On the increments of the principal value of Brownian local time

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Summary. Let W be a one-dimensional Brownian motion starting from 0. Define $Y(t) = \int_0^t \frac{\mathrm{d}s}{W(s)} := \lim_{\epsilon \to 0} \int_0^t 1_{(|W(s)|>\epsilon)} \frac{\mathrm{d}s}{W(s)}$ as Cauchy's principal value related to local time. We prove limsup and liminf results for the increments of Y.

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1. Introduction

Let $\{W(t);\, t \geq 0\}$ be a one-dimensional standard Brownian motion with $W(0) = 0$, and let $\{L(t,x);\, t \geq 0,\, x \in \mathbb{R}\}$ denote its jointly continuous local time process. That is, for any Borel function $f \geq 0$,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x, \qquad t \geq 0.$$

We are interested in the process

$$Y(t) := \int_0^t \frac{\mathrm{d}s}{W(s)}, \qquad t \geq 0. \tag{1.1}$$

Rigorously speaking, the integral $\int_0^t \mathrm{d}s/W(s)$ should be considered in the sense of Cauchy's principal value, i.e., $Y(t)$ is defined by

$$Y(t) := \lim_{\varepsilon\to 0^+} \int_0^t \frac{\mathrm{d}s}{W(s)} \mathbb{1}_{\{|W(s)|\geq\varepsilon\}} = \int_0^\infty \frac{L(t,x) - L(t,-x)}{x}\,\mathrm{d}x. \tag{1.2}$$

Since $x \mapsto L(t,x)$ is Hölder continuous of order ν, for any $\nu < 1/2$, the integral on the extreme right in (1.2) is almost surely absolutely convergent for all $t > 0$. The process $\{Y(t),\, t \geq 0\}$ is called the principal value of Brownian local time.

It is easily seen that $Y(\cdot)$ inherits a scaling property from Brownian motion, namely, for any fixed $a > 0$, $t \mapsto a^{-1/2}Y(at)$ has the same law as $t \mapsto Y(t)$. Although some properties distinguish $Y(\cdot)$ from Brownian motion (in particular, $Y(\cdot)$ is not a semimartingale), it is a kind of folklore that Y behaves somewhat like a Brownian motion. For detailed studies and surveys on principal value, and relation to Hilbert transform see Biane and Yor [4], Fitzsimmons and Getoor [13], Bertoin [2], [3], Yamada [20], Boufoussi et al. [5], Ait Ouahra and Eddahbi [1], Csáki et al. [11] and a collection of papers [22] together with their references. Biane and Yor [4] presented a detailed study on Y and determined a number of distributions for principal values and related processes.

Concerning almost sure limit theorems for Y and its increments, we summarize the relevant results in the literature. It was shown in [17] that the following law of the iterated logarithm holds:

Theorem A. (Hu and Shi [17])

$$\limsup_{T\to\infty} \frac{Y(T)}{\sqrt{T\log\log T}} = \sqrt{8}\,, \qquad \text{a.s.} \tag{1.3}$$

This was extended in [10] to a Strassen-type [18] functional law of the iterated logarithm.

Theorem B. (Csáki et al. [10]) With probability one the set

$$\left\{ \frac{Y(xT)}{\sqrt{8T\log\log T}},\, 0 \leq x \leq 1 \right\}_{T\geq 3} \tag{1.4}$$

is relatively compact in $C[0,1]$ with limit set equal to

$$\mathcal{S} := \left\{ f \in C[0,1] :\ f(0)=0,\ f \text{ is absolutely continuous and } \int_0^1 (f'(x))^2\,\mathrm{d}x \le 1 \right\}. \tag{1.5}$$

Concerning Chung-type law of the iterated logarithm, we have the following result:

Theorem C. (Hu [16])

$$\liminf_{T\to\infty} \sqrt{\frac{\log\log T}{T}} \sup_{0\le s\le T} |Y(s)| = K_1\,, \qquad \text{a.s.} \tag{1.6}$$

with some (unknown) constant $K_1 > 0$.

The large increments were studied in [7] and [8]:

Theorem D. (Csáki et al. [7]) *Under the conditions*

$$\begin{cases} 0 < a_T \le T, \\ T \mapsto a_T \text{ and } T \mapsto T/a_T \text{ are both non-decreasing}, \\ \lim\limits_{T\to\infty} \dfrac{\log(T/a_T)}{\log\log T} = \infty, \end{cases} \tag{1.7}$$

we have

$$\lim_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{a_T\log(T/a_T)}} = 2, \qquad \text{a.s.} \tag{1.8}$$

Wen [19] studied the lag increments of Y and among others proved the following results.

Theorem E. (Wen [19])

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \frac{\sup_{t\le s\le T} |Y(s)-Y(s-t)|}{\sqrt{t(\log(T/t)+2\log\log t)}} = 2, \qquad \text{a.s.} \tag{1.9}$$

Under the conditions $0 < a_T \le T$, $a_T \to \infty$ *as* $T\to\infty$, *we have*

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{\sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{a_T(\log((t+a_T)/a_T)+2\log\log a_T)}} \le 2, \qquad \text{a.s.} \tag{1.10}$$

If a_T *is onto, then we have equality in* (1.10).

In this note our aim is to investigate further limsup and liminf behaviors of the increments of Y.

Theorem 1.1. *Assume that $T \mapsto a_T$ is a function such that $0 < a_T \le T$, and both a_T and T/a_T are non-decreasing. Then*

(i)

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{a_T\left(\log\sqrt{T/a_T}+\log\log T\right)}} = \sqrt{8}, \qquad \text{a.s.} \tag{1.11}$$

(iia) *If $a_T > T(\log T)^{-\alpha}$ for some $\alpha < 2$, then*

$$\liminf_{T\to\infty} \sqrt{\frac{\log\log T}{a_T}} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| = K_2, \qquad \text{a.s.} \tag{1.12}$$

(iib) *If $a_T \le T(\log T)^{-\alpha}$ for some $\alpha > 2$, then*

$$\liminf_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{a_T \log(T/a_T)}} = K_3, \qquad \text{a.s.} \tag{1.13}$$

with some positive constants K_2, K_3. If, moreover,

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty,$$

then $K_3 = 2$.

Theorem 1.2. *Assume that $T \mapsto a_T$ is a function such that $0 < a_T \le T$, and both a_T and T/a_T are non-decreasing. Then*

(i)

$$\liminf_{T\to\infty} \frac{\sqrt{T\log\log T}}{a_T} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| = K_4, \qquad \text{a.s.} \tag{1.14}$$

with some positive constant K_4. If, $\lim_{T\to\infty}(a_T/T) = 0$, then $K_4 = 1/\sqrt{2}$.

(iia) *If $0 < \rho \le 1$, then*

$$\limsup_{T\to\infty} \frac{\inf_{0\le t\le T-\rho T} \sup_{0\le s\le \rho T} |Y(t+s)-Y(t)|}{\sqrt{T\log\log T}} = \rho\sqrt{8}, \qquad \text{a.s.} \tag{1.15}$$

(iib) *If*

$$\lim_{T\to\infty} \frac{a_T(\log\log T)^2}{T} = 0,$$

then

$$\limsup_{T\to\infty} \frac{\sqrt{T}}{a_T\sqrt{\log\log T}} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| = K_5, \qquad \text{a.s.} \tag{1.16}$$

with some positive constant K_5.

