On the local time of random walk on the 2-dimensional comb

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Abstract

We study the path behaviour of general random walks, and that of their local times, on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the x-axis. We prove strong approximation results for such random walks and also for their local times. Concentrating mainly on the latter, we establish strong and weak limit theorems, including Strassen-type laws of the iterated logarithm, Hirsch-type laws, and weak convergence results in terms of functional convergence in distribution.

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1 Introduction and main results

In this paper we continue our study of a simple random walk $\mathbf{C}(n)$ on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal lines off the x-axis (cf. Csáki *et al.* [16]).

A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice $\mathbb{C}^2$ can be formulated via its transition probabilities as follows: for $(x,y) \in \mathbb{Z}^2$

$$\mathbf{P}(\mathbf{C}(n+1) = (x, y \pm 1) \mid \mathbf{C}(n) = (x,y)) = \frac{1}{2}, \quad \text{if } y \neq 0, \tag{1.1}$$

$$\mathbf{P}(\mathbf{C}(n+1) = (x \pm 1, 0) \mid \mathbf{C}(n) = (x,0)) = \mathbf{P}(\mathbf{C}(n+1) = (x, \pm 1) \mid \mathbf{C}(n) = (x,0)) = \frac{1}{4}. \tag{1.2}$$

The coordinates of the just defined vector valued simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ are denoted by $C_1(n), C_2(n)$, i.e., $\mathbf{C}(n) := (C_1(n), C_2(n))$.

A compact way of describing the just introduced transition probabilities for this simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ is via defining

$$p(\mathbf{u}, \mathbf{v}) := \mathbf{P}(\mathbf{C}(n+1) = \mathbf{v} \mid \mathbf{C}(n) = \mathbf{u}) = \frac{1}{\deg(\mathbf{u})}, \tag{1.3}$$

for locations $\mathbf{u}$ and $\mathbf{v}$ that are neighbors on $\mathbb{C}^2$, where $\deg(\mathbf{u})$ is the number of neighbors of $\mathbf{u}$, otherwise $p(\mathbf{u}, \mathbf{v}) := 0$. Consequently, the non-zero transition probabilities are equal to $1/4$ if $\mathbf{u}$ is on the horizontal axis and they are equal to $1/2$ otherwise.

This and related models have been studied intensively in the literature and have a number of applications in various problems in physics. See, for example, Arkhincheev [1], [2], [3], Cassi and Regina [12], Dean and Jansons [23], Durhuus *et al.* [25], Reynolds [38], Zahran [43], Zahran *et al.* [44], and the references in these papers. It was observed that the second component $C_2(n)$ behaves like ordinary Brownian motion, but the first component $C_1(n)$ exhibits some anomalous subdiffusion property of order $n^{1/4}$. Zahran [43] and Zahran et al. [44] applied Fokker–Planck equation to describe the properties of comb-like model. Weiss and Havlin [42] derived the asymptotic form for the probability that $\mathbf{C}(n) = (x,y)$ by appealing to a central limit argument. Bertacchi and Zucca [8] obtained space-time asymptotic estimates for the n-step transition probabilities $p^{(n)}(\mathbf{u}, \mathbf{v}) := \mathbf{P}(\mathbf{C}(n) = \mathbf{v} \mid \mathbf{C}(0) = \mathbf{u})$, $n \geq 0$, from $\mathbf{u} \in \mathbb{C}^2$ to $\mathbf{v} \in \mathbb{C}^2$, when $\mathbf{u} = (2k, 0)$ or $(0, 2k)$ and $\mathbf{v} = (0,0)$. Using their estimates, they concluded that, if k/n goes to zero with a certain speed, then $p^{(2n)}((2k,0),(0,0))/p^{(2n)}((0,2k),(0,0)) \to 0$, as $n \to \infty$, an indication that suggests that the particle in this random walk spends most of its time on some tooth of the comb. Bertacchi [7] noted that a Brownian motion is the right object to approximate $C_2(\cdot)$, but for the first component $C_1(\cdot)$ the right object is a Brownian motion time-changed by the local time of the second component. More precisely, Bertacchi [7] on defining the continous time process $\mathbf{C}(nt) = (C_1(nt), C_2(nt))$ by linear interpolation, established the following remarkable joint weak convergence result.

