Random walk local time approximated by a Wiener sheet combined with an independent Brownian motion

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Dedicated to the memory of Walter Philipp

Abstract

Let $\xi(k,n)$ be the local time of a simple symmetric random walk on the line. We give a strong approximation of the centered local time process $\xi(k,n)-\xi(0,n)$ in terms of a Wiener sheet and an independent Wiener process, time changed by an independent Brownian local time. Some related results and consequences are also established.

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1 Introduction and main results

Let X_i, $i = 1, 2, \ldots$, be i.i.d. random variables with the distribution $P(X_i = 1) = P(X_i = -1) = 1/2$ and put $S_0 := 0$, $S_i := X_1 + \ldots + X_i$, $i = 1, 2, \ldots$. Define the local time process of this simple symmetric random walk by

$$\xi(k, n) := \#\{i : 1 \le i \le n, S_i = k\}, \quad k = 0, \pm 1, \pm 2, \ldots, n = 1, 2, \ldots \tag{1.1}$$

We can also interpret $\xi(k, n)$ as the number of excursions away from k completed before n.

We now define some related quantities for further use. Let $\rho_0 := 0$ and

$$\rho_i := \min\{j > \rho_{i-1} : S_j = 0\}, \quad i = 1, 2, \ldots, \tag{1.2}$$

i.e., ρ_i is the time of the i-th return to zero, or, in other words, the endpoint of the i-th excursion away from 0. We say that $(S_a, S_{a+1}, \ldots, S_b)$ is an excursion away from k, if $S_a = S_b = k$, $S_i \neq k$, $a < i < b$. This excursion will be called upward if $S_i > k$, $a < i < b$ and downward if $S_i < k$, $a < i < b$. Define ρ_i^+ as the endpoint of the i-th upward excursion away from 0, and let $\xi(k, n, \uparrow)$ be the number of upward excursions away from k completed up to time n. Similarly, let $\xi(k, n, \downarrow)$ be the number of downward excursions away from k completed up to time n.

Let $\{W(t), t \ge 0\}$ be a standard Wiener process and consider its two-parameter local time process $\{\eta(x, t), x \in R, t \ge 0\}$ satisfying

$$\int_A \eta(x, t)\, dx = \lambda\{s : 0 \le s \le t, W(s) \in A\} \tag{1.3}$$

for any $t \ge 0$ and Borel set $A \subset R$, where $\lambda(\cdot)$ is the Lebesgue measure. In the sequel we simply call $\eta(\cdot, \cdot)$ a standard Brownian local time.

The study of the asymptotic behaviour of the centered local time processes $\xi(k, n) - \xi(0, n)$ and $\eta(x, t) - \eta(0, t)$ has played a significant role in the development of the local time theory of random walks and that of Brownian and iterated Brownian motions. The first of this kind of results we have in mind is due to Dobrushin [17]. Namely, in this landmark paper, a special case of one of his theorems for additive functionals of a simple symmetric random walk reads as follows.

Theorem A1 *For any* $k = 1, 2, \ldots$

$$\frac{\xi(k, n) - \xi(0, n)}{(4k-2)^{1/2} n^{1/4}} \to_d U\sqrt{|V|}, \quad n \to \infty, \tag{1.4}$$

where U *and* V *are two independent standard normal variables.*

Here and in the sequel $\to_d$ denotes convergence in distribution.

On the other hand, concerning now centered Brownian local times, a special case of a more general fundamental theorem of Skorokhod and Slobodenyuk [34], that is an analogue of Dobrushin's theorem as in [17], yields the following result.

Theorem B1 *For any* $x > 0$

$$\frac{\eta(x,t)-\eta(0,t)}{2x^{1/2}t^{1/4}} \to_d U\sqrt{|V|}, \quad t\to\infty, \tag{1.5}$$

where U *and* V *are two independent standard normal variables.*

While these two theorems are similar, we call attention to their intriguing difference in their scaling constants. For example, the respective scaling constant for $k = x = 1$ is $2^{1/2}$ in Theorem A1 and it is 2 in Theorem B1.

Dobrushin's result as in [17] was extended under various conditions by Kesten [25], Skorokhod and Slobodenyuk [35], Kasahara [23], [24] and Borodin [5]. For some details on the nature of these extensions we refer to the Introduction in [11].

In connection with the analogue of (1.4) as spelled out in (1.5), for further extensions along these lines we refer to Papanicolaou et al. [29], Ikeda and Watanabe [21] and the survey paper of Borodin [6]. For some details we again refer to [11].

The papers mentioned in the previous two paragraphs, in general, are concerned with studying additive functionals of the form $A_n := \sum_{i=1}^n f(S_i)$, and their integral forms $I_t := \int_0^t g(W(s))\,ds$, where $f(x)$, $x\in R$, and $g(x)$, $x\in R$, are real valued functions satisfying appropriate conditions. In particular, Csáki et al. [11] deals with strong approximations of these two types of additive functionals, together with their weak and strong convergence implications.

In view of (1.4) and (1.5) above, we now mention some corresponding iterated logarithm laws. For example, (4.1a) of Csáki et al. [11] yields

Theorem C1 *For* $k = 1,2,\ldots$ *we have*

$$\limsup_{n\to\infty}\frac{\xi(k,n)-\xi(0,n)}{(4k-2)^{1/2}n^{1/4}(\log\log n)^{3/4}} = \frac{2}{3}6^{1/4} \quad \text{a.s.} \tag{1.6}$$

While studying the local time process of a symmetric random walk standardized by its local time at zero, Csörgő and Révész [16] established the next result.

Theorem C2 *For* $k = 1,2,\ldots$ *we have*

$$\limsup_{n\to\infty}\frac{\xi(k,n)-\xi(0,n)}{(4k-2)^{1/2}(\xi(0,n)\log\log n)^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{1.7}$$

Moreover, Theorem 1 of Csáki and Földes [13] yields the next pair of Theorems.

Theorem D1 *For* $x > 0$ *we have*

$$\limsup_{t\to\infty}\frac{\eta(x,t)-\eta(0,t)}{2x^{1/2}t^{1/4}(\log\log t)^{3/4}} = \frac{2}{3}6^{1/4} \quad \text{a.s.} \tag{1.8}$$

and

Theorem D2 *For $x > 0$ we have*

$$\limsup_{t\to\infty} \frac{\eta(x,t)-\eta(0,t)}{2x^{1/2}(\eta(0,t)\log\log t)^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{1.9}$$

While these two pairs of theorems are similar, just like in case of (1.4) and (1.5), we call attention to their intriguing difference in their scaling constants.

In view of Theorems C2 and D2, we state the next two results.

Theorem A2 *For $k = 1, 2, \ldots$ we have*

$$\frac{\xi(k,n)-\xi(0,n)}{(4k-2)^{1/2}(\xi(0,n))^{1/2}} \to_d U, \quad n \to \infty, \tag{1.10}$$

where U is a standard normal random variable.

Theorem B2 *For $x > 0$ we have*

$$\frac{\eta(x,t)-\eta(0,t)}{2x^{1/2}(\eta(0,t))^{1/2}} \to_d U, \quad t \to \infty, \tag{1.11}$$

where U is a standard normal random variable.

Theorem A2 is argued *intuitively* on p. 90 of Csörgő and Révész [16], and it can be rigorously based on our results in Csáki et al. [11], while Theorem B2 is stated as one of the consequences of our results in Csáki et al. [10].

The next weak convergence result for fixed k follows from Kasahara [23].

Theorem E *For $k = 1, 2, \ldots$ we have*

$$\frac{\xi(k,[\lambda t])-\xi(0,[\lambda t])}{(4k-2)^{1/2}\lambda^{1/4}} \to_w W(\widetilde{\eta}(0,t)), \qquad \lambda \to \infty,$$

where $\widetilde{\eta}(\cdot,\cdot)$ is a standard Brownian local time, independent of the Wiener process $W(\cdot)$.

Here and in the sequel $\to_w$ denotes weak convergence in the respective function spaces in hand (here $D[0,\infty)$).

Moreover, for fixed x the next weak convergence result in $C[0,\infty)$ follows from Papanicolaou et al. [29].

Theorem F *For $x > 0$ we have*

$$\frac{\eta(x,\lambda t)-\eta(0,\lambda t)}{2x^{1/2}\lambda^{1/4}} \to_w W(\widetilde{\eta}(0,t)), \qquad \lambda \to \infty,$$

where $\widetilde{\eta}(0,t)$ is as in Theorem E.

When our paper [11] on strong approximations of additive functionals is interpreted in our present context, its general results also imply strong approximations for $\xi(k,n)-\xi(0,n)$ when k is fixed, as spelled out in the next theorem.

Theorem G *On an appropriate probability space for a simple symmetric random walk $\{S_i,\ i=0,1,\dots\}$, for any $k=1,2,\dots$, we can construct a standard Wiener process $\{W(t),\ t\geq 0\}$ and, independently of the latter, a standard Brownian local time $\{\widetilde{\eta}(0,t),\ t\geq 0\}$ such that, as $n\to\infty$, with sufficiently small $\varepsilon>0$ we have*

$$\xi(k,n)-\xi(0,n)=(4k-2)^{1/2}W(\widetilde{\eta}(0,n))+O(n^{1/4-\varepsilon})\quad \text{a.s.} \tag{1.12}$$

and

$$\xi(0,n)-\widetilde{\eta}(0,n)=O(n^{1/2-\varepsilon})\quad \text{a.s.} \tag{1.13}$$

Following the method of proof of Theorem 2 in Section 3 of [11], one can also establish the next theorem, which is also a consequence of our Theorem in [10], that is quoted below (cf. Theorem J).

Theorem H *On an appropriate probability space for the standard Brownian local time process $\{\eta(x,t),\ x\in R,\ t\geq 0\}$ of a standard Brownian motion, for any $x>0$, we can construct a standard Wiener process $\{W(t),\ t\geq 0\}$ and, independently of the latter, a standard Brownian local time $\{\widetilde{\eta}(0,t),\ t\geq 0\}$ such that, as $t\to\infty$, with sufficiently small $\varepsilon>0$ we have*

$$\eta(x,t)-\eta(0,t)=2x^{1/2}W(\widetilde{\eta}(0,t))+O(t^{1/4-\varepsilon})\quad \text{a.s.} \tag{1.14}$$

and

$$\eta(0,t)-\widetilde{\eta}(0,t)=O(t^{1/2-\varepsilon})\quad \text{a.s.} \tag{1.15}$$

A common property of the above quoted theorems is that they treat the two-time parameter processes $\xi(k,n)$ and $\eta(x,t)$ for k, respectively x, fixed, i.e., as if they were one-time parameter stochastic processes. (In (1.12), resp. (1.14), both W and the O term may depend on k, resp. x.) Clearly, studying them as two-time parameter processes is of cardinal interest. A significant first step along these lines was made by Yor [38], who established the following weak convergence result.

Theorem I *As $\lambda\to\infty$,*

$$\left(\frac{1}{\lambda}W(\lambda^2 t),\frac{1}{\lambda}\eta(x,\lambda^2 t),\frac{1}{2\sqrt{\lambda}}(\eta(x,\lambda^2 t)-\eta(0,\lambda^2 t))\right)$$
$$\to_w (W(t),\eta(x,t),W^*(x,\eta(0,t))),$$

where $W^*(\cdot,\cdot)$ is a Wiener sheet, independent of the standard Wiener process $W(\cdot)$, $\eta(\cdot,\cdot)$ is the local time of $W(\cdot)$, and $\to_w$ denotes weak convergence over the space of all continuous functions from R_+^2 to R^3, endowed with the topology of compact uniform convergence.

By a Wiener sheet we mean a two-parameter Gaussian process

$$\{W(x,y),\ x\geq 0,\ y\geq 0\}$$

with mean 0 and covariance function

$$EW(x_1,y_1)W(x_2,y_2)=(x_1\wedge x_2)(y_1\wedge y_2)$$

(cf., e.g., Section 1.11 in Csörgő and Révész [15]).

In Csáki et al. [10] we proved the following strong approximation of Brownian local time by a Wiener sheet.