Remark 1. *The exact values of the constants K_i, $i = 2, 3, 4, 5$ are unknown. It seems difficult to determine the exact values of these constants. In the proofs we establish upper and lower bounds with possibly different constants. It follows however by 0-1 law for Brownian motion that the limsup's and liminf's considered here are non-random constants.*

Remark 2. *Plainly we recover some previous results on the path properties of Y by considering particular cases of Theorems 1.1 and 1.2. For instance, Theorems A and C follow from (1.11) and (1.12) respectively by taking $a_T = T$, and (1.8) follows from (1.11) combining with (1.13). However in Theorem 1.1(ii) and Theorem 1.2(ii) there are still small gaps in a_T.*

The organization of the paper is as follows: In Section 2 some facts are presented needed in the proofs. Section 3 contains the necessary probability estimates. Theorem 1.1(i) and Theorem 1.1(iia,b) are proved in Sections 4 and 5, resp., while Theorem 1.2(i) and Theorem 1.2(iia,b) are proved in Sections 6 and 7, resp.

Throughout the paper, the letter K with subscripts will denote some important but unknown finite positive constants, while the letter c with subscripts denotes some finite and positive universal constants not important in our investigations. When the constants depend on a parameter, say δ, they are denoted by $c(\delta)$ with subscripts.

2. Facts

Let $\{W(t),\, t \geq 0\}$ be a standard Brownian motion and define the following objects:

$$g := \sup\{t : \; t \leq 1,\, W(t) = 0\} \tag{2.1}$$

$$B(s) := \frac{W(sg)}{\sqrt{g}}, \qquad 0 \leq s \leq 1, \tag{2.2}$$

$$m(s) := \frac{|W(g + s(1-g))|}{\sqrt{1-g}}, \qquad 0 \leq s \leq 1. \tag{2.3}$$

Here we summarize some well-known facts needed in our proofs.

Fact 2.1. (Biane and Yor [4])

$$\frac{\mathbb{P}(Y(1) \in \, \mathrm{d}x)}{\mathrm{d}x} = \sqrt{\frac{2}{\pi^3}} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{(2k+1)^2 x^2}{8}\right), \quad x \in \mathbb{R}. \tag{2.4}$$

Consequently we have the estimate: for $\delta > 0$

$$c_1 \exp\left(-\frac{z^2}{8(1-\delta)}\right) \leq \mathbb{P}(Y(1) \geq z) \leq \exp\left(-\frac{z^2}{8}\right), \qquad z \geq 1 \tag{2.5}$$

with some positive constant $c_1 = c_1(\delta)$. Moreover, g, $\{B(s),\, 0 \le s \le 1\}$ and $\{m(s),\, 0 \le s \le 1\}$ are independent, g has arcsine distribution, B is a Brownian bridge and m is a Brownian meander.

$$\begin{aligned} &\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} < z \,\Big|\, m(1) = 0\right) \\ &= \sum_{k=-\infty}^{\infty} (1 - k^2 z^2) \exp\left(-\frac{k^2 z^2}{2}\right) = \frac{8\pi^2\sqrt{2\pi}}{z^3} \sum_{k=1}^{\infty} \exp\left(-\frac{2k^2\pi^2}{z^2}\right), \quad z > 0. \end{aligned} \tag{2.6}$$

$$\mathbb{P}(m(1) > x) = e^{-x^2/2}, \qquad x > 0. \tag{2.7}$$

Fact 2.2. (Yor [21, Exercise 3.4 and pp. 44]) *Let $Q^{\delta}_{x\to 0}$ be the law of square of Bessel bridge from x to 0 of dimension $\delta > 0$ during time interval $[0,1]$. The process $(m^2(1-v), 0 \le v \le 1)$ conditioned on $\{m^2(1) = x\}$ is distributed as $Q^3_{x\to 0}$. Furthermore, we have*

$$Q^{\delta}_{x\to 0} = Q^{\delta}_{0\to 0} * Q^{0}_{x\to 0}, \quad \forall\, \delta > 0,\ x > 0, \tag{2.8}$$

where $*$ denotes convolution operator. Consequently, for any $x > 0$

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} < z \,\Big|\, m(1) = x\right) \ge \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} < z \,\Big|\, m(1) = 0\right). \tag{2.9}$$

Fact 2.3. (Hu [16]) *For $0 < z \le 1$*

$$c_2 \exp\left(-\frac{c_3}{z^2}\right) \le \mathbb{P}\left(\sup_{0\le s\le 1} |Y(s)| < z\right) \le c_4 \exp\left(-\frac{c_5}{z^2}\right) \tag{2.10}$$

with some positive constants c_2, c_3, c_4, c_5.

Fact 2.4. (Csörgő and Révész [12]) *Assume that $T \mapsto a_T$ is a function such that $0 < a_T \le T$, and both a_T and T/a_T are non-decreasing. Then*

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |W(t+s) - W(t)|}{\sqrt{a_T\,(\log(T/a_T) + \log\log T)}} = \sqrt{2}, \qquad \text{a.s.} \tag{2.11}$$

Fact 2.5. (Strassen [18]) *If $f \in \mathcal{S}$ defined by (1.5), then for any partition $x_0 = 0 < x_1 < \ldots < x_k < x_{k+1} = 1$ we have*

$$\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} \le 1. \tag{2.12}$$

Fact 2.6. (Chung [6])

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \sup_{0\le s\le t} |W(s)| = \frac{\pi}{\sqrt{8}}, \qquad \text{a.s.} \tag{2.13}$$

Define $g(T) := \max\{s \le T : W(s) = 0\}$. A joint lower class result for $g(T)$ and $M(T) := \sup_{0\le s\le T} |W(s)|$ reads as follows.

Fact 2.7. (Grill [15]) *Let $\beta(t)$, $\gamma(t)$ be positive functions slowly varying at infinity, such that $0 < \beta(t) \le 1$, $0 < \gamma(t) \le 1$, $\beta(t)$ is non-increasing, $\beta(t)\sqrt{t} \uparrow \infty$, $\gamma(t)$ is monotone, $\gamma(t)t \uparrow \infty$, $\gamma(t)/\beta^2(t)$ is monotone. Then*

$$\mathbb{P}\left(M(T) \le \beta(T)\sqrt{T},\ g(T) \le \gamma(T)T \quad \text{i.o.}\right) = 0 \quad \text{or} \quad 1$$

according as $I(\beta,\gamma) < \infty$ or $= \infty$, where

$$I(\beta,\gamma) = \int_1^\infty \frac{1}{t\beta^2(t)}\left(1 + \frac{\beta^2(t)}{\gamma(t)}\right)^{-1/2} \exp\left(-\frac{(4-3\gamma(t))\pi^2}{8\beta^2(t)}\right) \mathrm{d}t.$$

Now define $d(T) := \min\{s \ge T :\ W(s) = 0\}$. Since $\{d(T) > t\} = \{g(t) < T\}$, we deduce from Fact 2.7 the following estimate on $d(T)$ when $T \to \infty$.

Fact 2.8. *With probability 1*

$$d(T) = O(T(\log T)^3), \qquad T \to \infty.$$

3. Probability estimates

Lemma 3.1. *For $T \ge 1$, $\delta, z > 0$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le T-1}\ \sup_{0\le s\le 1} |Y(t+s) - Y(t)| > z\right) \le c_6\left(\sqrt{T}\exp\left(-\frac{z^2}{8(1+\delta)}\right) + T\exp\left(-\frac{z^2}{2(1+\delta)}\right)\right) \tag{3.1}$$

with some positive constant $c_6 = c_6(\delta)$.