Theorem A *For the* $\mathbb{R}^2$ *valued random elements* $\mathbf{C}(nt)$ *of* $C[0,\infty)$ *we have*

$$\left(\frac{C_1(nt)}{n^{1/4}}, \frac{C_2(nt)}{n^{1/2}}; t \geq 0\right) \xrightarrow{\text{Law}} (W_1(\eta_2(0,t)), W_2(t); t \geq 0), \quad n \to \infty, \tag{1.4}$$

where W_1, W_2 *are two independent Brownian motions and* $\eta_2(0,t)$ *is the local time process of* W_2 *at zero, and* $\xrightarrow{\text{Law}}$ *denotes weak convergence on* $C([0,\infty),\mathbb{R}^2)$ *endowed with the topology of uniform convergence on compact subsets.*

Here, and throughout as well, $C(I,\mathbb{R}^d)$, respectively $D(I,\mathbb{R}^d)$, stand for the space of $\mathbb{R}^d$-valued, $d = 1,2$, continuous, respectively *càdlàg*, functions defined on an interval $I \subseteq [0,\infty)$. $\mathbb{R}^1$ will throughout be denoted by $\mathbb{R}$.

In Csáki *et al.* [16] we established the corresponding strong approximation that reads as follows.

Recall that a standard Brownian motion $\{W(t),\, t \geq 0\}$ (called also Wiener process in the literature) is a mean zero Gaussian process with covariance $\mathbf{E}(W(t_1)W(t_2)) = \min(t_1,t_2)$. Its two-parameter local time process $\{\eta(x,t),\, x \in \mathbb{R},\, t \geq 0\}$ can be defined via

$$\int_A \eta(x,t)\,dx = \lambda\{s :\, 0 \leq s \leq t,\, W(s) \in A\} \tag{1.5}$$

for any $t \geq 0$ and Borel set $A \subset \mathbb{R}$, where $\lambda(\cdot)$ is the Lebesgue measure, and $\eta(\cdot,\cdot)$ is frequently referred to as Brownian local time.

Theorem B *On an appropriate probability space for the simple random walk* $\{\mathbf{C}(n) = (C_1(n), C_2(n)); n = 0,1,2,\ldots\}$ *on* $\mathbb{C}^2$*, one can construct two independent standard Brownian motions* $\{W_1(t);\, t \geq 0\}$*,* $\{W_2(t);\, t \geq 0\}$ *so that, as* $n \to \infty$*, we have with any* $\varepsilon > 0$

$$n^{-1/4}|C_1(n) - W_1(\eta_2(0,n))| + n^{-1/2}|C_2(n) - W_2(n)| = O(n^{-1/8+\varepsilon}) \quad a.s.,$$

where $\eta_2(0,\cdot)$ *is the local time process at zero of* $W_2(\cdot)$*.*

The strong approximation nature of Theorem B enabled us to establish some Strassen type almost sure set of limit points for the simple random walk $\mathbf{C}(n) = (C_1(n), C_2(n))$ on the 2-dimensional comb lattice, as well as the Hirsch type liminf behaviour (cf. Hirsch [29]) of its components.

Here we extend Theorem B for more general distributions along the horizontal and vertical directions. More precisely, let $X_j(n)$, $n = 1,2,...$, $j = 1,2$, be two independent sequences of i.i.d. integer valued random variables having disributions $\mathcal{P}_1 = \{p_1(k),\, k \in \mathbb{Z}\}$ and $\mathcal{P}_2 = \{p_2(k),\, k \in \mathbb{Z}\}$ respectively, satisfying the following conditions:

- (i) $\sum_{k=-\infty}^{\infty} kp_j(k) = 0, \quad j = 1,2,$
- (ii) $\sum_{k=-\infty}^{\infty} |k|^3 p_j(k) < \infty, \quad j = 1,2,$
- (iii) $\psi(\theta) := \sum_{k=-\infty}^{\infty} e^{i\theta k} p_j(k) = 1, \quad j = 1,2,$ if and only if θ is an integer multiple of 2π.