Theorem J *On an appropriate probability space for the standard Brownian local time process* $\{\eta(x,t),\ x\in R,\ t\geq 0\}$ *of a standard Brownian motion, we can construct a Wiener sheet* $\{W(x,u),$ $x,u\geq 0\}$ *and, independently of the latter, a standard Brownian local time* $\{\widetilde{\eta}(0,t),\ t\geq 0\}$ *such that, as* $t\to\infty$*, for sufficiently small* $\varepsilon>0$ *there exists* $\delta>0$ *for which we have*

$$\sup_{0\leq x\leq t^\delta}|\eta(x,t)-\eta(0,t)-2W(x,\widetilde{\eta}(0,t))|=O(t^{1/4-\varepsilon})\quad \text{a.s.}$$

and

$$\eta(0,t)-\widetilde{\eta}(0,t)=O(t^{1/2-\varepsilon})\quad \text{a.s.}$$

In Csáki et al. [10] we also proved the analogue of Theorem J for t replaced by the inverse local time $\alpha(\cdot)$ defined by

$$\alpha(u):=\inf\{t\geq 0:\ \eta(0,t)\geq u\}.$$

Proposition A *On an appropriate probability space for the standard Brownian local time process* $\eta(x,t),\ x\in R,\ t\geq 0\}$ *of a standard Brownian motion, we can construct a Wiener sheet* $\{W(x,u),\ x,u\geq 0\}$ *and, independently of the latter, an inverse local time process* $\{\widetilde{\alpha}(u),\ u\geq 0\}$ *such that for sufficiently small* $\varepsilon>0$ *there exists* $\delta>0$ *for which as* $u\to\infty$*, we have*

$$\sup_{0\leq x\leq u^\delta}|\eta(x,\alpha(u))-u-2W(x,u)|=O(u^{1/2-\varepsilon})\quad \text{a.s.}$$

and

$$\alpha(u)-\widetilde{\alpha}(u)=O(u^{2-\varepsilon})\quad \text{a.s.}$$

Concerning weak convergence of increments of random walk local time, Eisenbaum [19] established a two-parameter result for symmetric Markov chains at inverse local times, which for a simple symmetric random walk reads as follows.

Proposition B *As* $\lambda\to\infty$*,*

$$\frac{\xi(k,\rho_{[\lambda t]})-[\lambda t]}{\sqrt{\lambda}}\to_w G(k,t),$$

where $\{G(k,t), k = 1,2,\ldots, t \geq 0\}$ *is a mean zero Gaussian process with covariance*

$$EG(k,s)G(\ell,t) = (s \wedge t)(4(k \wedge \ell) - 1_{\{k=\ell\}} - 1),$$

where weak convergence is meant on the function space D *that is defined in Section* 2.1 *below.*

In view of Theorem J and Propositions A, B the present paper establishes several strong approximation results in a similar vein for random walk local times, appropriately uniformly in k, in both of the cases when the time is random or deterministic.

In the next three theorems we study the asymptotic Gaussian behaviour of the centered two-time parameter local time process $\{\xi(k,n) - \xi(0,n),\ k = 0,1,\ldots,\ n = 1,2,\ldots\}$ via appropriate strong approximations in terms of a Wiener sheet and a standard Brownian motion.

Theorem 1.1 *On an appropriate probability space for a symmetric random walk* $\{S_j,\ j = 0,1,\ldots\}$, *we can construct a Wiener sheet* $\{W(x,y),\ x \geq 0, y \geq 0\}$ *and, independently, a standard Brownian motion* $\{W^*(y),\ y \geq 0\}$ *such that, as* $n \to \infty$, *with* $\varepsilon > 0$ *we have*

$$\xi(k,n) - \xi(0,n) = G(k, \xi(0,n)) + O(k^{5/4} n^{1/8+5\varepsilon/8}) \quad \text{a.s.} \tag{1.16}$$

where, for a given $\varepsilon > 0$, *the* $O(\cdot)$ *term is uniform in* $k \in [1, n^{1/6-\varepsilon}]$ *and*

$$G(x,y) := W(x,y) + W(x-1,y) - W^*(y), \quad x \geq 1,\ y \geq 0. \tag{1.17}$$

The just introduced notation in (1.17) for $G(\cdot,\cdot)$ will be used throughout this exposition. We note that it is in fact the same process as that of Proposition B, i.e., the two Gaussian processes agree in distribution, but here $G(\cdot,\cdot)$ is to be constructed of course, and so that we should have (1.16) holding true.

It will be seen via our construction of $W(\cdot,\cdot)$ and $W^*(\cdot)$ for establishing (1.16) that $\xi(0,n)$ cannot be independent of the latter Gaussian processes. This, in turn, limits its immediate use. For the sake of making it more accessible for applications, we also establish the next two companion conclusions to Theorem 1.1.

For further use we introduce the notation $=_d$ for designating equality in distribution of appropriately indicated stochastic processes.

Theorem 1.2 *The probability space of* Theorem 1.1 *can be extended to accommodate a random walk local time* $\widetilde{\xi}(0,n)$ *such that*

(i) $\{\widetilde{\xi}(0,n),\ n = 1,2,\ldots\} =_d \{\xi(0,n),\ n = 1,2,\ldots\}$,

(ii) $\widetilde{\xi}(0,\cdot)$ *is independent of* $G(\cdot,\cdot)$

and, as $n \to \infty$, *with* $\varepsilon > 0$ *we have for some* $\delta > 0$

(iii) $\xi(0,n) - \widetilde{\xi}(0,n) = O(n^{1/2-\delta}) \quad$ a.s.,

(iv) $\xi(k,n)-\xi(0,n)=G(k,\widetilde{\xi}(0,n))$
$+O(k^{5/4}n^{1/8+5\varepsilon/8}+kn^{1/6+\varepsilon/4}+k^{1/2}n^{1/4-\delta})$ a.s.,

where, for a given $\varepsilon>0$, *the latter* $O(\cdot)$ *term is uniform in* $k\in[1,n^{1/6-\varepsilon}]$.

Theorem 1.3 *The probability space of* Theorem 1.1 *can be extended to accommodate a standard Brownian local time process* $\{\eta(0,t),\, t\geq 0\}$ *such that*

(i) $\eta(0,\cdot)$ *is independent of* $G(\cdot,\cdot)$

and, as $n\to\infty$, *with* $\varepsilon>0$ *we have for some* $\delta>0$

(ii) $\xi(0,n)-\eta(0,n)=O(n^{1/2-\delta})$ a.s.,

(iii) $\xi(k,n)-\xi(0,n)=G(k,\eta(0,n))$
$+O(k^{5/4}n^{1/8+5\varepsilon/8}+kn^{1/6+\varepsilon/4}+k^{1/2}n^{1/4-\delta})$ a.s.,

where, for a given $\varepsilon>0$, *the latter* $O(\cdot)$ *term is uniform in* $k\in[1,n^{1/6-\varepsilon}]$.

The proofs of the above theorems will be based on the following propositions.

Proposition 1.1 *On an appropriate probability space for the symmetric random walk* $\{S_k,\, k=1,2,\ldots\}$ *one can construct a Wiener sheet* $\{W(\cdot,\cdot)\}$ *such that as* $N\to\infty$, *with* $\varepsilon>0$ *we have*

$$\xi(k,\rho_N^+,\uparrow)-\xi(0,\rho_N^+,\uparrow)=W(k,2N)+O(k^{5/4}N^{1/4+\varepsilon/2})\quad \text{a.s.,} \tag{1.18}$$

where, for a given $\varepsilon>0$, *the* O *term is uniform in* $k\in[1,N^{1/3-\varepsilon})$.

Proposition 1.2 *The probability space of* Proposition 1.1 *can be so extended that as* $N\to\infty$, *with* $\varepsilon>0$ *and* $G(\cdot,\cdot)$ *as in* Theorem 1.1 *we have*

$$\xi(k,\rho_N^+)-\xi(0,\rho_N^+)=G(k,2N)+O(k^{5/4}N^{1/4+\varepsilon/2})\quad \text{a.s.,} \tag{1.19}$$

where, for a given $\varepsilon>0$, *the* O *term is uniform in* $k\in[1,N^{1/3-\varepsilon})$.

Proposition 1.3 *On the probability space of* Proposition 1.1, *as* $N\to\infty$, *with* $\varepsilon>0$ *we have*

$$\xi(k,\rho_N)-\xi(0,\rho_N)=G(k,N)+O(k^{5/4}N^{1/4+\varepsilon/2})\quad \text{a.s.,} \tag{1.20}$$

where for a given $\varepsilon>0$, *the* O *term is uniform in* $k\in[1,N^{1/3-\varepsilon})$.

From now on the outline of this paper is as follows. In Section 2 we mention and prove some consequences of our just stated theorems and propositions. In Section 3 we collect preliminary results that are needed to prove these theorems and propositions. Theorem 1.1 and Propositions 1.1-1.3 are proved in Section 4, while Theorems 1.2 and 1.3 in Section 5.

2 Consequences

Here we establish a few consequences of our theorems and propositions, concerning weak convergence and laws of the iterated logarithm.

2.1 Weak convergence

We start with convenient strong approximations for the sake of concluding corresponding weak convergence.

Theorem 2.1 *Let $\xi(\cdot,\cdot)$, $\eta(\cdot,\cdot)$, and $G(\cdot,\cdot)$ be as in* Theorem 1.3. *As $\lambda \to \infty$, we have*

$$\max_{1\le k\le K} \sup_{0\le t\le T} \left| \frac{\xi(k,[\lambda t]) - \xi(0,[\lambda t])}{\lambda^{1/4}} - \frac{G(k,\eta(0,\lambda t))}{\lambda^{1/4}} \right| \to 0 \quad \text{a.s.} \tag{2.1}$$

and

$$\max_{1\le k\le K} \sup_{0\le t\le T} \left| \frac{\xi(k,\rho_{[\lambda t]}) - \lambda t}{\lambda^{1/2}} - \frac{G(k,\lambda t)}{\lambda^{1/2}} \right| \to 0 \quad \text{a.s.} \tag{2.2}$$

for all fixed integer $K \ge 1$ and $T > 0$.

Proof. In view of Theorem 1.3 and Proposition 1.3 the respective statements of (2.1) and (2.2) are seen to be true. □

Let $N^+ := [1, 2, \ldots)$, and define the space of real valued bivariate functions

$$f(k,t) \in D := D(N^+ \times [0,\infty))$$

that are cadlag in $t \in [0,\infty)$. Define also

$$\Delta = \Delta(f_1, f_2) = \max_{1\le k\le K} \sup_{0\le t\le T} |f_1(k,t) - f_2(k,t)|$$

with any fixed $(K,T) \in N^+ \times [0,\infty)$, and the measurable space $(D,\mathcal{D})$, where $\mathcal{D}$ is the σ-field generated by the Δ-open balls of D.

On account of having for each $\lambda > 0$

$$\left\{ \frac{G(k,\eta(0,\lambda t))}{\lambda^{1/4}},\ (k,t) \in N^+ \times [0,\infty) \right\} =_d \{G(k,\eta(0,t)),\ (k,t) \in N^+ \times [0,\infty)\}$$

and

$$\left\{ \frac{G(k,\lambda t)}{\lambda^{1/2}},\ (k,t) \in N^+ \times [0,\infty) \right\} =_d \{G(k,t),\ (k,t) \in N^+ \times [0,\infty)\},$$

Theorem 2.1 yields the following weak convergence results.

Corollary 2.1 *Let $\xi(\cdot,\cdot)$, $\eta(\cdot,\cdot)$, and $G(\cdot,\cdot)$ be as in* Theorem 1.3. *As $\lambda \to \infty$, we have*

$$h\left(\frac{\xi(k,[\lambda t]) - \xi(0,[\lambda t])}{\lambda^{1/4}}\right) \to_d h(G(k,\eta(0,t)))$$

and

$$h\left(\frac{\xi(k,\rho_{[\lambda t]}) - \lambda t}{\lambda^{1/2}}\right) \to_d h(G(k,t))$$

for all $h : D \to R$ that are $(D,\mathcal{D})$ measurable and Δ-continuous, or Δ-continuous except at points forming a set of measure zero on $(D,\mathcal{D})$ with respect to $G(\cdot,\cdot)$, over all compact sets in $\mathcal{D}$.