For the proof see Csáki et al. [7], Lemma 2.8.

Lemma 3.2. *For $T > 1$, $0 < \delta < 1/2$, $z > 1$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le T-1} (Y(t+1) - Y(t)) \ge z\right) \ge \min\left(\frac{1}{2},\ \frac{c\sqrt{T-1}}{z}\exp\left(-\frac{z^2}{8(1-\delta)}\right)\right) - \exp\left(-z^2\right) \tag{3.2}$$

with some positive constant $c_7 = c_7(\delta) > 0$.

Proof. Let us construct an increasing sequence of stopping times by $\eta_0 := 0$ and

$$\eta_{k+1} := \inf\{t > \eta_k + 1 :\ W(t) = 0\}, \qquad k = 0, 1, 2, \ldots$$

Let

$$\nu_t := \min\{i \geq 1 : \eta_i > t\}$$

$$Z_i := Y(\eta_{i-1}+1) - Y(\eta_{i-1}), \qquad i = 1, 2, \ldots$$

Then $(Z_i, \eta_i - \eta_{i-1})_{i\geq 1}$ are i.i.d. random vectors with

$$\eta_i - \eta_{i-1} \overset{\text{law}}{=} 1 + \tau^2, \qquad Z_i \overset{\text{law}}{=} Y(1),$$

where τ has Cauchy distribution. Clearly, for $t > 0$,

$$\sup_{0\leq s\leq t} (Y(s+1) - Y(s)) \geq \max_{1\leq i\leq \nu_t} Z_i = \overline{Z}_{\nu_t},$$

with $\overline{Z}_k := \max_{1\leq i\leq k} Z_i$. First consider the Laplace transform ($\lambda > 0$):

$$\begin{aligned}
&\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}\left(\overline{Z}_{\nu_u} < z\right) \,\mathrm{d}u \\
&= \lambda \sum_{k=1}^\infty \mathbb{E} \int_0^\infty e^{-\lambda u} 1_{\{\eta_{k-1}\leq u<\eta_k\}} 1_{\{\overline{Z}_k<z\}} \,\mathrm{d}u \\
&= \sum_{k=1}^\infty \mathbb{E}\left(\left[e^{-\lambda\eta_{k-1}} - e^{-\lambda\eta_k}\right] 1_{\{\overline{Z}_k<z\}}\right) \\
&= \sum_{k=1}^\infty \left(\mathbb{E}\Big[1_{\{\overline{Z}_k<z\}} e^{-\lambda\eta_{k-1}}\Big] - \mathbb{E}\Big[1_{\{\overline{Z}_k<z\}} e^{-\lambda\eta_k}\Big]\right) \\
&= \sum_{k=1}^\infty \left(\mathbb{E}\Big[1_{\{\overline{Z}_{k-1}<z\}} e^{-\lambda\eta_{k-1}}\Big] - \mathbb{E}\Big[1_{\{\overline{Z}_{k-1}<z,\, Z_k\geq z\}} e^{-\lambda\eta_{k-1}}\Big] - \mathbb{E}\Big[1_{\{\overline{Z}_k<z\}} e^{-\lambda\eta_k}\Big]\right) \\
&= 1 - \sum_{k=1}^\infty \mathbb{E}\Big[1_{\{\overline{Z}_{k-1}<z,\, Z_k\geq z\}} e^{-\lambda\eta_{k-1}}\Big] \\
&= 1 - \sum_{k=1}^\infty \mathbb{E}\Big[1_{\{\overline{Z}_{k-1}<z\}} e^{-\lambda\eta_{k-1}}\Big] \mathbb{P}(Y(1)\geq z) \\
&= 1 - \sum_{k=1}^\infty \left(\mathbb{E}\Big[1_{\{Z_1<z\}} e^{-\lambda\eta_1}\Big]\right)^{k-1} \mathbb{P}(Y(1)\geq z) \\
&= 1 - \frac{\mathbb{P}(Y(1)\geq z)}{1 - \mathbb{E}\Big[1_{\{Z_1<z\}} e^{-\lambda\eta_1}\Big]},
\end{aligned}$$

i.e.,

$$\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}\left(\overline{Z}_{\nu_u} \geq z\right) \,\mathrm{d}u = \frac{\mathbb{P}(Y(1)\geq z)}{1 - \mathbb{E}\Big[1_{\{Z_1<z\}} e^{-\lambda\eta_1}\Big]}. \tag{3.3}$$

But (recalling that $Z_1 = Y(1)$)

$$1 - \mathbb{E}\Big[1_{\{Z_1<z\}} e^{-\lambda\eta_1}\Big] \leq 1 - \mathbb{E}(e^{-\lambda\eta_1}) + \mathbb{P}(Y(1)\geq z)$$

and (cf. [14], 3.466/1)

$$1-\mathbb{E}e^{-\lambda\eta_1}=1-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\lambda(1+x^2)}}{1+x^2}\,\mathrm{d}x=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\lambda}}e^{-x^2}\,\mathrm{d}x\le 2\sqrt{\lambda},$$

hence

$$\lambda\int_0^\infty e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u}\ge z\right)\,\mathrm{d}u\ge\frac{\mathbb{P}(Y(1)\ge z)}{2\sqrt{\lambda}+\mathbb{P}(Y(1)\ge z)}.$$

On the other hand, for any $u_0>0$ we have

$$\begin{aligned}\lambda\int_0^\infty e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u}\ge z\right)\,\mathrm{d}u&=\lambda\int_0^{u_0} e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u}\ge z\right)\,\mathrm{d}u+\lambda\int_{u_0}^\infty e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u}\ge z\right)\,\mathrm{d}u\\&\le\mathbb{P}\left(\overline{Z}_{\nu_{u_0}}\ge z\right)+e^{-\lambda u_0}.\end{aligned}$$

It turns out that

$$\mathbb{P}\left(\overline{Z}_{\nu_{u_0}}\ge z\right)\ge\frac{\mathbb{P}(Y(1)\ge z)}{2\sqrt{\lambda}+\mathbb{P}(Y(1)\ge z)}-e^{-\lambda u_0}\ge\min\left(\frac12,\ \frac{\mathbb{P}(Y(1)\ge z)}{4\sqrt{\lambda}}\right)-e^{-\lambda u_0},$$

where the inequality

$$\frac{x}{y+x}\ge\min\left(\frac12,\ \frac{x}{2y}\right),\qquad x>0,\ y>0$$

was used. Choosing $u_0=T-1$, $\lambda=z^2/u_0$, and applying (2.5) of Fact 2.1, we finally get

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T-1}(Y(t+1)-Y(t))\ge z\right)\\&\ge\min\left(\frac12,\ \frac{c_8(\delta)\sqrt{T-1}}{z}\exp\left(-\frac{z^2}{8(1-\delta)}\right)\right)-\exp\left(-z^2\right).\end{aligned}\tag{3.4}$$

This proves Lemma 3.2. □

Lemma 3.3. *For $T\ge 2$, $0\le\kappa<1$ and $\delta,z>0$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le T-1}(Y(t+1)-Y(t))<z\right)\le\frac{5}{T^{\kappa/2}}+\exp\left(-c_9T^{(1-\kappa)/2}e^{-(1+\delta)z^2/8}\right)\tag{3.6}$$

with some positive constant $c_9=c_9(\delta)$.

