm{Age}(M) = \mathcal{C}$*. If N is another homogeneous structure whose age is $\mathcal{C}$, then $M \cong N$.*

Model-theoretic properties:

- M is $\aleph_0$-categorical (i.e. any two models of the theory of M of size $\aleph_0$ are isomorphic).
- Equivalently: $\mathrm{Aut}(M)$ acts oligomorphically on M (i.e. has finitely many orbits on M^n for all positive integers n).
- For each $n > 0$, there are finitely many formulas $\varphi(x_0, \ldots, x_{n-1})$ up to $\mathrm{Th}(M)$-provable equivalence.

Fraïssé limits give many more examples:

Some frequent amalgamation classes $\mathcal{C}$:

- finite graphs ($\rightarrow$ the random graph)
- finite digraphs ($\rightarrow$ the random digraph)
- finite tournaments
- finite linear orders ($\rightarrow$ the structure $\langle \mathbb{Q}, < \rangle$)
- K_n-free graphs
- finite groups
- finite vector spaces over a common finite field

Reducts

Structure of definable sets

Definition

$X \subseteq M^n$ is definable if

$$X = \{\bar{x} \in M^n : \ M \models \varphi(\bar{x})\}$$

is the set of solutions in M^n of some formula φ. (If φ contains parameters from $A \subseteq M$, then we say A-definable).

Definition

A first order theory T has quantifier elimination if for every formula $\varphi(\bar{x})$ there is a qunatifier-free formula $\psi(\bar{x})$ such that

$$T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Structure of definable sets

Quantifier elimination makes it feasible to understand definable sets in models of T.

Theorem

T.F.A.E.

- *M is $\aleph_0$-categorical and* $\mathrm{Th}(M)$ *has quantifier elimination.*
- *M is homogeneous. (recall: the language is finite).*

Remark: If M is $\aleph_0$-categorical and $A \subset M$ is finite, then $X \subseteq M^n$ is A-definable if and only if X is a union of $\mathrm{Aut}(M)_{(A)}$-orbits on M^n.

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But what is the "structure" of definable sets?

Reducts

Definition

A definable reduct of a countably infinite relational structure M if a structure M' with the same universe, whose relations are definable without parameters in M

It is improper if M is also a reduct of M' and trivial is M' is a reduct of a pure set.

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Definition

If M and M' has the same universe, M' is a group-reduct if

$$\mathrm{Aut}(M) \leq \mathrm{Aut}(M'),$$

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Remark: if M is $\aleph_0$-categorical (in particular: homogeneous), then the two notions of reduct coincide (and we call them 'reducts').

Theorem (Cameron, 1976)

The only proper non-trivial reducts of $\langle \mathbb{Q}, < \rangle$ are

- $\langle \mathbb{Q}, B \rangle$ *(ternary betweenness, 'x is between y and z')*
- $\langle \mathbb{Q}, K \rangle$ *(ternary circular order induced from <')*
- $\langle \mathbb{Q}, S \rangle$ *(quaternary separation relation, up to reversal).*

Theorem (Thomas, 1991)

(i) *The only non-trivial reducts of the random graph $\langle V, R \rangle$ are*
- $\langle V, R^{(3)} \rangle$ *(a triple satisfies $R^{(3)}$ if its entries are distinct and it contains an odd number of graph edges)*
- $\langle V, R^{(4)} \rangle$ *(ternary, similar def.)*
- $\langle V, R^{(3)}, R^{(4)} \rangle$.

(ii) *For $n \geq 3$ the generic K_n-free graph has no proper non-trivial reducts.*

Theorem (Junker, Ziegler, 2008)

$\langle \mathbb{Q}, <, 0 \rangle$ *has* 116 *reducts.*

Theorem (Pach, Pinsker, Pluhár, Pongrácz, Szabó, 2013)

The generic poset has 5 *reducts.*

Theorem (Bodirsky, Pinsker, Pongrácz, 2013)

The random ordered graph has 42 *reducts.*

Cases where reducts are describable:

- Strictly minimals sets (Cherlin–Harrington–Lachlan, Zilber).
- Smoothly approximable rank 1 Lie geometries.