2.2 Law of the iterated logarithm

Theorem 2.2 *Let $K = K(t)$, $t \geq 0$ be an integer valued non-decreasing function of t such that $K(t) \geq 1$ and*

$$\lim_{\alpha\to 1}\lim_{\ell\to\infty}\frac{K(\alpha^{\ell})}{K(\alpha^{\ell-1})} = 1.$$

If $K(N) \leq N^{1/3-\varepsilon}$ for some $\varepsilon > 0$, then

$$\limsup_{N\to\infty}\frac{\sup_{1\leq k\leq K}|\xi(k,\rho_N) - N|}{(4K-2)^{1/2}(N\log\log N)^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{2.3}$$

If, however, $K(n) \leq n^{1/6-\varepsilon}$ for some $\varepsilon > 0$, then

$$\limsup_{n\to\infty}\frac{\sup_{1\leq k\leq K}|\xi(k,n) - \xi(0,n)|}{(4K-2)^{1/2}n^{1/4}(\log\log n)^{3/4}} = \frac{2}{3}6^{1/4} \quad \text{a.s.} \tag{2.4}$$

The proof of Theorem 2.2 is based on the following result.

Lemma 2.1 *For any $\alpha > 1$, $K \geq 1$, $t > 0$ we have the following inequalities:*

$$P\left(\max_{1\leq k\leq K}\sup_{0\leq s\leq t}|G(k,s)| > u\right) \tag{2.5}$$

$$\leq C\exp\left(-\frac{u^2}{2\alpha t(4K-2)}\right), \quad u > 0,$$

$$P\left(\max_{1\leq k\leq K}\sup_{0\leq s\leq \eta(0,t)}|G(k,s)| > u\right) \tag{2.6}$$

$$\leq C\exp\left(-\frac{3u^{4/3}}{2^{5/3}\alpha t^{1/3}(4K-2)^{2/3}}\right), \quad u > 0$$

with a certain positive constant C depending on α.

Proof. Consider the process $\{Y(s) = \max_{1\le k\le K} G(k,s),\ s\ge 0\}$. $Y(s)$ is a submartingale with respect to $\mathcal{F}_s$, the sigma algebra generated by $G(k,u)$, $1\le k\le K$, $0\le u\le s$, since if k_0 is defined by $G(k_0,s)=\max_{1\le k\le K} G(k,s)$, then obviously

$$E(G(k_0,t)\mid \mathcal{F}_s) = G(k_0,s)$$

and

$$E(Y(t)\mid \mathcal{F}_s) \ge E(G(k_0,t)\mid \mathcal{F}_s) = G(k_0,s) = Y(s).$$

Consequently,

$$\left\{\sup_{0\le s\le t}\max_{1\le k\le K} G(k,s),\ t\ge 0\right\}$$

and for $\lambda>0$

$$\left\{\sup_{0\le s\le t}\max_{1\le k\le K} \exp(\lambda G(k,s)),\ t\ge 0\right\}$$

are submartingales. Using Doob inequalities (twice) we get

$$\begin{aligned}
&P\left(\sup_{0\le s\le t}\max_{1\le k\le K} G(k,s)\ge u\right)\\
&= P\left(\sup_{0\le s\le t}\max_{1\le k\le K} \exp(\lambda G(k,s))\ge \exp(\lambda u)\right)\\
&\le e^{-\lambda u} E\left(\max_{1\le k\le K}\exp(\lambda G(k,t)\right)\\
&\le \left(\frac{\hat\alpha}{\hat\alpha-1}\right)^{\hat\alpha} e^{-\lambda u}E(\exp(\hat\alpha\lambda G(K,t))) \qquad (2.7)
\end{aligned}$$

for any $\hat\alpha>1$. $G(K,t)$ has normal distribution with mean zero and variance $(4K-2)t$, hence

$$E(\exp(\hat\alpha\lambda G(K,t))) = \exp\left(\frac{\lambda^2\hat\alpha^2}{2}(4K-2)t\right)$$

and putting

$$\lambda = \frac{u}{\hat\alpha^2(4K-2)t},\qquad \alpha=\hat\alpha^2$$

into (2.7), we get (2.5).

On the other hand, if $\eta(0,\cdot)$ is a Brownian local time, independent of $G(\cdot,\cdot)$, we get from (2.7)

$$P\left(\sup_{0\le s\le \eta(0,t)}\max_{1\le k\le K} G(k,s)\ge u\right)\le Ce^{-\lambda u}E(\exp(\tilde\alpha\lambda G(K,\eta(0,t)))).$$

But

$$\frac{G(K,\eta(0,t))}{(4K-2)^{1/2}t^{1/4}} =_d \mathcal{N}_1|\mathcal{N}_2|^{1/2},$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent standard normal variables. Hence

$$P\left(\max_{1\le k\le K}\sup_{0\le s\le \eta(0,t)} G(k,s)\ge u\right)\le Ce^{-\lambda u}E\left(\frac{\tilde{\alpha}^2\lambda^2}{2}|\mathcal{N}_2|(4K-2)t^{1/2}\right)$$

$$\le 2Ce^{-\lambda u}E\left(\frac{\tilde{\alpha}^2\lambda^2}{2}\mathcal{N}_2(4K-2)t^{1/2}\right)=2Ce^{-\lambda u}\exp\left(\frac{\tilde{\alpha}^4\lambda^4}{8}(4K-2)^2t\right).$$

Putting $\lambda=(2u)^{1/3}\tilde{\alpha}^{-4/3}(4K-2)^{-2/3}t^{-1/3}$, $\alpha=\tilde{\alpha}^{4/3}$, we get (2.6). □

Proof of Theorem 2.2. Let $t_\ell=\alpha^\ell,\ \alpha>1$. Putting

$$u=(2\alpha^2(4K(t_\ell)-2)t_\ell\log\log t_\ell)^{1/2},\quad t=t_\ell, K=K(t_\ell)$$

into (2.5), using Borel-Cantelli lemma and interpolating between $t_{\ell-1}$ and t_ℓ, the usual procedure gives for all large t

$$\max_{1\le k\le K}|G(k,t)|\le\alpha^{3/2}(2(4K-2)t\log\log t)^{1/2}. \tag{2.8}$$

By Proposition 1.3 we also have for large N

$$\max_{1\le k\le K}|\xi(k,\rho_N)-N|\le\alpha^{3/2}(2(4K-2)N\log\log N)^{1/2}+O(K^{5/4}N^{1/4+5\varepsilon/8}).$$

Since

$$\lim_{N\to\infty}\frac{K^{5/4}N^{1/4+\varepsilon/2}}{(KN\log\log N)^{1/2}}=0,$$

if $K\le N^{1/3-\varepsilon}$, and $\alpha>1$ is arbitrary, we have an upper bound in (2.3).

The upper bound in (2.4) is similar. Put

$$u=2^{5/4}3^{-3/4}\alpha^{3/2}(4K-2)^{1/2}t^{1/4}(\log\log t)^{3/4}$$

into (2.6). Then, as before, we conclude that almost surely

$$\max_{1\le k\le K}|G(k,\eta(0,t))|\le\alpha^2 2^{5/4}3^{-3/4}(4K-2)^{1/2}t^{1/4}(\log\log t)^{3/4}$$

for t large enough. Since $\alpha>1$ is arbitrary, using Theorem 1.3, we get an upper bound in (2.4).

To prove the lower bound in (2.3), for $0<\delta<1$ define the events

$$A_\ell=\{G(K_\ell,t_\ell)-G(K_\ell,t_{\ell-1})\ge(1-\delta)(2(4K_\ell-2)t_\ell\log\log t_\ell)^{1/2}\},$$

$\ell = 1, 2, \ldots$, where $t_\ell = \delta^{-\ell}$ and $K_\ell = K(t_\ell)$. Since $G(K_\ell, t_\ell) - G(K_\ell, t_{\ell-1})$ has normal distribution with mean zero and variance $(4K_\ell - 2)(t_\ell - t_{\ell-1})$, an easy calculation shows

$$P(A_\ell) \geq \frac{C}{(\log t_\ell)^{1-\delta}} = \frac{C}{\ell^{1-\delta}}.$$

Since A_ℓ are independent, Borel-Cantelli lemma implies $P(A_\ell \, \text{i.o.}) = 1$. But

$$G(K_\ell, t_{\ell-1}) \leq (1+\delta)(4K_\ell - 2)^{1/2}(2t_\ell \log\log t_\ell)^{1/2}\delta^{1/2}$$

for all large ℓ, we have also

$$G(K_\ell, t_\ell) \geq ((1-\delta) - (1+\delta)\delta^{1/2})(2(4K_\ell - 2)t_\ell \log\log t_\ell)^{1/2}$$

infinitely often with probability 1. Since $\delta > 0$ is arbitrary, we conclude

$$\limsup_{t\to\infty} \frac{G(K,t)}{(2(4K-2)t\log\log t)^{1/2}} \geq 1.$$

Using Proposition 1.3, this also gives a lower bound in (2.3).

To show the lower bound in (2.4), we follow Burdzy [8] with some modifications. Define $t_\ell = \exp(\ell \log \ell)$, and the events

$$A_\ell^{(1)} = \{(1-\delta)a_\ell \leq \eta(t_\ell) \leq 2(1-\delta)a_\ell\}$$

and

$$A_\ell^{(2)} = \{\inf_{s\in I_\ell} G(K_\ell, s) - G(K_\ell, \gamma a_\ell) \geq (1-2\beta)(4K_\ell - 2)^{1/2}u_\ell\},$$

where $K_\ell = K(t_\ell)$, $\eta(t_\ell) = \eta(0, t_\ell)$,

$$a_\ell = \left(\frac{2}{3}t_\ell \log\log t_\ell\right)^{1/2},$$

$$u_\ell = \frac{2}{3^{1/2}}((1-2\delta)a_\ell \log\log a_\ell)^{1/2},$$

$$I_\ell = [(1-2\delta)a_\ell, 3(1-2\delta)a_\ell],$$

and β, δ, γ are certain small constants to be choosen later on. Obviously, the events $\{A_\ell^{(1)}, \ell = 1, 2, \ldots\}$ and $\{A_\ell^{(2)}, \ell = 1, 2, \ldots\}$ are independent. Let

$$A_\ell = A_\ell^{(1)} A_\ell^{(2)}.$$

We show that for certain values of the above constants, $P(A_\ell \, \text{i.o.}) = 1$.

Since $t_\ell^{-1/2}\eta(t_\ell)$ is distributed as the absolute value of a standard normal variable, an easy calculation shows

$$P(A_\ell^{(1)}) \geq \frac{C}{(\log t_\ell)^{(1-\delta)/3}}$$

with some $C > 0$.

Converting the inequality of Lemma 2 in Burdzy [8] from small time to large time, the following inequality can be concluded for large enough u:

$$P(\inf_{s\in[u,3u]} W(s) - W(\gamma u) \geq (1-2\beta)(2/3^{1/2})(u \log\log u)^{1/2})$$

$$\geq (\log u)^{(-2/3)(1-\beta)/(1-\gamma)},$$

where $W(\cdot)$ is a standard Wiener process. From this we get

$$P(A_\ell^{(2)}) \geq \frac{C}{(\log u_\ell)^{2(1-\beta)/(3(1-\gamma))}}$$

with $C > 0$.