Remark 1 *Condition* (iii) *is equivalent to the aperiodicity of the random walks* $\{S_j(n) := \sum_{l=1}^n X_j(l); n = 1, 2, \ldots\}$, $j = 1, 2$ (*see* Spitzer [39], p. 67).

The local time process of a random walk $\{S(n) := \sum_{l=1}^n X(l);\ n = 0, 1, 2, \ldots\}$ with values on $\mathbb{Z}$, is defined by

$$\xi(k, n) := \#\{i :\ 1 \le i \le n,\ S(i) = k\}, \quad k \in \mathbb{Z},\ \ n = 1, 2, \ldots \tag{1.6}$$

Keeping our previous notation in the context of conditions (i), (ii) and (iii) as well, from now on $\mathbf{C}(n)$ will denote a random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ with the following transition probabilities:

$$\mathbf{P}(\mathbf{C}(n+1) = (x, y+k) \mid \mathbf{C}(n) = (x, y)) = p_2(k), \quad (x, y, k) \in \mathbb{Z}^3,\ y \neq 0, \tag{1.7}$$

$$\mathbf{P}(\mathbf{C}(n+1) = (x, k) \mid \mathbf{C}(n) = (x, 0)) = \frac{1}{2} p_2(k), \quad (x, k) \in \mathbb{Z}^2, \tag{1.8}$$

$$\mathbf{P}(\mathbf{C}(n+1) = (x+k, 0) \mid \mathbf{C}(n) = (x, 0)) = \frac{1}{2} p_1(k), \quad (x, k) \in \mathbb{Z}^2. \tag{1.9}$$

Unless otherwise stated, we assume that $\mathbf{C}(0) = \mathbf{0} = (0, 0)$.

Theorem 1.1 *Suppose that the conditions* (i) – (iii) *are met. Assume that, on an appropriate probability space for two independent random walks* $S_j(n) = \sum_{l=1}^n X_j(l)$ *with their respective local time processes* $\xi_j(\cdot, \cdot)$, $j = 1, 2$, *one can construct two independent Brownian motions* $\{W_j(t),\ t \ge 0\}$ *with their respective local time processes* $\eta_j(\cdot, \cdot)$, $j = 1, 2$ *such that for any* $\varepsilon > 0$

$$\lim_{n\to\infty} n^{-\alpha-\varepsilon} |S_j(n) - \sigma_j W_j(n)| = 0 \quad a.s. \tag{1.10}$$

and

$$\lim_{n\to\infty} n^{-\beta-\varepsilon} \sup_{x\in\mathbb{Z}} \left| \xi_j(x, n) - \frac{1}{\sigma_j^2} \eta_j(x, n\sigma_j^2) \right| = 0 \quad a.s. \tag{1.11}$$

hold simultaneously with some $0 < \alpha, \beta < 1/2$, *as* $n \to \infty$. *Then, on a possibly larger probability space for* $\{\mathbf{C}(n) = (C_1(n), C_2(n)); n = 0, 1, 2, \ldots\}$ *on* $\mathbb{C}^2$, *as* $n \to \infty$, *we have with any* $\varepsilon > 0$

$$\left| C_1(n) - \sigma_1 W_1\left(\frac{1}{\sigma_2^2} \eta_2(0, n\sigma_2^2)\right) \right| = O(n^{\vartheta/2+\varepsilon}) \quad a.s.$$

and

$$|C_2(n) - \sigma_2 W_2(n)| = O(n^{\alpha^*+\varepsilon}) \quad a.s.$$

simultaneously, where $\sigma_j^2 := \sum_{k=-\infty}^{\infty} k^2 p_j(k)$, $j = 1, 2$,

$$\alpha^* := \max(\alpha, 1/4) \quad and \quad \vartheta := \max(\alpha^*, \beta).$$

We note in passing that under various random walk conditions the assumptions (1.10) and (1.11) hold true. A few of such examples are listed in Section 6.