See Csáki et al. [7], Lemma 3.1.

Lemma 3.4. *For $T>1$, $0<z\le 1/2$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le T-1}\sup_{0\le s\le 1}|Y(t+s)-Y(t)|<z\right)\ge\frac{c_{10}}{\sqrt{T}}\exp\left(-\frac{c_{11}}{z^2}\right)$$

with some positive constants c_{10}, c_{11}.

Proof. Define the events

$$A := \left\{ \sup_{0\le s\le 1} |Y(s)| < \frac{z}{4},\ W(1) \ge \frac{4}{z},\ \inf_{1\le u\le T} W(u) \ge \frac{2}{z} \right\}$$

and

$$\widetilde{A} := \left\{ \sup_{0\le t\le T-1} \sup_{0\le s\le 1} |Y(t+s) - Y(t)| < z \right\}.$$

Then $A \subset \widetilde{A}$, since if A occurs and $t < 1$, $t + s \le 1$, then

$$|Y(t+s) - Y(t)| \le 2 \sup_{0\le s\le 1} |Y(s)| \le \frac{z}{2} < z.$$

If A occurs and $t < 1$, $s \le 1$, $1 < t + s \le T$, then

$$|Y(t+s) - Y(t)| \le Y(t+s) - Y(1) + |Y(t) - Y(1)| \le \int_1^{t+s} \frac{\mathrm{d}u}{W(u)} + \frac{z}{2} < z.$$

Moreover, if A occurs and $1 \le t$, $s \le 1$, $t + s \le T$, then

$$|Y(t+s) - Y(t)| = \int_t^{t+s} \frac{\mathrm{d}u}{W(u)} \le \frac{z}{2} < z.$$

Hence $A \subset \widetilde{A}$ as claimed. But by the Markov property of W,

$$\mathbb{P}(A) = \int_{4/z}^{\infty} \mathbb{P}\left(\sup_{0\le s\le 1} |Y(s)| < \frac{z}{4} \,\middle|\, W(1) = x \right) \mathbb{P}\left(\inf_{1\le u\le T} W(u) \ge \frac{2}{z} \,\middle|\, W(1) = x \right) \varphi(x)\,\mathrm{d}x, \tag{3.8}$$

where φ denotes the standard normal density function.

Using reflection principle and $x \ge 4/z$, $z \le 1/2$, we get

$$\begin{aligned} \mathbb{P}\left(\inf_{1\le u\le T} W(u) \ge \frac{2}{z} \,\middle|\, W(1) = x \right) &= 2\Phi\left(\frac{x - 2/z}{\sqrt{T-1}} \right) - 1 \\ \ge 2\Phi\left(\frac{2}{z\sqrt{T-1}} \right) - 1 \ge 2\Phi\left(\frac{4}{\sqrt{T}} \right) - 1 &\ge \frac{c_{12}}{\sqrt{T}}, \end{aligned} \tag{3.9}$$

with some constant $c > 0$, where $\Phi(\cdot)$ is the standard normal distribution function. Hence

$$\mathbb{P}(\widetilde{A}) \ge \mathbb{P}(A) \ge \frac{c_{12}}{\sqrt{T}} \mathbb{P}\left(\sup_{0\le s\le 1} |Y(s)| \le \frac{z}{4},\ W(1) \ge \frac{4}{z} \right). \tag{3.10}$$

To get a lower bound of the probability on the right-hand side, define g, $(m(v), 0 \le v \le 1)$, $(B(u), 0 \le u \le 1)$ by (2.1), (2.2) and (2.3), respectively. Recall (see Fact 2.1) that these three objects are independent, g has arc sine distribution, m is a Brownian meander and B is a Brownian bridge. Moreover, (g, m, B) are independent of $\texttt{sgn}(W(1))$ which is a Bernoulli variable. Observe that

$$\sup_{0\le s\le g} |Y(s)| = \sqrt{g} \sup_{0\le s\le 1} \left| \int_0^s \frac{\mathrm{d}u}{B(u)} \right|,$$
$$\sup_{g\le s\le 1} |Y(s)| = |Y(1) - Y(g)| = \sqrt{1-g} \int_0^1 \frac{\mathrm{d}v}{m(v)},$$
$$|W(1)| = \sqrt{1-g}\, m(1).$$

Then

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le s\le 1} |Y(s)| \le \frac{z}{4},\, W(1) \ge \frac{4}{z} \right) \\
&\ge \mathbb{P}\left(\sup_{0\le s\le g} |Y(s)| \le \frac{z}{8},\, Y(1) - Y(g) \le \frac{z}{8},\, W(1) \ge \frac{4}{z} \right) \\
&\ge \mathbb{P}\left(\sqrt{g} \sup_{0\le s\le 1} \left| \int_0^s \frac{\mathrm{d}u}{B(u)} \right| \le \frac{z}{8},\, \sqrt{1-g} \int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8},\, \sqrt{1-g}\, m(1) \ge \frac{4}{z},\, W(1) > 0,\, g < z^2 \right) \\
&\ge \mathbb{P}\left(\sup_{0\le s\le 1} \left| \int_0^s \frac{\mathrm{d}u}{B(u)} \right| \le \frac{1}{8},\, \int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8},\, m(1) \ge \frac{4}{z\sqrt{1-z^2}},\, W(1) > 0,\, g < z^2 \right) \\
&= \mathbb{P}\left(\sup_{0\le s\le 1} \left| \int_0^s \frac{\mathrm{d}u}{B(u)} \right| \le \frac{1}{8} \right) \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8},\, m(1) \ge \frac{4}{z\sqrt{1-z^2}} \right) \mathbb{P}(W(1) > 0) \mathbb{P}(g < z^2) \\
&\ge c_{13} z \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8},\, m(1) \ge \frac{4}{z\sqrt{1-z^2}} \right) \\
&= c_{13} z \int_{4/(z\sqrt{1-z^2})}^{\infty} \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8} \,\Big|\, m(1) = x \right) \mathbb{P}(m(1) \in \mathrm{d}x).
\end{aligned}$$

It follows from Facts 2.1 and 2.2 that for $x > 0$, $z > 0$

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8} \,\Big|\, m(1) = x \right) \ge \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} \le \frac{z}{8} \,\Big|\, m(1) = 0 \right) \ge \frac{c_{14}}{z^3} \exp\left(-\frac{c_{15}}{z^2} \right) \tag{3.11}$$

and

$$\mathbb{P}\left(m(1) > \frac{4}{z\sqrt{1-z^2}} \right) = \exp\left(-\frac{8}{z^2(1-z^2)} \right). \tag{3.12}$$

Putting (3.10), (3.11), (3.12) together, we get (3.7). □

Lemma 3.5. *For $T > 1$, $0 < z \le 1/2$, $0 < \delta \le 1/2$ we have*

$$\begin{aligned}
&\mathbb{P}\left(\inf_{0\le t\le T-1} \sup_{0\le s\le 1} |Y(t+s) - Y(t)| < z \right) \\
&\le c_{16} \left(\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2 z^2 T} \right) + \exp\left(-\frac{c_5 \delta}{4(1+\delta)^2 z^2} \right) + \exp\left(\frac{c_{17}}{z^2} - \frac{c_{18} z^2}{T} e^{c_{19}/z^2} \right) \right)
\end{aligned} \tag{3.13}$$

with some positive constants c_{16}, $c_{17} = c_{17}(\delta)$, $c_{18} = c_{18}(\delta)$, $c_{19} = c_{19}(\delta)$.