We can choose the constants β, γ, δ appropriately to have $\sum_\ell P(A_\ell) = \sum_\ell P(A_\ell^{(1)})P(A_\ell^{(2)}) = \infty$. The events A_ℓ however are not independent. Next we show $P(A_j A_\ell) \leq CP(A_j)P(A_\ell)$ with some constant C. It can be seen that for large ℓ we have $3(1-2\delta)a_\ell \leq \gamma a_{\ell+1}$, therefore $A_\ell^{(2)}$ are independent events for $\ell \geq \ell_0$ with a certain ℓ_0. We have

$$P(A_j A_\ell) = P(A_j^{(1)} A_\ell^{(1)})P(A_j^{(2)})P(A_\ell^{(2}).$$

It suffices to show that $P(A_j^{(1)} A_\ell^{(1)}) \leq CP(A_j^{(1)})P(A_\ell^{(1)})$. For this purpose it is more convenient to work with $M(t)$, the supremum of the Wiener process, since according to Lévy's theorem, the process $\{\eta(0,t),\, t \geq 0\}$ is identical in distribution with $\{M(t),\, t \geq 0\}$. So let $\{\widehat{W}(t),\, t \geq 0\}$ be a standard Wiener process and $M(t) = \sup_{0\leq s\leq t} \widehat{W}(s)$. Denote by $g_t(y)$ the density of $M(t)$ and by $g_{t_1,t_2}(y_1, y_2)$ the joint density of $M(t_1), M(t_2)$. It is well known that

$$g_t(y) = \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$

Then with $h(t_1, z)$, the joint density of $M(t_1)$ and $\widehat{W}(t_1)$, we can write for $t_1 < t_2$,

$$g_{t_1,t_2}(y_1, y_2) = \int_{-\infty}^{y_1} h(t_1, z) g_{t_2-t_1}(y_2 - z)\, dz.$$

It can be seen that for $z \leq y_1$

$$g_{t_2-t_1}(y_2 - z) \leq \sqrt{\frac{t_2}{t_2-t_1}}\, g_{t_2}(y_2) \exp\left(\frac{y_1 y_2}{t_2-t_1}\right).$$

Hence

$$g_{t_1,t_2}(y_1,y_2) \le g_{t_1}(y_1)g_{t_2}(y_2)\sqrt{\frac{t_2}{t_2-t_1}}\exp\left(\frac{y_1y_2}{t_2-t_1}\right).$$

Returning to the probability of the events $A^{(1)}$, we have for $j<\ell$

$$\begin{aligned} P(A_j^{(1)}A_\ell^{(1)}) &\le \sqrt{\frac{t_\ell}{t_\ell-t_j}}\exp\left(\frac{4(1-\delta)^2a_ja_\ell}{t_\ell-t_j}\right)P(A_j^{(1)})P(A_\ell^{(1)}) \\ &\le CP(A_j^{(1)})P(A_\ell^{(1)}), \end{aligned}$$

where $C>1$ can be choosen arbitrarily close to 1 by choosing $\ell-j$ sufficiently large. Hence for any $\varepsilon>0$ there exists m_0 such that

$$P(A_jA_\ell) \le (1+\varepsilon)P(A_j)P(A_\ell)$$

if $\ell-j\ge m_0$. It follows that

$$\sum_{\ell=1}^{n}\sum_{j=1}^{\ell}P(A_jA_\ell) \le (1+\varepsilon)\sum_{\ell=1}^{n}\sum_{j=1}^{\ell-m_0}P(A_j)P(A_\ell)+m_0\sum_{\ell=1}^{n}P(A_\ell).$$

By Borel-Cantelli lemma (cf. [36], p. 317) $P(A_\ell \text{ i.o.})\ge 1/(1+\varepsilon)$. Since $\varepsilon>0$ is arbitrary, we also have $P(A_\ell \text{ i.o.})=1$. $A_\ell^{(1)}$ implies

$$\eta(t_\ell)\in[(1-\delta)a_\ell, 2(1-\delta)a_\ell]\subset I_\ell,$$

consequently, if both $A_\ell^{(1)}$ and $A_\ell^{(2)}$ occur, then

$$\begin{aligned} &G(K_\ell,\eta(t_\ell)) \\ &\ge (1-2\beta)(4K_\ell-2)^{1/2}2((1-2\delta)a_\ell\log\log a_\ell)^{1/2}/3^{1/2}+G(K_\ell,\gamma a_\ell). \end{aligned}$$

It follows from (2.8) that

$$G(K_\ell,\gamma a_\ell)\ge -(4K_\ell-2)^{1/2}(\gamma a_\ell\log\log a_\ell)^{1/2}$$

for all large ℓ with probability 1, i.e.

$$\limsup_{\ell\to\infty}\frac{G(K_\ell,\eta(t_\ell))}{(4K_\ell-2)^{1/2}(a_\ell\log\log a_\ell)^{1/2}}\ge(1-2\beta)(1-2\delta)^{1/2}2/3^{1/2}-\gamma^{1/2}.$$

But

$$\lim_{\ell\to\infty}\frac{a_\ell\log\log a_\ell}{t_\ell^{1/2}(\log\log t_\ell)^{3/2}}=(2/3)^{1/2},$$

implying

$$\limsup_{t\to\infty} \frac{\max_{1\le k\le K} G(k,\eta(0,t))}{(4K-2)^{1/2}t^{1/4}(\log\log t)^{3/4}}$$
$$\ge 2^{5/4}3^{-3/4}(1-2\beta)(1-2\delta)^{1/2} - 2^{1/4}3^{-1/4}\gamma^{1/2} \quad \text{a.s.}$$

Since it is possible to choose β, δ, γ arbitrarily small, combining this with Theorem 1.3, gives a lower bound in (2.4). □

3 Preliminaries

In this Section we collect the results needed to prove our theorems and propositions. The proofs will use the branching property (Ray-Knight description) of the random walk local time. For more details in this respect we refer to Knight [26], Dwass [18], Rogers [32] and Tóth [37].

Introduce the following notations for $k = 1, 2, \ldots,\ i = 1, 2, \ldots$.

$$\tau_i^{(k)} := \min\{j > \tau_{i-1}^{(k)} : S_{j-1} = k, S_j = k-1\}, \tag{3.1}$$

with $\tau_0^{(k)} := 0$,

$$T_i^{(k)} := \xi(k, \tau_i^{(k)}) - \xi(k, \tau_{i-1}^{(k)}). \tag{3.2}$$

With probability 1, there is such a double infinite sequence of $\tau_i^{(k)}$ and hence also of $T_i^{(k)}$, $i = 1, 2, \ldots,\ k = 1, 2, \ldots$

Lemma 3.1 *The random variables $\{T_i^{(k)},\ k = 1, 2, \ldots, i = 1, 2, \ldots\}$ are completely independent and distributed as*

$$P(T_i^{(k)} = j) = \frac{1}{2^j}, \quad j = 1, 2, \ldots, \tag{3.3}$$

$$E(T_i^{(k)}) = 2, \quad Var(T_i^{(k)}) = 2. \tag{3.4}$$

Proof. Obvious.

Introduce

$$U^{(k)}(j) := T_1^{(k)} + \ldots + T_j^{(k)} - 2j, \quad k = 1, 2, \ldots,\ j = 1, 2, \ldots . \tag{3.5}$$

For the following inequality we refer to Tóth [37].

Lemma 3.2

$$P\left(\max_{1\le i\le n} |U^{(k)}(i)| > z\right) \le 2\exp\left(-\frac{z^2}{8n}\right), \quad 0 < z < a_0 n$$

for some $a_0 > 0$.

We need Hoeffding's inequality [20] for binomial distribution (cf. also Shorack and Wellner [33], pp. 440).

Lemma 3.3 *Let ν_N have binomial distribution with parameters $(N, 1/2)$. Then*

$$P(|2\nu_N - N| \geq u) \leq 2\exp\left(-\frac{u^2}{2N}\right), \quad 0 < u.$$

To establish our results, we make use of one of the celebrated KMT strong invariance principles (cf. Komlós et al. [28]).

Lemma 3.4 *Let $\{Y_i\}_{i=1}^{\infty}$ be i.i.d. random variables with expectation zero, variance σ^2 and having moment generating function in a neighbourhood of zero. On an appropriate probability space one can construct $\{Y_i\}_{i=1}^{\infty}$ and a Wiener process $\{W(t),\, t \geq 0\}$ such that for all $x > 0$ and $n = 1, 2, \ldots$*

$$P\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} Y_j - W(i\sigma^2)\right| > C_1 \log n + x\right) \leq C_2 e^{-C_3 x},$$

where C_1, C_2, C_3 are positive constants, and C_3 can be choosen arbitrarily large by choosing C_1 sufficiently large.

There are several papers on strong invariance principles for local times, initiated by Révész [30], and further developed by Borodin [5], [6], Bass and Khoshnevisan [2], and others, as in the references of these papers. The best rate via Révész's Skorokhod type construction was given by Csörgő and Horváth [14].

Lemma 3.5 *On a rich enough probability space one can define a simple symmetric random walk with local time $\xi(\cdot,\cdot)$ and a standard Brownian local time $\eta(\cdot,\cdot)$ such that as $n \to \infty$*

$$\sup_{k\in Z}|\xi(k,n) - \eta(k,n)| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad \text{a.s.} \tag{3.6}$$

We note in passing that having (3.6) with $O(n^{1/4}(\log\log n)^{3/4})$ is best possible for any construction (cf. Csörgő and Horváth [14]), i.e., only the $(\log n)^{1/2}$ term of (3.6) could be changed, and only to $(\log\log n)^{1/2}$, by any other construction. It remains an open problem to find such a construction that would achieve this best possible minimal gain.

Lemma 3.6 *Let $\{W_i(\cdot),\, i = 1, \ldots, k\}$ be independent Wiener processes and $t > 0$. The following inequality holds.*

$$P\left(\sup_{0\leq t_i\leq t,\, i=1,\ldots,k}\left|\sum_{i=1}^{k} W_i(t_i)\right| \geq z\right) \leq 2k e^{-z^2/(2k^2 t)}, \quad 0 < z.$$

Proof. Since

$$\sup_{0\le t_i\le t,\, i=1,\dots,k}\left|\sum_{i=1}^{k} W_i(t_i)\right| \le k \max_{1\le i\le k}\sup_{0\le t_i\le t}|W_i(t_i)|,$$

we have

$$P\left(\sup_{0\le t_i\le t,\, i=1,\dots,k}\left|\sum_{i=1}^{k} W_i(t_i)\right|\ge z\right)\le P\left(\max_{1\le i\le k}\sup_{0\le t_i\le t}|W_i(t_i)|\ge z/k\right)$$
$$\le kP\left(\sup_{0\le s\le t}|W(s)|\ge z/k\right)\le 2ke^{-z^2/(2k^2t)}.$$

□

Lemma 3.7 *The following identities hold.*

$$\xi(k,\rho_N^+) = U^{(k)}(\xi(k-1,\rho_N^+,\uparrow)) + 2\xi(k-1,\rho_N^+,\uparrow), \tag{3.7}$$

$$\xi(k,\rho_N^+,\uparrow) = \sum_{i=1}^{\xi(k-1,\rho_N^+,\uparrow)} (T_i^{(k)}-1), \tag{3.8}$$

$$\xi(k,\rho_N^+,\downarrow) = \xi(k-1,\rho_N^+,\uparrow). \tag{3.9}$$

Proof. Obvious.

Equation (3.8) amounts to saying that $\xi(k,\rho_N^+,\uparrow)$, $k=0,1,\dots$ is a critical branching process with geometric offspring distribution.

Lemma 3.8 *For $K\ge 1$*

$$P(\max_{1\le k\le K}\xi(k,\rho_N^+)\ge 5N)\le K\exp\left(-\frac{N}{4K}\right) \tag{3.10}$$

Proof. For the distribution of $\xi(k,\rho_1^+)$ we have (cf. Révész [31])

$$P(\xi(k,\rho_1^+)=m)=\begin{cases} 1-\frac{1}{k} & \text{if } \ m=0,\\ \frac{1}{2k^2}\left(1-\frac{1}{2k}\right)^{m-1} & \text{if } \ m=1,2,\dots \end{cases} \tag{3.11}$$

Hence for the moment generating function we have

$$g(k,t)=E\left(e^{t\xi(k,\rho_1^+)}\right)=\frac{1-(2k-2)(1-e^{-t})}{1-2k(1-e^{-t})}\le 1+\frac{2t}{1-2kt}.$$

Selecting $t=1/(4K)$, we arrive at

$$g(k,t)\le 1+\frac{1}{2K-k}\le 1+\frac{1}{K}\le e^{1/K}.$$

Since

$$E\left(e^{t\xi(k,\rho_N^+)}\right) = (g(k,t))^N,$$

we have by Markov's inequality

$$P(\xi(k,\rho_N^+) \geq 5N) \leq \left(g(k,t)e^{-5t}\right)^N \leq e^{\left(\frac{1}{K}-\frac{5}{4K}\right)N} = e^{-N/(4K)}.$$

□

We need inequalities for increments of the Wiener process (Csörgő and Révész [15]), Brownian local time (Csáki et al. [9]), and random walk local time (Csáki and Földes [12]).