The intrinsic nature of random walks is usually highlighted by studying their local time behaviour (cf., e.g., Borodin [10], Révész [37], Csáki *et al.* [15], and references in these works). The study of local time is also of interest concerning some random walk problems in physics. In this regard we refer to Ferraro and Zaninetti [27], who deal with various statistics of the "number of times a site is visited by a walker", called "local time" in the present paper. Building on their previous paper [26], in [27] they present a formula for the probability that a site was visited exactly r times after n steps, and then derive all moments of this distribution. Naturally, these moments depend on the type of random walk in hand, and specific formulas are given in [27] for the mean and variance in case of simple symmetric random walks on lattices with various boundary conditions. Here we are to continue our exposition with studying the asymptotic local time behavior of a walker on 2-dimensional comb lattice as detailed on the next few pages of this section.

Define now the local time process $\Xi(\cdot,\cdot)$ of the random walk $\{\mathbf{C}(n); n = 0, 1, \ldots\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$ by

$$\Xi(\mathbf{x}, n) := \#\{0 < k \leq n : \mathbf{C}(k) = \mathbf{x}\}, \quad \mathbf{x} \in \mathbb{Z}^2,\, n = 1, 2, \ldots \tag{1.12}$$

The next result concludes a strong approximation of the local time process $\Xi((x, 0), n)$.

Theorem 1.2 *On the probability space of* Theorem 1.1, *as $n \to \infty$, we have for any $\delta > 0$*

$$\sup_{x\in\mathbb{Z}} \left| \Xi((x,0),n) - \frac{2}{\sigma_1^2}\eta_1\left(x, \frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2 n)\right)\right| = O(n^{\beta^*/2+\delta}) \quad a.s., \tag{1.13}$$

where $\beta^* = \max(\beta, 1/4)$.

Corollary 1.1 below establishes iterated local time approximations for $\Xi((x, 0), n)$ and $\Xi((x, y), n)$ over increasing subintervals for $(x, y) \in \mathbb{Z}^2$ via Theorem 1.3.

Theorem 1.3 *On the probability space of* Theorem 1.1, *as $n \to \infty$, we have for any $0 < \varepsilon < 1/4$*

$$\max_{|x|\leq n^{1/4-\varepsilon}} |\Xi((x,0),n) - \Xi((0,0),n)| = O(n^{1/4-\delta}) \quad a.s. \tag{1.14}$$

and

$$\max_{0<|y|\leq n^{1/4-\varepsilon}} \max_{|x|\leq n^{1/4-\varepsilon}} |\Xi((x,y),n) - \frac{1}{2}\Xi((0,0),n)| = O(n^{1/4-\delta}) \quad a.s., \tag{1.15}$$

for any $0 < \delta < \varepsilon/2$, where max *in* (1.14) *and* (1.15) *is taken on the integers.*

Corollary 1.1 *On the probability space of* Theorem 1.1, *as $n \to \infty$, we have for any $0 < \varepsilon < 1/4$*

$$\max_{|x|\leq n^{1/4-\varepsilon}} \left| \Xi((x,0),n) - \frac{2}{\sigma_1^2}\eta_1\left(0, \frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2 n)\right)\right| = O(n^{\beta^*/2+\varepsilon}) \quad a.s. \tag{1.16}$$

and

$$\max_{0<|y|\le n^{1/4-\varepsilon}}\ \max_{|x|\le n^{1/4-\varepsilon}}\left|\Xi((x,y),n)-\frac{1}{\sigma_1^2}\eta_1\left(0,\frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2 n)\right)\right| = O(n^{\beta^*/2+\varepsilon}) \quad a.s. \tag{1.17}$$

where $\beta^*=\max(\beta,1/4)$ *and* max *in* (1.16) *and* (1.17) *is taken on the integers.*

Remark 2 *We call attention to the fact that on the x-axis as in* (1.16)*, the local time is approximately twice as much as in* (1.17)*, where* $y\neq 0$ (*cf. also* (1.14) *versus* (1.15) *in this regard*).