Proof. Consider a positive integer N to be given later, $h = (T-1)/N$, $t_k = kh$, $k = 0, 1, 2, \dots, N$. Then for $0 < \delta \le 1/2$ we have

$$\begin{aligned}
&\mathbb{P}\left(\inf_{0\le t\le T-1}\sup_{0\le s\le 1}|Y(t+s)-Y(t)|<z\right)\\
&\le \mathbb{P}\left(\inf_{0\le k\le N}\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)|\le (1+\delta)z\right)+\mathbb{P}\left(\sup_{0\le t\le T-1}\sup_{0\le s\le h}|Y(t+s)-Y(t)|>\delta z\right)\\
&=: P_1+P_2.
\end{aligned}$$

By scaling and Lemma 3.1

$$\begin{aligned}
P_2 &= \mathbb{P}\left(\sup_{0\le t\le (T-1)/h}\sup_{0\le s\le 1}|Y(t+s)-Y(t)|>\frac{\delta z}{\sqrt{h}}\right)\\
&\le c_6\left(\sqrt{\frac{T-1}{h}+1}\,\exp\left(-\frac{\delta^2 z^2}{8h(1+\delta)}\right)+\left(\frac{T-1}{h}+1\right)\exp\left(-\frac{\delta^2 z^2}{2h(1+\delta)}\right)\right)\\
&\le 2c_6(N+1)\exp\left(-\frac{\delta^2 z^2}{8h(1+\delta)}\right).
\end{aligned}$$

To bound P_1, we denote by $d(t) := \inf\{s \ge t : W(s) = 0\}$ the first zero of W after t. Consider those k for which $\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z$. If, moreover, $d(t_k) \ge t_k + 1 - \delta$, which means that the Brownian motion W does not change sign over $[t_k, t_k+1-\delta)$, then

$$(1+\delta)z \ge |Y(t_k+1-\delta)-Y(t_k)| = \int_0^{1-\delta}\frac{\mathrm{d}s}{|W(t_k+s)|} \ge \frac{1-\delta}{\sup_{0\le s\le T}|W(s)|},$$

and it follows that

$$\begin{aligned}
P_1 &\le \mathbb{P}\left(\sup_{0\le s\le T}|W(s)|>\frac{(1-\delta)}{z(1+\delta)}\right)\\
&+\mathbb{P}\left(\exists k\le N:\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)|\le(1+\delta)z; d(t_k)<t_k+1-\delta\right)\\
&\le 4\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2 z^2 T}\right)\\
&+\sum_{k=0}^{N}\mathbb{P}\left(\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)|\le(1+\delta)z; d(t_k)<t_k+1-\delta\right).
\end{aligned}$$

Let $\widehat{W}(s) = W(d(t_k)+s)$ for $s \ge 0$ and $\widehat{Y}(s)$ be the associated principal values. Observe that on $\{\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z; d(t_k) < t_k+1-\delta\}$, we have $\sup_{0\le u\le\delta}|\widehat{Y}(u) + (Y(d(t_k)) - Y(t_k))| < (1+\delta)z$, and $|Y(d(t_k)) - Y(t_k)| \le (1+\delta)z$ which implies that

$$\sup_{0\le u\le\delta}|\widehat{Y}(u)| < 2(1+\delta)z.$$

By scaling and Fact 2.3 we have

$$\mathbb{P}\left(\sup_{0\le u\le\delta}|\widehat{Y}(u)|<2(1+\delta)z\right)\le c_4\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right).$$

Therefore, we obtain:

$$P_1\le 4\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+c_4(N+1)\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right).$$

Hence

$$P_1+P_2\le 4\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+c_4(N+1)\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right)$$
$$+2c_6(N+1)\exp\left(-\frac{\delta^2z^2}{8h(1+\delta)}\right).$$

By taking $N=[e^{c_5\delta/(4(1+\delta)^2z^2)}]+1$, we get

$$P_1+P_2$$
$$\le c_{16}\left(\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right)+\exp\left(\frac{c_{17}}{z^2}-\frac{c_{18}z^2}{T}e^{c_{19}/z^2}\right)\right)$$

with relevant constants c_{16}, c_{17}, c_{18}, c_{19}, proving (3.13). □

4. Proof of Theorem 1.1(i)

The upper estimation, i.e.

$$\limsup_{T\to\infty}\frac{\sup_{0\le t\le T-a_T}\ \sup_{0\le s\le a_T}\ |Y(t+s)-Y(t)|}{\sqrt{8a_T\left(\log\sqrt{T/a_T}+\log\log T\right)}}\le 1,\qquad \text{a.s.}\tag{4.1}$$

follows easily from Wen's Theorem E.

Now we prove the lower bound, i.e.

$$\limsup_{T\to\infty}\frac{\sup_{0\le t\le T-a_T}\ \sup_{0\le s\le a_T}\ |Y(t+s)-Y(t)|}{\sqrt{8a_T\left(\log\sqrt{T/a_T}+\log\log T\right)}}\ge 1,\qquad \text{a.s.}\tag{4.2}$$

In the case when $a_T=T$, (4.2) follows from the law of the iterated logarithm (1.3) of Theorem A. Now we assume that $a_T/T\le\rho<1$, with some constant ρ for all $T>0$.

By scaling, (3.2) of Lemma 3.2 is equivalent to

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T-a}(Y(t+a)-Y(t))\ge z\sqrt{a}\right)\\ &\qquad\ge\min\left(\frac{1}{2},\frac{c_7\sqrt{T/a-1}}{z}\exp\left(-\frac{z^2}{8(1-\delta)}\right)\right)-\exp\left(-z^2\right)\end{aligned}\tag{4.3}$$

for $0 < a < T$, $0 < \delta < 1/2$, $z > 1$.

Define the sequences

$$t_k := e^{7k \log k}, \qquad k = 1, 2, \dots \tag{4.4}$$

and $\theta_0 := 0$,

$$\theta_k := \inf\{t > T_k : W(t) = 0\}, \qquad k = 1, 2, \dots, \tag{4.5}$$

where $T_k := \theta_{k-1} + t_k$. For $0 < \delta < \min(1/2, 1-\rho)$ define the events

$$A_k := \left\{ \sup_{0 \le t \le t_k(1-\delta) - a_{t_k}} (Y(\theta_{k-1} + t + a_{t_k}) - Y(\theta_{k-1} + t)) \ge (1-\delta)\beta_k \right\}, \quad k = 1, 2, \dots$$

with

$$\beta_k := \sqrt{8 a_{t_k} \left(\log \sqrt{\frac{t_k}{a_{t_k}}} + \log\log t_k \right)}.$$

Applying (4.3) with $T = t_k(1-\delta)$, $a = a_{t_k}$, $z = (1-\delta)\sqrt{8(\log \sqrt{t_k/a_{t_k}} + \log\log t_k)}$, we have for k large

$$\begin{aligned} \mathbb{P}(A_k) &= \mathbb{P}\left(\sup_{0 \le t \le t_k(1-\delta) - a_{t_k}} (Y(t + a_{t_k}) - Y(t)) \ge (1-\delta)\beta_k \right) \\ &\ge \min\left(\frac{1}{2}, \frac{b_k}{(\log t_k)^{1-\delta}} \right) - \frac{1}{(\log t_k)^{8(1-\delta)^2}} \end{aligned}$$

with

$$b_k = \frac{c_7 \sqrt{t_k(1-\delta)/a_{t_k} - 1}}{(t_k/a_{t_k})^{(1-\delta)/2} \sqrt{\log \sqrt{t_k/a_{t_k}} + \log\log t_k}} \ge \frac{c_{20}}{\sqrt{\log k}}.$$