Lemma 3.9 *With any constant* $C_2 < 1/2$ *and some* $C_1 > 0$ *we have*

$$P\left(\sup_{0\leq s\leq T-h}\sup_{0\leq t\leq h}|W(s+t)-W(s)| \geq v\sqrt{h}\right) \leq \frac{C_1 T}{h}e^{-C_2 v^2},$$

$$P\left(\sup_{0\leq s\leq t-h}(\eta(0,h+s)-\eta(0,s)) \geq x\sqrt{h}\right) \leq C_1\left(\frac{t}{h}\right)^{1/2} e^{-C_2 x^2},$$

and

$$P\left(\max_{0\leq j\leq t-a}(\xi(0,a+j)-\xi(0,j)) \geq x\sqrt{a}\right) \leq C_1\left(\frac{t}{a}\right)^{1/2} e^{-C_2 x^2}.$$

Note that we may have the same constants C_1, C_2 in the above inequalities. In fact, in our proofs the values of these constants are not important, and it is indifferent whether they are the same or not. We continue using these notations for constants of no interest that may differ from line to line.

Lemma 3.10 *For* $1 \leq u$ *we have*

$$P(\rho_N \geq uN^2) \leq \frac{1}{\sqrt{u}}$$

and

$$E(\rho_1 I\{\rho_1 \leq u\}) \leq 3\sqrt{u}.$$

Proof. For the distribution of ρ_N we have (cf. Révész [31], pp. 98)

$$P(\rho_N > 2n) = \frac{1}{2^{2n}}\sum_{j=0}^{N-1} 2^j \binom{2n-j}{n}, \quad n = 1, 2, \ldots.$$

An elementary calculation shows that the largest term in the sum above is for $j = 0$, hence

$$P(\rho_N > 2n) \leq \frac{N}{2^{2n}}\binom{2n}{n}.$$

Moreover, it can be easily seen that

$$\frac{(2n+2)^{1/2}}{2^{2n}}\binom{2n}{n}$$

is decreasing in $n = 1, 2, \ldots$, hence it is less than 1 for all n, and thus implying

$$P(\rho_N > 2n) \le \frac{N}{(2n+2)^{1/2}}.$$

For a given $u \ge 1$ choose n so that $2n < uN^2 \le 2n+2$. Then

$$P(\rho_N \ge uN^2) \le P(\rho_N > 2n) \le \frac{N}{(2n+2)^{1/2}} \le \frac{1}{\sqrt{u}}.$$

Moreover,

$$E(\rho_1 I\{\rho_1 \le u\}) = \sum_{1\le j\le u} jP(\rho_1 = j) \le \sum_{0\le j\le u} P(\rho_1 \ge u)$$

$$\le 1 + \sum_{1\le j\le u} \frac{1}{\sqrt{j}} \le 1 + \int_0^u \frac{dx}{\sqrt{x}} = 1 + 2\sqrt{u} \le 3\sqrt{u}.$$

□

Lemma 3.11 *Define* $\tau_0 := 0$,

$$\tau_n := \inf\{t:\ t > \tau_{n-1},\ |W(t) - W(\tau_{n-1})| = 1\}, \quad n = 1, 2, \ldots$$

Then τ_n *is a sum of* n *i.i.d. random variables,* $E(\tau_1) = 1$ *and*

$$E(e^{\theta\tau_1}) = \frac{1}{\cosh(\sqrt{2\theta})}. \tag{3.12}$$

Moreover,

$$P(|\tau_n - n)| \ge u\sqrt{n}) \le 2e^{-3u^2/8}, \qquad 0 < u < 2\sqrt{n}/3. \tag{3.13}$$

Proof. For (3.12) see, e.g., Borodin and Salminen [7]. To show (3.13), we use exponential Markov's inequality:

$$P(|\tau_n - n| \ge u\sqrt{n}) \le e^{-u\theta\sqrt{n}}\left((g(\theta))^n + (g(-\theta))^n\right),$$

for $0 < \theta \le 1/2$, where

$$g(\theta) := E(e^{\theta(\tau_1 - 1)}) = \frac{1}{e^\theta \cosh(\sqrt{2\theta})}.$$

By the series expansion of $\log\cos x$ (cf. Abramowitz and Stegun [1], pp. 75) and putting $\cosh x = \cos(ix)$, we get

$$\log\cosh x = \sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)B_{2k}}{k(2k)!} x^{2k}, \quad |x| \le \frac{\pi}{2},$$

where B_i are Bernoulli numbers, and using that $B_2 = 1/6$, $B_4 = -1/30$ and the inequality (cf. [1], pp. 805)

$$|B_{2n}| \le \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})}$$

for $n > 2$, one can easily see that

$$\log g(\theta) \le \frac{\theta^2}{3}(1+\theta+\theta^2+\ldots) = \frac{\theta^2}{3(1-\theta)} \le 2\theta^2/3$$

if $0 \le \theta \le 1/2$. Similarly,

$$\log g(-\theta) \le 2\theta^2/3, \quad 0 \le \theta \le 1/2,$$

hence putting $\theta = 3u/(4\sqrt{n})$, we get (3.13). □

Lemma 3.12 *Let Y_i, $i = 1, 2, \ldots$ be i.i.d. random variables having exponential distribution with parameter 1. Then*

$$P\left(\max_{1\le j\le n}\left|\sum_{i=0}^{j}(Y_i-1)\right| \ge u\sqrt{n}\right) \le 2e^{-u^2/8}, \qquad 0 < u < 2\sqrt{n}. \tag{3.14}$$

Moreover, with any $C > 0$,

$$P\left(\max_{1\le i\le n} Y_i \ge C\log n\right) \le n^{1-C}. \tag{3.15}$$

Proof. By exponential Kolmogorov's inequality (see Tóth [37]) we have for $0 < \theta \le 1/2$

$$P\left(\max_{1\le j\le n}\left|\sum_{i=0}^{j}(Y_i-1)\right| \ge u\sqrt{n}\right)$$

$$\le e^{-\theta u\sqrt{n}}\left((f(\theta))^n + (f(-\theta))^n\right),$$

where

$$f(\theta) = E\left(e^{\theta(Y_1-1)}\right) = \frac{1}{e^{\theta}(1-\theta)} \le e^{2\theta^2}$$

and, similarly,

$$f(-\theta) = \frac{e^{\theta}}{1+\theta} \le e^{2\theta^2}.$$

Now (3.14) can be obtained by putting $\theta = u/(4\sqrt{n})$, and (3.15) is easily seen as follows.

$$P\left(\max_{1\le i\le N} Y_i \ge C\log N\right) \le NP(Y_1 \ge C\log N) = N^{1-C}.$$

□

Finally, we quote the following lemma from Berkes and Philipp [3].

Lemma 3.13 *Let B_i, $i = 1, 2, 3$ be separable Banach spaces. Let F be a distribution on $B_1 \times B_2$ and let G be a distribution on $B_2 \times B_3$ such that the second marginal of F equals the first marginal of G. Then there exists a probability space and three random variables Z_i, $i = 1, 2, 3$, defined on it such that the joint distribution of Z_1 and Z_2 is F and the joint distribution of Z_2 and Z_3 is G.*

4 Proof of Theorem 1.1

4.1 Proof of Proposition 1.1

First we prove the next lemma, which is a consequence of Lemma 3.4.

Lemma 4.1 *On an appropriate probability space one can construct independent random variables $\{T_i^{(k)}\}_{i,k=1}^{\infty}$ with distribution (3.3) and a sequence of independent Wiener processes $\{W_k(t),\, t \geq 0\}_{k=1}^{\infty}$ such that, as $N \to \infty$, we have*

$$\max_{1 \leq k \leq N} \max_{1 \leq j \leq N} |U^{(k)}(j) - W_k(2j)| = O(\log N) \quad \text{a.s.,} \tag{4.1}$$

where $U^{(k)}(j)$ are defined by (3.5).

Proof. By Lemma 3.4 for each fixed $k = 1, 2, \ldots$ on a probability space one can construct $T_j^{(k)}$ and W_k satisfying

$$P\left(\max_{1 \leq j \leq N} |U^{(k)}(j) - W_k(2j)| \geq (C_1 + 1) \log N\right) \leq C_2 e^{-C_3 \log N}. \tag{4.2}$$

Note that the constants C_1, C_2, C_3 depend only on the distribution of $T_j^{(k)}$, hence they do not depend on k. Now consider the product space so that we have (4.2) for all $k = 1, 2, \ldots$ on it. Then

$$P\left(\max_{1 \leq k \leq N} \max_{1 \leq j \leq N} |U^{(k)}(j) - W_k(2j)| \geq (C_1 + 1) \log N\right)$$
$$\leq N C_2 e^{-C_3 \log N} = C_2 e^{-(C_3 - 1) \log N} = \frac{C_2}{N^{C_3 - 1}}.$$

Choosing $C_3 > 2$, (4.1) follows by Borel-Cantelli lemma. □

Now on the probability space of Lemma 4.1 a Wiener sheet $W(\cdot, \cdot)$ is constructed from the independent Wiener processes W_k, $k = 1, 2, \ldots$ as above in such a way that for integer k we have (cf. Section 1.11 of [15])

$$W(k, y) = \sum_{i=1}^{k} W_i(y). \tag{4.3}$$

By Lemma 3.13 this can be extended to a Wiener sheet $\{W(x, y),\, x, y \geq 0\}$ on the probability space of Lemma 4.1, so that on the same probability space we have a simple symmetric random walk $\{S_i\}_{i=0}^{\infty}$ as defined in the Introduction, satisfying (3.1) and (3.2).

To show Proposition 1.1, we start from the identity

$$\xi(k,\rho_N^+,\uparrow)=\xi(k,\rho_N^+)-\xi(k,\rho_N^+,\downarrow)=U^{(k)}(\xi(k-1,\rho_N^+,\uparrow))+\xi(k-1,\rho_N^+,\uparrow).$$

Repeating this procedure several times, we arrive at

$$\xi(k,\rho_N^+,\uparrow)=\sum_{i=1}^{k}U^{(i)}(\xi(i-1,\rho_N^+,\uparrow))+N.$$

For brevity, from here on in this proof we use the notation

$$\xi_i=\xi(i,\rho_N^+,\uparrow).$$

Continuing accordingly, using Lemma 4.1 and the fact that as $N\to\infty$

$$\max_{1\le i\le N^{1-\varepsilon}}\log\xi_i=O(\log N)\qquad \text{a.s.},$$

which follows from Lemma 3.8, we get for $k=1,2,\dots$

$$\xi_k=\sum_{i=1}^{k}W_i(2\xi_{i-1})+N+O(k\log N)$$

$$=W(k,2N)+N+O(k\log N)+\sum_{i=1}^{k}(W_i(2\xi_{i-1})-W_i(2N))\quad \text{a.s.}$$

Now we are to estimate the last term in our next lemma.

Lemma 4.2 *As* $N\to\infty$*,*

$$\sum_{i=1}^{k}(W_i(2\xi_{i-1})-W_i(2N))=O(k^{5/4}N^{1/4+\varepsilon/2})\quad \text{a.s.},$$

where the O *term is uniform in* $k\in[1,N^{1/3-\varepsilon}]$.

Proof. Observe that

$$\sum_{i=1}^{k}(W_i(2\xi_{i-1})-W_i(2N))=\sum_{i=1}^{k}\widetilde{W}_i(2|\xi_{i-1}-N|),$$

where $\widetilde{W}_i(\cdot)$, $i=1,2,\dots$ are independent Wiener processes.