From these strong approximation results one can easily conclude almost sure limit theorems for the path behaviour of $\mathbf{C}(\cdot)$ and its local times $\Xi(\cdot,\cdot)$ in hand. In this paper we concentrate on almost sure local time path behaviour, and only note that the almost sure path behaviour of the random walk $\mathbf{C}(\cdot)$ on the 2-dimensional comb lattice $\mathbb{C}^2$ under the conditions of Theorem 1.1 can be studied similarly to that of a simple random walk $\mathbf{C}(\cdot)$ on $\mathbb{C}^2$ as in Csáki *et al.* [16].

Since, by Theorem E below, the iterated local time process $\{\eta_1(0,\eta_2(0,t));\ t\ge 0\}$ has the same distribution as $\{\sup_{0\le s\le t}W_1(\eta_2(0,s));\ t\ge 0\}$, the following result follows from Theorem 2.2 in [19].

Corollary 1.2 *The net*

$$\left\{\frac{\eta_1(0,\eta_2(0,zt))}{2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}};\ 0\le z\le 1\right\}_{t\ge 3},$$

as $t\to\infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$, and the set of its limit points is the class of nondecreasing absolutely continuous functions (*with respect to the Lebesgue measure*) *on $[0,1]$ for which*

$$f(0)=0 \qquad \textit{and} \qquad \int_0^1 |\dot f(x)|^{4/3}\,dx\le 1. \tag{1.18}$$

In what follows we will need the following scaling properties of Brownian local time

$$\eta(x,t)\stackrel{d}{=}\frac{1}{\sqrt{a}}\,\eta(x\sqrt{a},at), \quad a>0,\ t>0, \quad x\in\mathbb{R},$$

where $\stackrel{d}{=}$ means equality in distribution. Consequently, we also have

$$\left\{\frac{1}{\sigma_1^2}\eta_1\left(0,\frac{\sigma_1^2}{\sigma_2^2}\,\eta_2(0,\sigma_2^2 t)\right),t\ge 0\right\}\stackrel{d}{=}\left\{\frac{1}{\sigma_1\sqrt{\sigma_2}}\,\eta_1(0,\eta_2(0,t)),t\ge 0\right\}. \tag{1.19}$$

For the next results we consider a continuous version of the local times $\Xi(\cdot,\cdot)$ in (1.12), obtained by linear interpolation. A combination of Corollaries 1.1 and 1.2 yields the following conclusions.

Corollary 1.3 *Under the conditions of* Theorem 1.1*, for fixed* $(x,y)\in\mathbb{Z}^2$*, the sequences*

$$\left\{\frac{\sigma_1\sqrt{\sigma_2}\,\Xi((x,0),zn)}{2^{9/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}};0\le z\le 1\right\}_{n\ge 3},$$

and

$$\left\{\frac{\sigma_1\sqrt{\sigma_2}\,\Xi((x,y),zn)}{2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}};\ 0\le z\le 1\right\}_{n\ge 3},\quad y\ne 0,$$

as $n\to\infty$*, are almost surely relatively compact in the space* $C([0,1],\mathbb{R})$*, and the set of their limit points coincides with that in* Corollary 1.2*.*

In particular, we have the following laws of the iterated logarithm for fixed $(x,y)\in\mathbb{Z}^2$:

$$\limsup_{t\to\infty}\frac{\eta_1(0,\eta_2(0,t))}{t^{1/4}(\log\log t)^{3/4}}=\frac{2^{5/4}}{3^{3/4}}\quad\text{a.s.,}\tag{1.20}$$

$$\limsup_{n\to\infty}\frac{\Xi((x,0),n)}{n^{1/4}(\log\log n)^{3/4}}=\frac{2^{9/4}}{3^{3/4}\sigma_1\sqrt{\sigma_2}}\quad\text{a.s.,}\tag{1.21}$$

$$\limsup_{n\to\infty}\frac{\Xi((x,y),n)}{n^{1/4}(\log\log n)^{3/4}}=\frac{2^{5/4}}{3^{3/4}\sigma_1\sqrt{\sigma_2}}\quad\text{a.s.}\quad y\ne 0.\tag{1.22}$$