Conjecture (Thomas, 1991)

If M is a homogeneous structure over a finite relational language, then M has finitely many reducts (up to first order interdefinability).

In Thomas' conjecture, finite language cannot be replaced with countable language and homogeneity cannot be weakened to $\aleph_0$-categoricity.

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Theorem (Horváth–Sági–Gy)

- *There are continuum many pairwise non-isomorphic countable homogeneous structures on a countably infinite language that have infinitely many non-equivalent reducts.*
- *There are continuum many pairwise non-isomorphic countable, $\aleph_0$-categorical structures on a finite language that have infinitely many non-equivalent reducts.*

In addition:

Theorem (Horváth–Sági–Gy)

- *For every countable, $\aleph_0$-categorical structure A there exists a countable $\aleph_0$-categorical structure B in a finite language such that the lattice $R(A)$ of reducts ordered by inclusion is isomorphic to an interval in $R(B)$.*
- *There exist continuum many pairwise non-isomorphic countable $\aleph_0$-categorical structure A in a finite language such that A has continuum many reducts and A has a reduct which is not interdefinable with any structure in a finite language.*

Related nice open problems:

1. Non-relational languages? E.g. Does $\langle \mathbb{Q}, f \rangle$, where $f(x, y, z) = x - y + z$, have any proper non-trivial reducts?

2. Does $\mathrm{Sym}(\omega)$ have any countable maximal-closed subgroups?

Recall: M has no proper non-trivial group-reducts iff $\mathrm{Aut}(M)$ is maximal-closed in $\mathrm{Sym}(M)$. In the $\aleph_0$-categorical case each closed group-extension $\mathrm{Aut}(M) \leq G \leq \mathrm{Sym}(M)$ corresponds to a reduct of M.

Note: There are many ($2^{2^{\aleph_0}}$, up to conjugacy) maximal subgroups of $\mathrm{Sym}(\omega)$, e.g. stabilizers of ultrafilters on ω.

3. Is every closed proper subgroup of $\mathrm{Sym}(\omega)$ contained in a maximal-closed subgroup of $\mathrm{Sym}(\omega)$?

Generic automorphisms

Generic automorphisms - Motivation

G is a permutation group on a set of cardinality $\aleph_0$. What should be regarded as a typical member of G?

- G is Polish group ($\rightarrow$ Baire Category Theorem!)
- Basic open sets of the form $[p] = \{g \in G : \ g \text{ extends } p\}$, for p partial map.
- Typical elements should have any property which it would be unreasonable to prohibit on the basis of finitely many of its values.
- Typical elements should lie in certain dense open sets.

Any two generic elements should look alike: they should be conjugates.

Definition (Truss) $g \in G$ is strongly generic if it lies in a comeagre conjugacy class, that is, the complement of the conjugacy class

$$g^G = \{h^{-1}gh : \ h \in G\}$$

is the union of countably many nowhere dense sets.

Equivalently, g^G is dense G_δ.

Definition. $g \in G$ is called generic if its conjugacy class

$$g^G = \{h^{-1}gh : \ h \in G\}$$

is dense.

In the literature, usually generic is what we call strongly generic and topological Rokhlin is what we call generic.

Examples

Groups having a dense conjugacy class:

- Automorphism group of a standard Borel space (Rokhlin),
- Unitary group of a separable infinite-dimensional Hilbert space (Choksi, Nadkarni),
- Isometry group of the Urysohn space,
- Homeomorphism group of the Hilbert cube, or the Cantor space

Examples

Groups having a strongly generic element

- $Aut(\mathbb{N}, =)$ (the symmetric group on a countable set).
- $Aut(\mathbb{Q}, <)$.
- $Aut(R)$, where R is the countable random graph.
- $Aut(P)$, where P is the countable random poset.

Chromatic number of the direct product

The chromatic number $\chi(D)$ of a graph D is the least cardinal κ for which there is a coloring with κ colors so that no adjacent vertices have the same color.

$H(\kappa, \lambda)$ is the statement: if $\chi(D), \chi(D') \geq \kappa$ then $\chi(D \times D') \geq \lambda$.