Let $K=[N^{1/3-\varepsilon}]$, $w_k=k^{1/2}N^{1/2+\varepsilon/2}$, $z_k=k^{5/4}N^{1/4+\varepsilon/2}$. Then

$$P\left(\bigcup_{k=1}^{K}\left\{\left|\sum_{i=1}^{k}(W_i(2\xi_{i-1})-W_i(2N))\right|\ge z_k\right\}\right)$$

$$
\begin{aligned}
&= P\left(\bigcup_{k=1}^{K}\left\{\left|\sum_{i=1}^{k}\widetilde{W}_i(2|\xi_{i-1}-N|)\right| \geq z_k\right\}\right)\\
&\leq \sum_{k=1}^{K} P\left(\left|\sum_{i=1}^{k}\widetilde{W}_i(2|\xi_{i-1}-N|)\right| \geq z_k\right)\\
&\leq \sum_{k=1}^{K}\left(P\left(\max_{1\leq i\leq k}|\xi_i-N|\geq w_k\right)+P\left(\sup_{0\leq t_i\leq 2w_k,\, i=1,\dots,k}\left|\sum_{i=1}^{k}\widetilde{W}_i(t_i)\right|\geq z_k\right)\right).
\end{aligned}
$$

It follows from (3.8) by telescoping that

$$
\xi_i - N = \sum_{j=1}^{\xi_{i-1}+\xi_{i-2}+\dots\xi_1+N}(T_j-2), \quad i=1,2,\dots
$$

where T_j are i.i.d. random variables distributed as $T_i^{(k)}$. From Lemma 3.2 and Lemma 3.8 we obtain

$$
\begin{aligned}
&P\left(\max_{1\leq i\leq k}|\xi_i-N|\geq w_k\right)\\
&\leq P\left(\max_{1\leq i\leq k}\xi_i\geq 5N\right)+kP\left(\max_{1\leq n\leq 5Nk}\left|\sum_{j=1}^{n}(T_j-2)\right|\geq w_k\right)\\
&\leq ke^{-N/(4k)}+2ke^{-w_k^2/(40kN)}.
\end{aligned}
$$

From this, together with Lemma 3.6, we finally get

$$
\begin{aligned}
&P\left(\bigcup_{k=1}^{K}\left\{\left|\sum_{i=1}^{k}(W_i(2\xi_{i-1})-W_i(2N))\right|\geq z_k\right\}\right)\\
&\leq \sum_{k=1}^{K}\left(ke^{-N/(4k)}+2ke^{-w_k^2/(40kN)}+2ke^{-z_k^2/(4k^2w_k)}\right)\\
&\leq N^{2/3}e^{-N^{2/3+\varepsilon}/4}+2N^{2/3}e^{-N^{\varepsilon}/40}+2N^{2/3}e^{-N^{\varepsilon/2}/4}.
\end{aligned}
$$

This is summable in N, so the lemma follows by Borel-Cantelli lemma. □

Since $\xi(0,\rho_N^+,\uparrow)=N$, this also proves Proposition 1.1. □

4.2 Proof of Proposition 1.2

According to (3.9) and Proposition 1.1, as $N\to\infty$,

$$
\begin{aligned}
&\xi(k,\rho_N^+)=\xi(k,\rho_N^+,\uparrow)+\xi(k-1,\rho_N^+,\uparrow)\\
&=2N+W(k,2N)+W(k-1,2N)+O(k^{5/4}N^{1/4+\varepsilon/2}) \quad \text{a.s.}
\end{aligned}
\tag{4.4}
$$

On the other hand,

$$\xi(0,\rho_N^+) = \xi(0,\rho_N^+,\uparrow) + \xi(0,\rho_N^+,\downarrow) = N + \xi(0,\rho_N^+,\downarrow). \tag{4.5}$$

But

$$\xi(0,\rho_N^+,\downarrow) = T_1^* + \ldots + T_N^*, \tag{4.6}$$

where T_i^* represents the number of downward excursions away from 0 between the ith and $(i+1)$st upward excursions away from 0. Hence T_i^* are i.i.d. random variables with geometric distribution

$$P(T_i^* = j) = \frac{1}{2^{j+1}}, \quad j = 0, 1, 2, \ldots$$

and also independent of $\{T_i^{(k)},\, i, k = 1, 2, \ldots\}$. Hence from KMT Lemma 3.4 and by Lemma 3.13, on the probability space of Proposition 1.1 one can construct a Wiener process $W^*(\cdot)$, independent of $W(\cdot,\cdot)$ such that, as $N \to \infty$,

$$T_1^* + \ldots T_N^* = N + W^*(2N) + O(\log N) \quad \text{a.s.}$$

This together with (4.4), (4.5) and (4.6) proves Proposition 1.2. □

4.3 Proof of Proposition 1.3

Consider N excursions away from 0, out of which ν_N are upward excursions, and $N - \nu_N$ are downward excursions. According to Proposition 1.2, as $N \to \infty$,

$$\xi(k,\rho_{\nu_N}^+) - \xi(0,\rho_{\nu_N}^+) = G(k, 2\nu_N) + O(k^{5/4}(\nu_N)^{1/4+\varepsilon/2})$$

$$= G(k, 2\nu_N) + O(k^{5/4}N^{1/4+\varepsilon/2}) \quad \text{a.s.}$$

Since $\xi(k,\rho_N) = \xi(k,\rho_{\nu_N}^+)$ for $k > 0$, it is enough to verify the next Lemma.

Lemma 4.3 *As $N \to \infty$ we have*

$$W(k, 2\nu_N) - W(k, N) = O(k^{1/2}N^{1/4+\varepsilon/2}) \quad \text{a.s.,} \tag{4.7}$$

where O is uniform in $k \in [1, N]$. Moreover,

$$W^*(2\nu_N) - W^*(N) = O(N^{1/4+\varepsilon/2}) \quad \text{a.s.} \tag{4.8}$$

$$\xi(0,\rho_N) - \xi(0,\rho_{\nu_N}^+) = O(\log N) \quad \text{a.s.} \tag{4.9}$$

Proof.

$$
\begin{aligned}
&P\left(\bigcup_{k=1}^{N}\left\{|W(k,2\nu_N)-W(k,N)|\geq k^{1/2}N^{1/4+\varepsilon/2}\right\}\right)\\
&\leq \sum_{k=1}^{N} P\left(\frac{|W(k,2\nu_N)-W(k,N)|}{k^{1/2}}\geq N^{1/4+\varepsilon/2}\right)\\
&\leq NP\left(\widetilde{W}(|2\nu_N-N|)\geq N^{1/4+\varepsilon/2}\right)\\
&\leq NP\left(\sup_{0\leq u\leq N^{1/2+\varepsilon/2}}|\widetilde{W}(u)|\geq N^{1/4+\varepsilon/2}\right)+NP(|2\nu_N-N|\geq N^{1/2+\varepsilon/2})\\
&\leq 2N\exp(-N^{\varepsilon/2}/2)+2N\exp(-N^{\varepsilon}/2),
\end{aligned}
$$

where $\widetilde{W}(\cdot)$ is a standard Wiener process and we used Lemmas 3.3 and 3.6 (with $k=1$).

Hence (4.7) follows by Borel-Cantelli lemma, and (4.8) follows from (4.7) by putting $k=1$ there. To show (4.9), observe that

$$
P(\xi(0,\rho_N)-\xi(0,\rho^{+}_{\nu_N})\geq j)=\frac{1}{2^j},
$$

since the event $\{\xi(0,\rho_N)-\xi(0,\rho^{+}_{\nu_N})\geq j\}$ means that the last j excursions out of N are downward and, looking at the random walk from ρ_N backward, this event is equivalent to the event that the first j excursions are downward, which has the probability $1/2^j$. Putting $j=2\log N$, (4.9) follows by Borel-Cantelli lemma. □

This also completes the proof of Proposition 1.3. □

Now we are ready to prove Theorem 1.1. Put $N=\xi(0,n)$ into (1.20). By Proposition 1.3 we have

$$
\xi(k,\kappa_n)-\xi(0,n)=G(k,\xi(0,n))+O(k^{5/4}(\xi(0,n))^{1/4+\varepsilon/2})\quad \text{a.s.,} \tag{4.10}
$$

where $\kappa_n=\max\{i\leq n:\ S_i=0\}$, i.e., the last zero before n of the random walk and O is uniform for $k\in[1,(\xi(0,n))^{1/3-\varepsilon})$.

Lemma 4.4 *For any $\delta>0$, as $n\to\infty$,*

$$
\xi(k,n)-\xi(k,\kappa_n)=O(kn^{\delta})\quad \text{a.s.,} \tag{4.11}
$$

where O is uniform in $k\in[1,n]$.

Proof. We have

$$
\xi(k,n)-\xi(k,\kappa_n)\leq \max_{0\leq i\leq \xi(0,n)}(\xi(k,\rho_{i+1})-\xi(k,\rho_i)),
$$

therefore

$$P\left(\bigcup_{k=1}^{n}\{\xi(k,n)-\xi(k,\kappa_n)\geq kn^\delta\}\right)$$
$$\leq\sum_{k=1}^{n}P\left(\max_{0\leq i\leq\xi(0,n)}(\xi(k,\rho_{i+1})-\xi(k,\rho_i))\geq kn^\delta\right)$$
$$\leq P(\xi(0,n)\geq n^{1/2+\delta})+\sum_{k=1}^{n}P\left(\max_{0\leq i\leq n^{1/2+\delta}}(\xi(k,\rho_{i+1})-\xi(k,\rho_i))\geq kn^\delta\right)$$
$$\leq P(\xi(0,n)\geq n^{1/2+\delta})+n^{1/2+\delta}\sum_{k=1}^{n}P(\xi(k,\rho_1)\geq kn^\delta).$$

Lemma 3.9 implies

$$P(\xi(0,n)\geq n^{1/2+\delta})\leq C_1e^{-C_2n^{2\delta}}. \tag{4.12}$$

Moreover, from the distribution of $\xi(k,\rho_1)$ (cf. Révész [31] pp. 100, Theorem 9.7), we get

$$P(\xi(k,\rho_1)\geq j)=\frac{1}{2k}\left(1-\frac{1}{2k}\right)^{j-1}\leq e^{-j/(2k)}. \tag{4.13}$$

Putting $j=kn^\delta$, (4.11) follows from (4.12) and (4.13) by applying Borel-Cantelli lemma. □

To complete the proof of Theorem 1.1, observe that for any $\delta>0$, almost surely

$$n^{1/2-\delta}\leq\xi(0,n)\leq n^{1/2+\delta}$$

for all n large enough. We have, as $n\to\infty$,

$$(\xi(0,n))^{1/4+\varepsilon/2}=O(n^{1/8+5\varepsilon/8})\quad\text{a.s.}$$

Now (1.16) follows from (4.10) and Lemma 4.4, since for large n the O term in (4.10) is uniform in $k\in[1,n^{1/6-\varepsilon})$, as stated. □

5 Proof of Theorems 1.2 and 1.3

In this section we show that the local time $\xi(0,n)$ in (1.16) can be changed to another random walk local time $\widetilde{\xi}(0,n)$ and also to a Brownian local time $\eta(0,n)$, both independent of $G(\cdot,\cdot)$, as claimed in Theorems 1.2 and 1.3, respectively. The method of proof is similar to that of [10], [11].

5.1 Proof of Theorem 1.2

Assume that on the same probability space we have two independent simple symmetric random walks $\{S_i^{(1)},\ i=1,2,\ldots\}$ and $\{S_i^{(2)},\ i=1,2,\ldots\}$, with respective local times $\xi^{(1)}(\cdot,\cdot)$ and $\xi^{(2)}(\cdot,\cdot)$. Assume furthermore that the above procedure has been performed for both random walks, i.e. we have Wiener sheets $W^{(1)}(\cdot,\cdot)$, $W^{(2)}(\cdot,\cdot)$ and Wiener processes $W^{*(1)}$, $W^{*(2)}$ satisfying Propositions 1.1-1.3 and Theorem 1.1. Based on these two random walks, we construct a new simple symmetric random walk $\{S_i,\ i=1,2,\ldots\}$ such that its local time $\xi(0,n)$ will be close to $\xi^{(1)}(0,n)$, while the increments $\xi(k,n)-\xi(0,n)$ will be close to $\xi^{(2)}(k,n)-\xi^{(2)}(0,n)$. This is achieved by taking "large" excursions from $S^{(1)}$ and "small" excursions from $S^{(2)}$. As a result, we shall conclude that $\xi(k,n)-\xi(0,n)$ can be approximated by $G^{(2)}(k,\xi^{(1)}(0,n))$.