Theorem C ([17])

$$\limsup_{t\to\infty}\frac{\sup_{x\in\mathbb{R}}\eta_1(x,\eta_2(0,t))}{t^{1/4}(\log\log t)^{3/4}}=\frac{2^{5/4}}{3^{3/4}}\quad a.s.$$

Concerning liminf results, using Theorem E below, and a Hirsch-type result of Bertoin [9], we conclude the following Hirsch-type law [29].

Corollary 1.4 *Let* $\beta(t)>0$*,* $t\ge 0$*, be a non-increasing function. Then we have almost surely that*

$$\liminf_{t\to\infty}\frac{\eta_1(0,\eta_2(0,t))}{t^{1/4}\beta(t)}=0\quad or\quad\infty$$

according as the integral $\int_1^\infty\beta(t)/t\,dt$ *diverges or converges.*

From Corollaries 1.1 and 1.4 we get the following results.

Corollary 1.5 *Let* $\beta(n), n=1,2,\dots$ *be a non-increasing sequence of positive numbers. Then, for fixed* $(x,y)\in\mathbb{Z}^2$*, under the conditions of* Theorem 1.1*, we have almost surely that*

$$\liminf_{n\to\infty}\frac{\Xi((x,y),n)}{n^{1/4}\beta(n)}=0\quad or\quad\infty$$

according as the series $\sum_1^\infty\beta(n)/n$ *diverges or converges.*

We conclude this Section by spelling out strong and weak convergence results that easily follow from Theorem 1.1, Theorem 1.2 and Corollary 1.1, respectively. To begin with, Theorem 1.1 yields a weak convergence for $\mathbf{C}([nt])$ on the space $D([0,\infty),\mathbb{R}^2)$ endowed with a uniform topology that is defined as follows.

For functions $(f_1(t), f_2(t))$, $(g_1(t), g_2(t))$ in the function space $D([0,\infty),\mathbb{R}^2)$, and for compact subsets A of $[0,\infty)$, we define

$$\Delta = \Delta(A,(f_1,f_2),(g_1,g_2)) := \sup_{t\in A} \|(f_1(t)-g_1(t), f_2(t)-g_2(t))\|,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^2$.

We also define the measurable space $(D([0,\infty),\mathbb{R}^2),\mathcal{D})$, where $\mathcal{D}$ is the σ-field generated by the collection of all Δ-open balls of $D([0,\infty),\mathbb{R}^2)$, where a ball is a subset of $D([0,\infty),\mathbb{R}^2)$ of the form

$$\{(f_1,f_2):\ \Delta(A,(f_1,f_2),(g_1,g_2)) < r\}$$

for some $(g_1,g_2)\in D([0,\infty),\mathbb{R}^2)$, some $r>0$, and some compact interval A of $[0,\infty)$.

In view of these two definitions, Theorem 1.1 yields a weak convergence result in terms of a functional convergence in distribution, as follows.

Corollary 1.6 *Under the conditions of* Theorem 1.1*, as $n\to\infty$, we have*

$$h\left(\frac{C_1([nt])}{n^{1/4}}, \frac{C_2([nt])}{n^{1/2}}\right) \xrightarrow{d} h\left(\frac{\sigma_1}{\sqrt{\sigma_2}} W_1(\eta_2(0,t)), \sigma_2 W_2(t)\right) \tag{1.23}$$

for all $h: D([0,\infty),\mathbb{R}^2)\longrightarrow\mathbb{R}^2$ that are $(D([0,\infty),\mathbb{R}^2),\mathcal{D})$ measurable and Δ-continuous, or Δ-continuous except at points forming a set of measure zero on $(D([0,\infty),\mathbb{R}^2),\mathcal{D})$ with respect to the measure generated by $(W_1(\eta_2(0,t)), W_2(t))$, where W_1, W_2 are two independent Brownian motions and $\eta_2(0,t)$ is the local time process of $W_2(\cdot)$ at zero, and $\xrightarrow{d}$ denotes convergence in distribution.