Hence $\sum_k \mathbb{P}(A_k) = \infty$ and since A_k are independent, Borel-Cantelli lemma yields

$$\mathbb{P}(A_k \text{ i.o.}) = 1.$$

It follows that

$$\limsup_{k \to \infty} \frac{\sup_{0 \le t \le t_k(1-\delta) - a_{t_k}} (Y(\theta_{k-1} + t + a_{t_k}) - Y(\theta_{k-1} + t))}{\sqrt{8 a_{t_k} \left(\log \sqrt{\frac{t_k}{a_{t_k}}} + \log\log t_k \right)}} \ge 1 - \delta, \qquad \text{a.s.} \tag{4.6}$$

It can be seen (cf. [9]) that we have almost surely for large enough k

$$t_k \le T_k \le t_k \left(1 + \frac{1}{k} \right),$$

consequently

$$\lim_{k\to\infty}\frac{t_k}{T_k}=1, \qquad \text{a.s.} \tag{4.7}$$

Since by our assumptions

$$\frac{t_k}{T_k}\le\frac{a_{t_k}}{a_{T_k}}\le 1,$$

we have also

$$\lim_{k\to\infty}\frac{a_{t_k}}{a_{T_k}}=1, \qquad \text{a.s.} \tag{4.8}$$

On the other hand, for any $\delta>0$ small enough we have almost surely for large k

$$a_{T_k}\le(1+\delta)a_{t_k}\le t_k\delta+a_{t_k},$$

thus

$$T_k-a_{T_k}\ge T_k-t_k\delta-a_{t_k},$$

consequently

$$\begin{aligned}&\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|\\ &\ge\sup_{0\le t\le t_k(1-\delta)-a_{t_k}}(Y(\theta_{k-1}+t+a_{t_k})-Y(\theta_{k-1}+t)),\end{aligned} \tag{4.9}$$

hence we have also

$$\limsup_{k\to\infty}\frac{\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|}{\sqrt{8a_{t_k}\left(\log\sqrt{\frac{t_k}{a_{t_k}}}+\log\log t_k\right)}}\ge 1-\delta, \qquad \text{a.s.} \tag{4.10}$$

and since $\delta>0$ can be arbitrary small, (4.2) follows by combining (4.7), (4.8), (4.9) and (4.10). □

5. Proof of Theorem 1.1(ii)

First assume that

$$a_T>\frac{T}{(\log T)^\alpha} \qquad \text{for some} \quad \alpha<2. \tag{5.1}$$

By Theorem C,

$$\begin{aligned}&\liminf_{T\to\infty}\sqrt{\frac{\log\log T}{a_T}}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|\\ &\ge\liminf_{T\to\infty}\sqrt{\frac{\log\log a_T}{a_T}}\sup_{0\le s\le a_T}|Y(s)|\ge K_1, \qquad \text{a.s.,}\end{aligned} \tag{5.2}$$

proving the lower bound in (1.12).

To get an upper bound, note that by scaling, (3.7) of Lemma 3.4 is equivalent to

$$\mathbb{P}\left(\sup_{0\le t\le T-a}\sup_{0\le s\le a}|Y(s+t)-Y(t)|<z\sqrt{a}\right)\ge c_{10}\sqrt{\frac{a}{T}}\exp\left(-\frac{c_{11}}{z^2}\right) \tag{5.3}$$

for $T\ge a$, $0<z\le 1/2$.

Let t_k and θ_k be defined by (4.4) and (4.5), resp., as in the proof of Theorem 1.1(i) and for any $\varepsilon>0$ and for $\delta>0$ such that $\alpha/2+c_{11}/\delta^2<1$, define the events

$$E_k:=\left\{\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(\theta_{k-1}+t+s)-Y(\theta_{k-1}+t)|\le\delta\sqrt{\frac{a_{t_k}}{\log\log t_k}}\right\}.$$

Then putting $T=(1+\varepsilon)t_k$, $a=a_{(1+\varepsilon)t_k}$, $z=\delta/\sqrt{\log\log t_k}$, into (5.3), we get

$$\begin{aligned}\mathbb{P}(E_k)&=\mathbb{P}\left(\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(t+s)-Y(t)|\le\delta\sqrt{\frac{a_{t_k}}{\log\log t_k}}\right)\\&\ge c_{10}\sqrt{\frac{a_{t_k}}{t_k}}\exp(-(c_{11}/\delta^2)\log\log((1+\varepsilon)t_k))\ge\frac{c_{10}}{(\log t_k)^{\alpha/2+c_{11}/\delta^2}},\end{aligned}$$

hence $\sum_k\mathbb{P}(E_k)=\infty$, and since E_k are independent, we have $\mathbb{P}(E_k\text{ i.o.})=1$, i.e.

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log t_k}{a_{t_k}}}\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(\theta_{k-1}+t+s)-Y(\theta_{k-1}+t)|\le\delta,\quad\text{a.s.} \tag{5.4}$$

for any ε. Put, as before, $T_k=\theta_{k-1}+t_k$. For large enough k by (4.7) and (4.8) we have $a_{T_k}\le(1+\varepsilon)a_{t_k}$, a.s. and $T_k-a_{T_k}\le\theta_{k-1}+(1+\varepsilon)t_k-(1+\varepsilon)a_{t_k}$, a.s. Thus given any $\varepsilon>0$, we have for large k

$$\begin{aligned}&\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|\\&\le 2\sup_{0\le t\le\theta_{k-1}}|Y(t)|+\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(\theta_{k-1}+t+s)-Y(\theta_{k-1}+t)|.\end{aligned} \tag{5.5}$$

By Theorem A, Fact 2.8, (4.7), (5.1) and simple calculation,

$$\begin{aligned}&\sup_{0\le t\le\theta_{k-1}}|Y(t)|=O(\theta_{k-1}\log\log\theta_{k-1})^{1/2}\\&=O(t_{k-1}(\log t_{k-1})^3\log\log t_{k-1})^{1/2}=o\left(\frac{a_{t_k}}{\log\log t_k}\right)^{1/2},\qquad\text{a.s.}\end{aligned} \tag{5.6}$$

as $k\to\infty$. Assembling (5.4), (5.5) and (5.6), we get

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log t_k}{a_{t_k}}}\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|$$

$$= \liminf_{k\to\infty} \sqrt{\frac{\log\log T_k}{a_{T_k}}} \sup_{0\le t\le T_k-a_{T_k}} \sup_{0\le s\le a_{T_k}} |Y(t+s)-Y(t)| \le \delta, \qquad \text{a.s.}$$

which together with (5.2) yields (1.12).

Now assume that

$$a_T \le \frac{T}{(\log T)^\alpha} \qquad \text{for some} \quad \alpha > 2. \tag{5.7}$$

By Theorem 1.1(i),

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|}{\sqrt{a_T\log(T/a_T)}}\\
&\le \limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|}{\sqrt{a_T\log(T/a_T)}}\\
&\le \limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|}{\sqrt{\frac{2\alpha a_T}{\alpha+2}\left(\log\sqrt{T/a_T}+\log\log T\right)}} \le 2\sqrt{\frac{\alpha+2}{\alpha}},
\end{aligned} \tag{5.8}$$

i.e., an upper bound in (1.13) follows.