Easy: $\chi(D \times D') \leq \chi(D), \chi(D')$.

Chromatic number of the direct product

Hajnal: $H(\aleph_0, \aleph_0)$

Conjecture: $\exists n < \omega\ H(n, 4)$ (weak Hedetniemi conjecture).

Schmerl: If $\mathcal{M} \not\models TA$ then TFAE:

1. $\mathcal{M}$ has an automorphism whose set of conjugates is dense;
2. There is $n < \omega$ such that $\mathcal{M} \models H(n, 4)$.

Schmerl: If $\mathcal{M} \not\models TA$ is arithmetically saturated then TFAE:

1. $\mathcal{M}$ has a generic automorphism;
2. There is $n < \omega$ such that $\mathcal{M} \models H(n, 4)$.

Hrushovski-type theorems

Hrushovksi: Each finite graph G can be enlarged to another finite graph H such that every partial isomorphism of G extends to an automorphism of H.

Same for relational structures:

Herwig: Each finite relational structure G can be enlarged to a finite H such that every partial isomorphism of G extends to an automorphism of H.

What if we want not just arbitrary relation structures extending G?

Hrushovski-type theorems

Let **K** be an amalgamation class and $\mathcal{M}$ be the Fraïssé limit of **K**. Then the following are equivalent

1. Each $\mathcal{A} \in$ **K** can be embedded into some $\mathcal{B} \in$ **K** such that partial isomorphisms of $\mathcal{A}$ extends to automorphisms of $\mathcal{B}$.
2. $\mathcal{M}$ has a generic automorphism such that each finite n-tuple has a finite orbit (under the diagonal action).

Generic example: Countable random graph.

Characterization – Context

- $\mathcal{M} = \langle M, \text{some relations} \rangle$ is a homogeneous, countable structure endowed with discrete topology.
- $G = Aut(\mathcal{M})$ automorphism group of $\mathcal{M}$.

$G \subseteq {}^M M$ is a Polish group with the topology inherited from the product topology of ${}^M M$.

g^G is dense if for any finite partial p there is $h \in G$ such that $h^{-1}gh \supseteq p$.

Characterization – Truss' theorem

Theorem (Truss)

For a countable, homogeneous structure $\mathcal{M}$ T.F.A.E

- $\mathcal{M}$ *has a generic element (there is a dense conjugacy class in* $\mathrm{Aut}(\mathcal{M})$*).*
- *For all finite partial isomorphisms p and q there is an automorphism h such that $h^{-1}ph$ and q are compatible (i.e. $h^{-1}ph \cup q$ is a partial isomorphism).*

Proof idea

Define the poset $(P, \leq)$:

- $P =$ finite partial isomorphisms,
- $p \leq q$ if $q \subseteq h^{-1}ph$ for some automorphism h.

Dense sets:

- $D_x = \{p : \ x \in \mathrm{dom}(p)\}$, $R_y = \{p : \ y \in \mathrm{ran}(p)\}$,
- $E_p = \{q : \ h^{-1}ph \subseteq q$ for some automorphism $h\}$.

There exists a generic filter F (in the forcing sense). $\bigcup F$ is an generic automorphism.

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Remark: Applying certain forcing axioms the proof can be extended to automorphism groups of not necessarily countable structures.

An n-type of $\mathcal{M}$ over a subset A is a set p of formulae in $\mathrm{Fm}(A)$ with the additional properties that

- p is finitely satisfiable in $\mathcal{M}$;
- elements of p have $\bar{x} = \langle x_0, \ldots, x_{n-1} \rangle$ as free variables;
- p is closed under conjunctions and consequences (if $\varphi(\bar{x}) \in p$, $\psi(\bar{x}) \in \mathrm{Fm}(A)$ and $\mathcal{M} \models \varphi(\bar{x}) \to \psi(\bar{x})$, then $\psi \in p$).

Typical example is the type of an n-tuple:

$$\mathrm{tp}(\bar{a}/A) = \{\varphi(\bar{x}, \bar{c}) : \mathcal{M} \models \varphi[\bar{a}, \bar{c}],\ \bar{c} \in A\}$$

which describes all the first order connections between $\bar{a}$ and elements of A.