As an example, on taking $t=1$ in Corollary 1.6, we obtain the following convergence in distribution result: as $n\to\infty$,

$$\left(\frac{\sqrt{\sigma_2}}{\sigma_1}\frac{C_1(n)}{n^{1/4}}, \frac{C_2(n)}{\sigma_2 n^{1/2}}\right) \xrightarrow{d} (W_1(\eta_2(0,1)), W_2(1)). \tag{1.24}$$

Concerning the joint distribution of the limiting vector valued random variable, we have

$$(W_1(\eta_2(0,1)), W_2(1)) \stackrel{d}{=} (X|Y|^{1/2}, Z),$$

where $(|Y|, Z)$ has the joint distribution of the vector $(\eta_2(0,1), W_2(1))$, X is equal in distribution to the random variable $W_1(1)$, and is independent of $(|Y|, Z)$.

As to the joint density of $(|Y|, Z)$, we have (cf. 1.3.8 on p. 127 in Borodin and Salminen [11])

$$\mathbf{P}(|Y| \in dy, Z \in dz) = \frac{1}{\sqrt{2\pi}}(y + |z|)e^{-\frac{(y+|z|)^2}{2}} dy\, dz, \quad y \geq 0,\ z \in \mathbb{R}.$$

Now, on account of the independence of X and $(|Y|, Z)$, the joint density function of the random variables $(X, |Y|, Z)$ reads as follows.

$$\mathbf{P}(X \in dx, |Y| \in dy, Z \in dz) = \frac{1}{2\pi}(y + |z|)e^{-\frac{x^2+(y+|z|)^2}{2}} dx\, dy\, dz, \quad y \geq 0,\ x, z \in \mathbb{R}.$$

By changing variables, via calculating the joint density function of the random variables $U := X|Y|^{1/2}, Y, Z$, and then integrating it out with respect to $y \in [0, \infty)$, we arrive at the joint density function of the random variables $(U = X|Y|^{1/2}, Z)$, which reads as follows.

$$\mathbf{P}(X|Y|^{1/2} \in du, Z \in dz) = \frac{1}{2\pi} \int_0^\infty \frac{y + |z|}{y^{1/2}} e^{-\frac{u^2}{2y} - \frac{(y+|z|)^2}{2}} dy\, du\, dz \quad u, z \in \mathbb{R}. \tag{1.25}$$

Clearly, Z is a standard normal random variable. The marginal distribution of $U = X|Y|^{1/2}$ is of special interest in that this random variable first appeared in the conclusion of Dobrushin's classical Theorem 2 of his fundamental paper [24], that was first to deal with the so-called second order limit law for additive functionals of a simple symmetric random walk on the real line. In view of the above joint density function in (1.25), on integrating it out with respect to z over the real line, we are now to also conclude Dobrushin's formula for the density function of this random variable.

$$\mathbf{P}(U \in du) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{y + z}{\sqrt{y}} e^{-\frac{u^2}{2y} - \frac{(y+z)^2}{2}} dy\, dz\, du$$
$$= \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{y}} e^{-\frac{u^2}{2y} - \frac{y^2}{2}} dy\, du = \frac{2}{\pi} \int_0^\infty e^{-\frac{u^2}{2v^2} - \frac{v^4}{2}} dv\, du.$$

As an immediate consequence of Theorem 1.2 now, on choosing $\delta > 0$ small enough, we conclude the following strong invariance principle.