To get a lower bound under (5.7), observe that by scaling, (3.6) of Lemma 3.3 is equivalent to

$$\mathbb{P}\left(\sup_{0\le t\le T-a}(Y(t+a)-Y(t)) < z\sqrt{a}\right) \le 5\left(\frac{a}{T}\right)^{\kappa/2} + \exp\left(-c_9\left(\frac{T}{a}\right)^{(1-\kappa)/2} e^{-(1+\delta)z^2/8}\right)$$

for $a \le T$, $0\le\kappa<1$, $0<\delta$, $0<z$. Using (5.7) we get further

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T-a}(Y(t+a)-Y(t)) < z\sqrt{a}\right)\\
&\le \frac{5}{(\log T)^{\alpha\kappa/2}} + \exp\left(-c_9\left(\log T\right)^{\alpha(1-\kappa)/2} e^{-(1+\delta)z^2/8}\right).
\end{aligned} \tag{5.9}$$

In the case when (1.7) holds, (1.13) was proved in [7]. In other cases the proof is similar. Let $T_k = e^k$ and define the events

$$F_k = \left\{\sup_{0\le t\le T_k-a_{T_k}}(Y(t+a_{T_k})-Y(t)) \le C_1\sqrt{a_{T_k}\log\frac{T_k}{a_{T_k}}}\right\}$$

with some constant C_1 to be given later. By (5.9)

$$\mathbb{P}(F_k) \le \frac{5}{k^{\alpha\kappa/2}} + \exp\left(-c_9 k^{\alpha((1-\kappa)/2-(1+\delta)C_1^2/8)}\right).$$

For given $\alpha>2$, choose small $\varepsilon>0$, $\kappa = 2/\alpha+\varepsilon$,

$$C_1 = 2\sqrt{\frac{\alpha-2-2\varepsilon(1+\alpha)}{(1+\varepsilon)\alpha}}.$$

One can easily see that with these choices $\sum_k \mathbb{P}(F_k) < \infty$, consequently

$$\liminf_{k\to\infty} \frac{\sup_{0\le t\le T_k - a_{T_k}} (Y(t+a_{T_k}) - Y(t))}{\sqrt{a_{T_k} \log \frac{T_k}{a_{T_k}}}} \ge C_1, \qquad \text{a.s.,}$$

implying also

$$\liminf_{k\to\infty} \frac{\sup_{0\le t\le T_k - a_{T_k}} \sup_{0\le s\le a_{T_k}} |Y(t+s) - Y(t)|}{\sqrt{a_{T_k} \log \frac{T_k}{a_{T_k}}}} \ge 2\sqrt{\frac{\alpha-2}{\alpha}}, \qquad \text{a.s.,}$$

for ε can be choosen arbitrary small.

Since $\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|$ is increasing in T, we obtain a lower bound in (1.13). This together with the 0-1 law for Brownian motion complete the proof of Theorem 1.1(ii). □

6. Proof of Theorem 1.2(i)

If $a_T = T$, then (1.14) is equivalent to Theorem C. Now assume that $\rho := \lim_{T\to\infty} a_T/T < 1$.

First we prove the lower bound, i.e.

$$\liminf_{T\to\infty} \frac{\sqrt{T \log\log T}}{a_T} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)| \ge c, \qquad \text{a.s.} \tag{6.1}$$

By scaling, (3.13) of Lemma 3.5 is equivalent to

$$\begin{aligned} &\mathbb{P}\left(\inf_{0\le t\le T-a} \sup_{0\le s\le a} |Y(t+s) - Y(t)| < z \right) \\ &\le c_{16} \left(\exp\left(-\frac{a(1-\delta)^2}{2(1+\delta)^2 z^2 T} \right) + \exp\left(-\frac{c_5 \delta}{4(1+\delta)^2 z^2} \right) + \exp\left(\frac{c_{17}}{z^2} - \frac{c_{18} a z^2}{T} e^{c_{19}/z^2} \right) \right) \end{aligned} \tag{6.2}$$

for $a < T$, $0 < z \le 1/2$, $0 < \delta \le 1/2$.

Define the events

$$G_k = \left\{ \inf_{0\le t\le T_{k+1} - a_{T_k}} \sup_{0\le s\le a_{T_k}} |Y(t+s) - Y(t)| < z_k \right\} \quad k = 1, 2, \ldots$$

Let $T_k = e^k$ and put $T = T_{k+1}$, $a = a_{T_k}$,

$$z = z_k = C_2 \sqrt{\frac{a_{T_k}}{T_{k+1} \log\log T_{k+1}}}$$

into (6.2). The constant C_2 will be choosen later. Denoting the terms on the right-hand side of (6.2) by I_1, I_2, I_3, resp., we have

$$\mathbb{P}(G_k) \le c_{16}(I_1^{(k)} + I_2^{(k)} + I_3^{(k)}),$$

where

$$I_1^{(k)} = \exp\left(-\frac{c_{21}}{C_2^2}\log\log T_{k+1}\right),$$

$$I_2^{(k)} = \exp\left(-\frac{c_{22}T_k}{C_2^2 a_{T_k}}\log\log T_{k+1}\right),$$

$$I_3^{(k)} = \exp\left(\frac{c_{23}T_k\log\log T_{k+1}}{C_2^2 a_{T_k}} - \frac{c_{24}C_2^2 a_{T_k}^2}{T_k^2\log\log T_{k+1}}(\log T_{k+1})^{\frac{c_{25}T_k}{C_2^2 a_{T_k}}}\right)$$

with some constants $c_{21} = c_{21}(\delta)$, $c_{22} = c_{22}(\delta)$, c_{23}, c_{24}, c_{25}.

One can see easily that for any choice of positive C_2 and for all possible a_T (satisfying our conditions) we have $\sum_k I_3^{(k)} < \infty$. So we show that for appropriate choice of C_2 we have also $\sum_k I_j^{(k)} < \infty$, $j = 1, 2$.

First consider the case $0 < \rho > 0$. Choosing a positive δ one can select $C_2 < \min(\sqrt{c_{21}}, \sqrt{\frac{c_{22}}{\rho}})$ and it is easy to verify that $\sum_k I_j^{(k)} < \infty$, $j = 1, 2$, hence also $\sum_k \mathbb{P}(G_k) < \infty$.

In the case $\rho = 0$ choose $C_2 < (1-\delta)/((1+\delta)\sqrt{2})$. With this choice we have $\sum_k I_1^{(k)} < \infty$ for arbitrary $\delta > 0$. Since $\lim_{k\to\infty}(T_k/a_{T_k}) = \infty$, we have also $\sum_k I_2^{(k)} < \infty$ and $\sum_k \mathbb{P}(G_k) < \infty$. Borell-Cantelli lemma and interpolation between T_k's finish the proof of (6.1). We have also verified that in the case $\rho = 0$ one can choose $C_2 = 1/\sqrt{2}$, since δ can be choosen arbitrary small.

Now we turn to the proof of the upper bound, i.e.

$$\liminf_{T\to\infty} \frac{\sqrt{T\log\log T}}{a_T} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)| \le C_3, \qquad \text{a.s.} \tag{6.3}$$

with some constant C_3.

If $\rho > 0$, then

$$\inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)| \le \sup_{0\le s\le a_T} |Y(s)| \le \sup_{0\le s\le T} |Y(s)|$$

and hence (6.3) with some positive constant C_3 follows from Theorem C.