This is done as follows (see [11]). Let $\rho_i^{(j)}$, $j=1,2$, $i=1,2,\ldots$ denote the consecutive return times to zero of the random walk $S^{(j)}$. Let furthermore $N_0=0$, $N_\ell=2^\ell$, $r_\ell=N_\ell-N_{\ell-1}=2^{\ell-1}$, $\ell=1,2,\ldots$, and consider the blocks out of which the ℓ-th block consisting of r_ℓ excursions as follows.

$$\left\{S^{(j)}_{\rho^{(j)}_{N_{\ell-1}}+1},\ldots,S^{(j)}_{\rho^{(j)}_{N_\ell}}\right\},\quad j=1,2,\quad \ell=1,2,\ldots$$

In this block call an excursion large if

$$\rho^{(j)}_{N_{\ell-1}+i}-\rho^{(j)}_{N_{\ell-1}+i-1}>r_\ell^{4/3},$$

and call it small otherwise. Now construct the block

$$\{S_{\rho_{N_{\ell-1}}+1},\ldots,S_{\rho_{N\ell}}\}$$

of the new random walk, the ℓ-th block having also r_ℓ excursions by keeping large excursions in the block of $S^{(1)}$ unaltered and replacing small excursions of $S^{(1)}$ by the small excursions of $S^{(2)}$, keeping also the order of small and large excursions as it was in $S^{(1)}$. It is possible that there are more small excursions in the block of $S^{(1)}$ than in the block of $S^{(2)}$. In this case replace as many small excursions as possible by those of $S^{(2)}$, and leave the other small excursions unaltered in $S^{(1)}$. One can easily see that, putting these blocks one after the other, the resulting $S_1,S_2,\ldots$ is a simple symmetric random walk. We denote by ξ,ρ, etc., without superfix, the corresponding quantities defined for this random walk, and continue with establishing the next five lemmas that will also lead to concluding Theorem 1.2.

Lemma 5.1 *The following inequalities hold:*

$$\max_{1\le i\le N_\ell}|\rho_i-\rho_i^{(1)}| \tag{5.1}$$
$$\le\sum_{j=1}^{2}\sum_{m=1}^{\ell}\sum_{i=1}^{r_m}\left(\rho^{(j)}_{N_{m-1}+i}-\rho^{(j)}_{N_{m-1}+i-1}\right)I\left\{\rho^{(j)}_{N_{m-1}+i}-\rho^{(j)}_{N_{m-1}+i-1}\le r_m^{4/3}\right\},$$

and

$$\max_{1\le i\le N_\ell}|\xi(k,\rho_i)-\xi^{(2)}(k,\rho_i^{(2)})|\le \xi^*(k)\sum_{m=1}^{\ell}(\mu_m^{(1)}+\mu_m^{(2)}), \tag{5.2}$$

where $I\{\cdot\}$ denotes the indicator of the event in the brackets,

$$\xi^*(k)=\max_{j=1,2}\max_{1\le i\le N_\ell}\left(\xi^{(j)}(k,\rho_i^{(j)})-\xi^{(j)}(k,\rho_{i-1}^{(j)})\right) \tag{5.3}$$

and $\mu_m^{(j)}$ is the number of large excursions in the m-th block of $S^{(j)}$.

Proof. Obviously, $\max_{1\le i\le N_\ell}|\rho_i-\rho_i^{(1)}|$ can be overestimated by the total length of small excursions of the two random walks up to time N_ℓ which is the right-hand side of (5.1).

Moreover, $|\xi(k,\rho_i)-\xi^{(2)}(k,\rho_i^{(2)})|$ can be overestimated by the total number of large excursions up to N_ℓ multiplied by the maximum of the local time of k over all excursions up to N_ℓ of the two random walks, which is the right-hand side of (5.2). □

Lemma 5.2 *For $n\le\rho_N^{(1)}$ we have*

$$\max_{1\le i\le n}|\xi(0,i)-\xi^{(1)}(0,i)|\le \max_{1\le j\le N}|\xi(0,\rho_j)-\xi(0,\rho_j^{(1)})|+1. \tag{5.4}$$

Proof. Since $\xi(0,\rho_j)=\xi^{(1)}(0,\rho_j^{(1)})=j$, we have for $\rho_{j-1}^{(1)}\le i<\rho_j^{(1)}$, $j\le N$,

$$\xi(0,i)-\xi^{(1)}(0,i)\le\xi(0,\rho_j^{(1)})-(j-1)$$

$$=\xi(0,\rho_j^{(1)})-\xi(0,\rho_j)+1\le\max_{1\le j\le N}|\xi(0,\rho_j)-\xi(0,\rho_j^{(1)})|+1.$$

On the other hand,

$$\xi^{(1)}(0,i)-\xi(0,i)\le j-1-\xi(0,\rho_{j-1}^{(1)})$$

$$=\xi(0,\rho_{j-1})-\xi(0,\rho_{j-1}^{(1)})\le\max_{1\le j\le N}|\xi(0,\rho_j)-\xi(0,\rho_j^{(1)})|+1.$$

□

Lemma 5.3 *For $C>0$, $K=1,2,\dots$ we have*

$$P\left(\bigcup_{k=1}^{K}\{\max_{1\le i\le N_\ell}|\xi(k,\rho_i)-\xi^{(2)}(k,\rho_i^{(2)})|\ge 3Ck\ell^2r_\ell^{1/3}\}\right)$$

$$\le N_\ell\sum_{k=1}^{K}\frac{1}{k}\left(1-\frac{1}{2k}\right)^{Ck\log N_\ell}+K\exp(2(e-3)\ell r_\ell^{1/3}). \tag{5.5}$$

Proof. Using (5.2) of Lemma 5.1, and $4\log 2 < 3$, we get

$$P\left(\bigcup_{k=1}^{K}\left\{\max_{1\le i\le N_\ell}|\xi(k,\rho_i)-\xi^{(2)}(k,\rho_i^{(2)})|\ge 3Ck\ell^2 r_\ell^{1/3}\right\}\right)$$

$$\le \sum_{k=1}^{K} P(\xi^*(k)\ge Ck\log N_\ell)+KP\left(\sum_{m=1}^{\ell}(\mu_m^{(1)}+\mu_m^{(2)})\ge 4\ell r_\ell^{1/3}\right).$$

Using again the distribution of $\xi(k,\rho_1)$ in [31], we get

$$P(\xi^*(k)\ge Ck\log N_\ell)\le 2N_\ell P(\xi(k,\rho_1)\ge Ck\log N_\ell)$$

$$\le \frac{N_\ell}{k}\left(1-\frac{1}{2k}\right)^{Ck\log N_\ell}.$$

Moreover, $\{\mu_m^{(j)},\ j=1,2,\ m=1,2,\ldots\}$ are independent random variables such that $\mu_m^{(1)}+\mu_m^{(2)}$ has binomial distribution with parameters $2r_m$ and $p_m=P(\rho_1\ge r_m^{4/3})\le r_m^{-2/3}$, where Lemma 3.10 was used for $N=1$. Using the moment generating function of the binomial distribution and exponential Markov's inequality, proceeding as in [11], we get

$$P\left(\sum_{m=1}^{\ell}(\mu_m^{(1)}+\mu_m^{(2)})\ge z\right)\le e^{-z}\prod_{m=1}^{\ell}(1+p_m(e-1))^{2r_m}$$

$$\le \exp\left(2(e-1)\sum_{m=1}^{\ell} r_m p_m - z\right)\le \exp(2(e-1)\ell r_\ell^{1/3}-z).$$

Putting $z=4\ell r_\ell^{1/3}$, we get (5.5). □

Lemma 5.4 *As* $N\to\infty$,

$$\xi(k,\rho_N)-\xi^{(2)}(k,\rho_N^{(2)})=O(kN^{1/3}\log^2 N)\quad \text{a.s.}, \tag{5.6}$$

where O *is uniform in* $k\in[1,N]$.

Proof. Applying the inequality (5.5) in Lemma 5.3 with $K=N_\ell$, the right hand side is summable for ℓ, provided that C is large enough. Hence

$$\max_{1\le i\le N_\ell}|\xi(k,\rho_i)-\xi^{(2)}(k,\rho_i^{(2)})|=O(k\ell^2 r_\ell^{1/3})=O(k(\log N_\ell)^2 N_\ell^{1/3})$$

almost surely, as $\ell\to\infty$, from which (5.6) follows. □

To verify Theorem 1.2, we start from (1.20) in Proposition 1.3, applying it for the random walk $S^{(2)}$. We have

$$\xi^{(2)}(k,\rho_N^{(2)})-\xi^{(2)}(0,\rho_N^{(2)})=G^{(2)}(k,N)+O(k^{5/4}N^{1/4+\varepsilon/2})\quad \text{a.s.}$$

as $N\to\infty$. Since $\xi^{(2)}(0,\rho_N^{(2)})=\xi(0,\rho_N)=N$, according to Lemma 5.4 we also have, as $N\to\infty$,

$$\xi(k,\rho_N)-\xi(0,\rho_N)=G^{(2)}(k,N)+O(k^{5/4}N^{1/4+\varepsilon/2}+kN^{1/3}\log^2 N)$$

almost surely. Now put $N=\xi(0,n)$. Using Lemma 4.4, we can see as before,

$$\xi(k,n)-\xi(0,n)=G^{(2)}(k,\xi(0,n))+O(k^{5/4}n^{1/8+5\varepsilon/8}+kn^{1/6+\varepsilon/4})$$

almost surely and uniformly in $k\in[1,n^{1/6-\varepsilon}]$, as $n\to\infty$. It remains to show that on the right-hand side $\xi(0,n)$ can be replaced by $\xi^{(1)}(0,n)$.

Lemma 5.5 *For any $\varepsilon>0$ there exists a $\delta>0$ such that, as $n\to\infty$,*

$$|G^{(2)}(k,\xi(0,n))-G^{(2)}(k,\xi^{(1)}(0,n))|=O(k^{1/2}n^{1/4-\delta})\quad \text{a.s.,} \tag{5.7}$$

where O is uniform in $k\in[1,n^{1/6-\varepsilon}]$.

Proof. Let $0<\varepsilon<1/6$ and $K_\ell=[2^{\ell(1/6-\varepsilon)}]$, $u_\ell=2^{\ell(1/4-\varepsilon/100)}$. Since $k^{-1/2}W(k,\cdot)$ is a standard Wiener process (denoted by $\widetilde{W}(\cdot)$), we have

$$\begin{aligned}
&P\left(\bigcup_{k=1}^{K_\ell}\left\{\max_{2^{\ell-1}\le n<2^\ell}|W^{(2)}(k,\xi(0,n))-W^{(2)}(k,\xi^{(1)}(0,n))|\ge k^{1/2}u_\ell\right\}\right)\\
&\le K_\ell P\left(\max_{2^{\ell-1}\le n<2^\ell}|\widetilde{W}(\xi(0,n))-\widetilde{W}(\xi^{(1)}(0,n)|\ge u_\ell\right)\\
&\le K_\ell P(\sup_{(u,v)\in A}|\widetilde{W}(u)-\widetilde{W}(v)|\ge u_\ell)\\
&+2K_\ell P(\xi(0,2^\ell)\ge 2^\ell)\\
&+K_\ell P\left(\max_{1\le n\le 2^\ell}|\xi(0,n)-\xi^{(1)}(0,n)|\ge 2^{\ell(1/2-\varepsilon/48)}\right),
\end{aligned}$$

where

$$A=\{(u,v):\,0\le u\le 2^\ell,\,0\le v\le 2^\ell,|u-v|\le 2^{\ell(1/2-\varepsilon/48)}\}.$$

First we estimate the last term. By Lemma 5.2

$$P\left(\max_{1\le n\le 2^\ell}|\xi(0,n)-\xi^{(1)}(0,n)|\ge 2^{\ell(1/2-\varepsilon/48)}\right)$$

$$\le P\left(\max_{1\le j\le 2^{\ell(1/2+\varepsilon/4)}} |\xi(0,\rho_j)-\xi(0,\rho_j^{(1)})| \ge 2^{\ell(1/2-\varepsilon/48)}-1\right)$$
$$+P\left(\rho^{(1)}_{[2^{\ell(1/2+\varepsilon/4)}]} \le 2^{\ell}\right)$$
$$\le P\left(\max_{(i,j)\in B} |\xi(0,i)-\xi(0,j)| \ge 2^{\ell(1/2-\varepsilon/48)}-1\right)$$
$$+P\left(\rho^{(1)}_{[2^{\ell(1/2+\varepsilon/4)}]} \le 2^{\ell}\right)+2P\left(\rho_{[2^{\ell(1/2+\varepsilon/4)}]} \ge 2^{\ell(4/3+\varepsilon)}\right)$$
$$+P\left(\max_{1\le j\le 2^{\ell(1/2+\varepsilon/4)}} |\rho_j-\rho_j^{(1)}| \ge 2^{\ell(1-\varepsilon/12)}\right),$$

where

$$B=\{(i,j):\, 1\le i\le 2^{\ell(4/3+\varepsilon)}, 1\le j\le 2^{\ell(4/3+\varepsilon)}, |i-j|\le 2^{\ell(1-\varepsilon/12)}\}.$$