Proposition: Let f, g be partial isomorphisms with disjoint range and domain, and

$$p = \mathrm{tp}(\mathrm{ran}(g)/\mathrm{dom}(g)),$$

$$q = f\left[\mathrm{tp}(\mathrm{dom}(g)/\mathrm{dom}(f))\right].$$

Then there is an automorphism h such that f and $h^{-1}gh$ are compatible, provided $p \cup q$ is realizable in $\mathcal{M}$.

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Thus, Truss' condition follows from the realization of unions of certain types.

How to guarantee this condition?

Definition (Robinson's property)

$\mathcal{M}$ has Robinson's property if whenever $p(\bar{x}) \subseteq \mathrm{Fm}(A)$ and $q(\bar{x}) \subseteq \mathrm{Fm}(B)$ are non-trivial partial types over the finite subsets $A, B \subseteq M$, then $p \cup q$ is realizable if and only if there is no formula $\vartheta \in \mathrm{Fm}(A \cap B)$ such that $\vartheta \in p$ and $\neg\vartheta \in q$.

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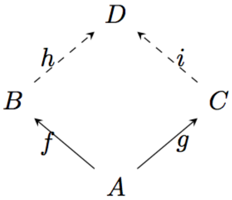
Recall Robinson's classical theorem: if T_1 and T_2 are consistent first order theories (maybe in different languages), then $T_1 \cup T_2$ is consistent if and only if there is no formula ϑ in the intersection of the languages of T_1 and T_2 such that $T_1 \models \vartheta$ and $T_2 \models \neg\vartheta$.

Recall the three most usual amalgamation properties for a class **K**.

(AP) Amalgamation property. If A, B and C are in **K** with embeddings $f : A \to B$ and $g : A \to C$ then there is $D \in$ **K** and embeddings h, i such that the following diagram commutes.



(SAP) Strong amalgamation property. In addition to AP we require $\mathrm{ran}(h) \cap \mathrm{ran}(i)$ to be equal to $\mathrm{ran}(hf)$ (which is necessarily $\mathrm{ran}(ig)$ too).

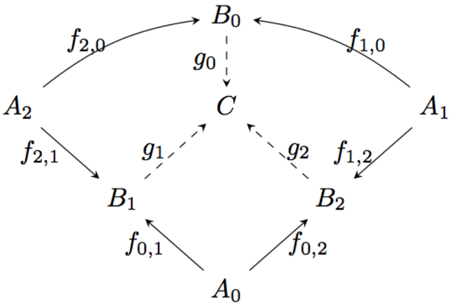
(FAP) Free amalgamation property. In addition to SAP we require that no tuple which meet both $h[B] - hf[A]$ and $i[C] - ig[A]$ (i.e. the non-overlapping parts) should satisfy any relations.

Definition

A class **K** of structures is said to have the *prescribed amalgamation property* (PAP) if for any $A_i,\ B_i \in \mathbf{K}$ and embeddings $f_{i,j} : A_i \to B_j$ $(i, j < 3,\ i \neq j)$ that satisfy

(P1) $f_{j,i}[A_j] \cap f_{k,i}[A_k] = \langle\!\langle \emptyset \rangle\!\rangle^{B_i}$ whenever $\{i, j, k\} = 3$,

there exists $C \in \mathbf{K}$ and embeddings g_j $(j < 3)$ such that the following diagram commutes.



Examples:

- $\mathbf{K}$ = finite graphs
- $\mathbf{K}$ = finite digraphs
- $\mathbf{K}$ = finite tournaments
- $\mathbf{K}$ = finite vector spaces (over a common finite field)
- $\mathbf{K}$ = finite groups

Counterexamples:

- $\mathbf{K}$ = finite K_n-free graphs
- $\mathbf{K}$ = finite linear orders

Theorem (Gy)

If $\mathcal{M}$ is countable homogeneous on a finite binary relational language, then T.F.A.E

- Age($\mathcal{M}$) *has PAP.*
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- Age($\mathcal{M}$) *has PAP.*
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This theorem has recently been extended to non-binary languages by Dávid Nyíri.

Thank you.