If $\rho = 0$, then let for any $\varepsilon > 0$

$$\lambda_T := \inf\{t : |W(t)| = \sup_{0\le s\le T(1-\varepsilon)} |W(s)|\}. \tag{6.4}$$

According to the law of the iterated logarithm, with probability one there exists a sequence $\{T_i,\ i \ge 1\}$ such that $\lim_{i\to\infty} T_i = \infty$ and

$$|W(\lambda_{T_i})| \ge \sqrt{2T_i(1-\varepsilon)\log\log T_i}. \tag{6.5}$$

But Fact 2.4 implies that for $\varepsilon > 0$

$$|W(\lambda_{T_i}) - W(s)| \le \sqrt{2(1+\varepsilon)\varepsilon T_i \log\log T_i}, \quad \lambda_{T_i} \le s \le \lambda_{T_i} + \varepsilon T_i, \quad i \ge 1. \tag{6.6}$$

Now assume that $W(\lambda_{T_i}) > 0$. The case when $W(\lambda_{T_i}) < 0$ is similar. Then (6.5) and (6.6) imply

$$W(s) \ge \left(\sqrt{1-\varepsilon} - \sqrt{\varepsilon(1+\varepsilon)}\right)\sqrt{2T_i \log\log T_i}, \quad \lambda_{T_i} \le s \le \lambda_{T_i} + \varepsilon T_i. \tag{6.7}$$

$\rho = 0$ implies that $a_T \le \varepsilon T$ for any $\varepsilon > 0$ and large enough T, hence we have from (6.7) for large i

$$\sup_{0\le s\le a_{T_i}} (Y(\lambda_{T_i}+s) - Y(\lambda_{T_i})) = Y(\lambda_{T_i} + a_{T_i}) - Y(\lambda_{T_i}) = \int_{\lambda_{T_i}}^{\lambda_{T_i}+a_{T_i}} \frac{\mathrm{d}s}{W(s)}$$

$$\le \frac{a_{T_i}}{\left(\sqrt{1-\varepsilon} - \sqrt{\varepsilon(1+\varepsilon)}\right)\sqrt{2T_i \log\log T_i}}.$$

Since $\varepsilon > 0$ is arbitrary, (6.3) follows with $C_3 = 1/\sqrt{2}$. This completes the proof of Theorem 1.2(i).
□

7. Proof of Theorem 1.2(ii)

If $\rho = 1$, then (1.15) is equivalent to (1.3) of Theorem A. So we may assume that $0 < \rho < 1$.

First we prove the upper bound

$$\limsup_{T\to\infty} \frac{\inf_{0\le t\le T-\rho T} \sup_{0\le s\le \rho T} |Y(t+s) - Y(t)|}{\sqrt{8T\log\log T}} \le \rho, \qquad \text{a.s.} \tag{7.1}$$

Let k be the largest integer for which $k\rho < 1$ and put $x_i = i\rho$, $i = 0, 1, \ldots, k$, $x_{k+1} = 1$. It suffices to show that if $f \in \mathcal{S}$ defined by (1.5), then

$$\min_{1\le i\le k+1} |f(x_i) - f(x_{i-1})| \le \rho.$$

Assume on the contrary that

$$|f(x_i) - f(x_{i-1})| > \rho, \qquad \forall i = 1, 2, \ldots, k+1.$$

Then

$$\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} > \sum_{i=1}^{k} \frac{\rho^2}{\rho} + \frac{\rho^2}{1-k\rho} = k\rho + \frac{\rho^2}{1-k\rho} \ge 1,$$

contradicting (2.12) of Fact 2.5. This proves (7.1).

The lower bound

$$\limsup_{T\to\infty} \frac{\inf_{0\le t\le T-\rho T}\ \sup_{0\le s\le \rho T}\ |Y(t+s)-Y(t)|}{\sqrt{8T\log\log T}} \ge \rho, \qquad \text{a.s.} \tag{7.2}$$

follows from the fact that by Theorem B the function $f(x)=x$, $0\le x\le 1$ is a limit point of

$$\frac{Y(xt)}{\sqrt{8T\log\log T}}$$

and for this function

$$\min_{0\le x\le 1-\rho} |f(x+\rho)-f(x)| = \rho.$$

This completes the proof of Theorem 1.2(iia). □

Now assume that

$$\lim_{T\to\infty} \frac{a_T(\log\log T)^2}{T} = 0. \tag{7.3}$$

Define λ_T as in (6.4). Then according to Chung's LIL (cf. Fact 2.6)

$$|W(\lambda_T)| \ge \frac{\pi}{\sqrt{8}}(1-\varepsilon)\sqrt{\frac{T}{\log\log T}} \tag{7.4}$$

for every T sufficiently large. But according to Fact 2.4,

$$\sup_{0\le s\le a_T} |W(\lambda_T+s)-W(\lambda_T)|$$

$$\le \sqrt{(2+\varepsilon)a_T(\log(T/a_T)+\log\log T)} \le \sqrt{\frac{(2+\varepsilon)\varepsilon T}{\log\log T}}.$$

Assuming $W(\lambda_T)>0$, we get

$$W(\lambda_T+s) \ge W(\lambda_T) - \sqrt{\frac{(2+\varepsilon)\varepsilon T}{\log\log T}} \ge c\sqrt{\frac{T}{\log\log T}}.$$

Hence

$$\inf_{0\le t\le T-a_T}\ \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| \le Y(\lambda_T+a_T)-Y(\lambda_T)$$

$$= \int_0^{a_T} \frac{\mathrm{d}s}{W(\lambda_T+s)} \le \frac{a_T}{c}\sqrt{\frac{\log\log T}{T}}$$

for all large T.

The case when $W(\lambda_T)<0$ is similar. This shows the upper bound in (1.16).

For the lower bound we use Fact 2.6: with probability one

$$g_T \le \frac{T}{(\log\log T)^2}, \quad \max_{0\le u\le T} |W(u)| \le \frac{\pi}{\sqrt{2}}\sqrt{\frac{T}{\log\log T}} \quad \text{i.o.} \tag{7.5}$$

According to Theorem 1.2(i) for every large T we have for any $\varepsilon > 0$ and sufficiently large T

$$\begin{aligned} &\inf_{0\le t\le T(\log\log T)^{-2}} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| \\ &\ge \frac{(K_4-\varepsilon)a_T}{\sqrt{\left(\frac{T}{(\log\log T)^2}+a_T\right)\log\log T}} \le \frac{(K_4-\varepsilon)a_T}{\sqrt{(1+\varepsilon)T\log\log T}}. \end{aligned} \tag{7.6}$$

On the other hand, if $T(\log\log T)^{-2} \le t \le T - a_T$, then by (7.5)

$$|Y(t+a_T)-Y(t)| = \int_t^{t+a_T} \frac{\mathrm{d}s}{|W(s)|} \ge \frac{a_T\sqrt{2\log\log T}}{\pi\sqrt{T}}.$$

Combining (7.6) and (7.7) we get for $\varepsilon > 0$ and all large T

$$\inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| \ge \min\left(\frac{K_4-\varepsilon}{\sqrt{1+\varepsilon}}, \frac{\sqrt{2}}{\pi}\right) \frac{a_T\sqrt{\log\log T}}{T}.$$

This shows the lower bound in (1.16). The proof of Theorem 1.2(iib) is complete by applying the 0-1 law for Brownian motion. □

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