Now we estimate the respective right-hand sides of the previous two inequalities term by term. Lemma 3.9 implies

$$P(\sup_{(u,v)\in A} |\widetilde{W}(u)-\widetilde{W}(v)| \ge 2^{\ell(1/4-\varepsilon/100)})$$
$$\le C_1 2^{\ell(1/2+\varepsilon/48)}\exp\left(-C_2 2^{\ell(\varepsilon/48-\varepsilon/50)}\right),$$

and

$$P\left(\max_{(i,j)\in B} |\xi(0,i)-\xi(0,j)| \ge 2^{\ell(1/2-\varepsilon/48)}-1\right)$$
$$\le C_1 2^{\ell(1/6+13\varepsilon/24)}\exp\left(-C_2(2^{\ell\varepsilon/24}-2)\right).$$

Observe that

$$P(\xi(0,2^{\ell})\ge 2^{\ell})=0$$

and

$$P(\rho^{(1)}_{[2^{\ell(1/2+\varepsilon/4)}]} \le 2^{\ell}) = P(\xi^{(1)}(0,2^{\ell})\ge 2^{\ell(1/2+\varepsilon/4)}) \le C_1 e^{-C_2 2^{\ell\varepsilon/2}}.$$

From Lemma 3.10 we have

$$P(\rho^{(1)}_{[2^{\ell(1/2+\varepsilon/4)}]} \ge 2^{\ell(4/3+\varepsilon)}) \le C 2^{-\ell(1/6+\varepsilon/4)}.$$

Finally, from (5.1) of Lemma 5.1, Lemma 3.10 and Markov's inequality

$$P\left(\max_{1\le j\le 2^{\ell(1/2+\varepsilon/4)}} |\rho_j-\rho_j^{(1)}| \ge 2^{\ell(1-\varepsilon/12)}\right)$$
$$\le \frac{2}{2^{\ell(1-\varepsilon/12)}} \sum_{m=1}^{\ell(1/2+\varepsilon/4)} r_m E(\rho_1 I(\rho_1\le r_m^{4/3}))$$

$$\le \frac{C}{2^{\ell(1-\varepsilon/12)}} \sum_{m=1}^{\ell(1/2+\varepsilon/4)} 2^{5(m-1)/3} \le C 2^{\ell(-1/6+\varepsilon/2)}.$$

Assembling all these estimations, we obtain

$$P\left(\bigcup_{k=1}^{K_\ell} \left\{ \max_{2^{\ell-1}\le n<2^\ell} |W^{(2)}(k,\xi(0,n)) - W^{(2)}(k,\xi^{(1)}(0,n))| \ge k^{1/2}u_\ell \right\}\right)$$

$$\le C_1 2^{2\ell/3} \exp\left(-C_2 2^{\ell\varepsilon(1/48-1/50)}\right) + C_3 2^{\ell/6} \exp(-C_2 2^{\ell\varepsilon/2})$$

$$+C_1 2^{\ell(1/3-11\varepsilon/24)} \exp\left(-C_2(2^{\ell\varepsilon/24}-2)\right) + C 2^{-5\ell\varepsilon/4}.$$

Since all these terms are summable in ℓ, by Borel-Cantelli lemma we have

$$\max_{2^{\ell-1}\le n<2^\ell} |W^{(2)}(k,\xi(0,n)) - W^{(2)}(k,\xi^{(1)}(0,n))| = O(k^{1/2} 2^{\ell(1/4-\varepsilon/100)})$$

almost surely, as $\ell \to \infty$, uniformly for $k \in [1, 2^{\ell(1/6-\varepsilon)}]$, i.e.,

$$|W^{(2)}(k,\xi(0,n)) - W^{(2)}(k,\xi^{(1)}(0,n))| = O(k^{1/2} n^{1/4-\varepsilon/100})$$

almost surely, as $n \to \infty$, uniformly for $k \in [1, n^{1/6-\varepsilon}]$. Similar estimations hold for the other terms of $G^{(2)}$, hence we have (5.7) with $\delta = \varepsilon/100$. □

Since the above estimations also imply

$$\xi(0,n) - \xi^{(1)}(0,n) = O(n^{1/2-\delta})$$

almost surely, when $n \to \infty$, with $\delta = \varepsilon/48$, on choosing $\widetilde{\xi}(0,\cdot) = \xi^{(1)}(0,\cdot)$, $G(\cdot,\cdot) = G^{(2)}(\cdot,\cdot)$, the proof of Theorem 1.2 is completed as well. □

5.2 Proof of Theorem 1.3

First, we give a coupling inequality for the invariance principle between random walk and Brownian local times at location zero. We use Skorokhod embedding as in [14], i.e., given a standard Wiener process $W(\cdot)$ with its local time $\eta(\cdot,\cdot)$, define a sequence of stopping times $\{\tau_i\}_{i=0}^{\infty}$ by $\tau_0 = 0$,

$$\tau_n := \inf\{t : t > \tau_{n-1}, |W(t) - W(\tau_{n-1})| = 1\}, \quad n = 1, 2, \ldots$$

Then $S_n = W(\tau_n)$, $n = 0, 1, 2, \ldots$ is a simple symmetric random walk. Denote by $\xi(\cdot,\cdot)$ its local time and by ρ_i the return times to zero. Moreover, define

$$\eta_i := \eta(0, \tau_{\rho_i+1}) - \eta(0, \tau_{\rho_i}),$$

i.e., the Brownian local time between the i-th return to zero and next stopping time τ. Then by Knight [27] the random variables η_i, $i = 1, 2,$ are i.i.d. having exponential distribution with parameter 1. There is no other contribution than η_i to the Brownian local time $\eta(0, \cdot)$. Moreover, we have

$$\left| \eta(0, \tau_n) - \sum_{i=1}^{\xi(0,n)} \eta_i \right| \le \eta_{\xi(0,n)},$$

the error term being zero if $S_n = W(\tau_n) \ne 0$. If $S_n = 0$, then the last term $\eta_{\xi(0,n)}$ is not counted in $\eta(0, \tau_n)$. Now we have

$$|\xi(0, n) - \eta(0, n)| \le |\eta(0, \tau_n) - \eta(0, n)| + \max_{1 \le j \le \xi(0,n)} \left| \sum_{i=1}^{j} (\eta_i - 1) \right| + \eta_{\xi(0,n)}.$$

Therefore, for $\delta > 0$

$$\begin{aligned} & P(|\xi(0, n) - \eta(0, n)| \ge 2n^{1/4+\delta} + C \log n) \\ & \le P(\xi(0, n) \ge n^{1/2+\delta}) + P\left(\max_{1 \le j \le n^{1/2+\delta}} \left| \sum_{i=1}^{j} (\eta_i - 1) \right| \ge n^{1/4+\delta} \right) \\ & + P\left(\max_{1 \le i \le n^{1/2+\delta}} \eta_i \ge C \log n \right) + P(|\tau_n - n| \ge n^{1/2+\delta}) \\ & + P\left(\sup_{|u-n| \le n^{1/2+\delta}} |\eta(0, u) - \eta(0, n)| \ge n^{1/4+\delta} \right). \end{aligned}$$

Estimating the above probabilities term by term, by Lemmas 3.9, 3.11 and 3.12,

$$\begin{aligned} & P(\xi(0, n) \ge n^{1/2+\delta}) \le C_1 e^{-C_2 n^{2\delta}}, \\ & P\left(\max_{1 \le j \le n^{1/2+\delta}} \left| \sum_{i=1}^{j} (\eta_i - 1) \right| \ge n^{1/4+\delta} \right) \le 2e^{-n^{\delta}/8}, \\ & P\left(\max_{1 \le i \le n^{1/2+\delta}} \eta_i \ge C \log n \right) \le n^{1/2+\delta-C}, \\ & P(|\tau_n - n| \ge n^{1/2+\delta}) \le 2e^{-3n^{2\delta}/8}, \\ & P\left(\sup_{|u-n| \le n^{1/2+\delta}} |\eta(0, u) - \eta(0, n)| \ge n^{1/4+\delta} \right) \le C_1 n^{1/4-\delta/2} e^{-C_2 n^{\delta}}. \end{aligned}$$

Hence, we arrive at the coupling inequality for the invariance principle between random walk and Brownian local times

$$\begin{aligned} & P(|\xi(0, n) - \eta(0, n)| \ge 2n^{1/4+\delta} + C \log n) \\ & \le C_1 n^{1/4-\delta/2} e^{-C_2 n^{\delta}} + n^{1/2+\delta-C}. \end{aligned} \tag{5.8}$$

By choosing $0 < \delta < 1/2$ and $C > 2$, this also implies

$$\xi(0,n) - \eta(0,n) = O(n^{1/4+\delta}) \quad \text{a.s.} \tag{5.9}$$

as $n \to \infty$.

For the proof of Theorem 1.3 we apply the above procedure for $\xi^{(1)}(0,\cdot)$, i.e., we construct a standard Brownian local time $\eta(0,\cdot)$ satisfying the above inequality with ξ replaced by $\xi^{(1)}$. We may assume that $\eta(0,\cdot)$ is also independent of $G(\cdot,\cdot)$ of Theorem 1.2. We show that in (iv) of Theorem 1.2, $\widetilde{\xi} = \xi^{(1)}$ can be replaced by η with the same O term.

Lemma 5.6 *As $n \to \infty$, we have for any $\delta > 0$*

$$\left| W^{(2)}(k, \xi^{(1)}(0,n)) - W^{(2)}(k, \eta(0,n)) \right| = O(k^{1/2} n^{1/8+\delta}) \quad \text{a.s.},$$

where O is uniform in $k \in [1, n^{1/6}]$.

Proof. Let $K_n = [n^{1/6}]$.

$$\begin{aligned}
&P\left(\bigcup_{k=1}^{K_n} \left| W^{(2)}(k, \xi^{(1)}(0,n)) - W^{(2)}(k, \eta(0,n))) \right| \geq k^{1/2} n^{1/8+\delta} \right) \\
&\leq K_n P(\sup_{(u,v)\in D} |\widetilde{W}(u) - \widetilde{W}(v)| \geq n^{1/8+\delta}) + K_n P(\xi^{(1)}(0,n) \geq n^{1/2+\delta}) \\
&+ K_n P(\eta(0,n) \geq n^{1/2+\delta}) \\
&+ K_n P(|\xi^{(1)}(0,n) - \eta(0,n)| \geq 2n^{1/4+\delta} + C\log n),
\end{aligned}$$

where

$$D = \{(u,v) :\, u \leq n^{1/2+\delta},\, v \leq n^{1/2+\delta},\, |u-v| \leq 2n^{1/4+\delta} + C\log n\}.$$

Now using Lemma 3.9, we get the inequalities

$$K_n P\left(\sup_{(u,v)\in D} |\widetilde{W}(u) - \widetilde{W}(v)| \geq n^{1/8+\delta} \right) \leq C_1 n^{1/2} e^{-C_2 n^{\delta}},$$

$$K_n P(\xi^{(1)}(0,n) \geq n^{1/2+\delta}) \leq C_1 n^{1/6} e^{-C_2 n^{2\delta}},$$

$$K_n P(\eta(0,n) \geq n^{1/2+\delta}) \leq C_1 n^{1/6} e^{-C_2 n^{2\delta}}.$$

By choosing C large enough in (5.8), the right-hand sides of that and also of the above inequalities are summable in n, Lemma 5.6 follows by Borel-Cantelli lemma. □

The same holds for other terms of $G^{(2)}$. Choosing $\delta < \varepsilon$, the error term in Lemma 5.6 is smaller than $kn^{1/6+\varepsilon}$ in (iv) of Theorem 1.2, hence we have also (iii) of Theorem 1.3 with $G = G^{(2)}$, $\widetilde{\xi} = \xi^{(1)}$. (ii) of Theorem 1.3 follows from (5.9) with ξ replaced by $\xi^{(1)} = \widetilde{\xi}$ and (iii) of Theorem 1.2. This completes the proof of Theorem 1.3. □

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