Corollary 1.7 *On the probability space of* Theorem 1.1*, we have almost surely, as* $n \to \infty$,

$$\sup_{t \in A} \sup_{x \in \mathbb{Z}} \frac{\left| \Xi((x, 0), [nt]) - \frac{2}{\sigma_1^2} \eta_1 \left(x, \frac{\sigma_1^2}{\sigma_2^2} \eta_2(0, \sigma_2^2\, nt) \right) \right|}{n^{1/4}} = o(1)$$

for all compact subsets A *of* $[0, \infty)$.

The next result concludes weak convergence for $\Xi(([x], 0), [nt])$ via in probability nearness of appropriate functionals on the function space $D(\mathbb{R} \times [0, \infty), \mathbb{R})$ with respect to the norm

$$\Delta_1 = \Delta_1(A, f(\cdot, \cdot), g(\cdot, \cdot)) := \sup_{t \in A} \sup_{x \in \mathbb{R}} |f(x, t) - g(x, t)|$$

for real valued functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and compact subsets A of $[0,\infty)$. Here $D(\mathbb{R}\times[0,\infty),\mathbb{R})$ stands for the space of real-valued bivariate *càdlàg* functions defined on $\mathbb{R}\times[0,\infty)$.

In order to state our result in this regard, we define the measurable space $(D(\mathbb{R}\times[0,\infty),\mathbb{R}),\mathcal{D})$, where $\mathcal{D}$ is the σ-field generated by the collection of all Δ_1-open balls of $D(\mathbb{R}\times[0,\infty),\mathbb{R})$, where a ball is a subset of $D(\mathbb{R}\times[0,\infty),\mathbb{R})$ of the form

$$\{f(\cdot,\cdot) : \Delta_1(A, f(\cdot,\cdot), g(\cdot,\cdot)) < r\}$$

for some $g(\cdot,\cdot)\in D(\mathbb{R}\times[0,\infty),\mathbb{R})$, some $r>0$, and some compact interval A of $[0,\infty)$.

In view of these definitions, Corollary 1.7 yields an in probability nearness of functionals, which reads as follows.

Corollary 1.8 *On the probability space of* Theorem 1.1, *as $n\to\infty$, we have*

$$\left| h\left(\frac{\Xi(([x],0),[nt])}{n^{1/4}}\right) - h\left(\frac{2}{\sigma_1^2}\eta_1\left(\frac{x}{\sqrt{n}},\frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2\,t)\right)\right)\right| = o_p(1) \tag{1.26}$$

for all $h : D(\mathbb{R}\times[0,\infty),\mathbb{R}) \longrightarrow \mathbb{R}$ that are $(D(\mathbb{R}\times[0,\infty),\mathbb{R}),\mathcal{D})$ measurable and Δ_1-continuous, or Δ_1-continuous except at points forming a set of measure zero on $(D(\mathbb{R}\times[0,\infty),\mathbb{R}),\mathcal{D})$ with respect to the measure generated by $\frac{2}{\sigma_1^2}\eta_1\left(x,\frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2\,t)\right)$ on this space, where $\eta_1(\cdot,\cdot)$, $\eta_2(\cdot,\cdot)$ are two independent Brownian local time processes.

Taking functionals of interest, corresponding convergence in distribution results can be easily deduced from (1.26). For example, for all h as in Corollary 1.8, as $n\to\infty$, we have

$$h\left(\frac{\sup_{x\in\mathbb{Z}}\Xi((x,0),[nt])}{n^{1/4}}\right) \xrightarrow{d} h\left(\frac{2}{\sigma_1^2}\sup_{x\in\mathbb{R}}\eta_1\left(x,\frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2\,t)\right)\right), \tag{1.27}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

We note in passing that taking $\sup_{x\in\mathbb{R}}$ instead of $\sup_{x\in\mathbb{Z}}$ on the right hand side in (1.27) is allowed in the limit, due to the modulus of continuity of Brownian local time in its space parameter (cf. Trotter [41], McKean [33] and Ray [35] as cited in Csáki *et al.* [13]).

In view of Corollary 1.1, on choosing $\varepsilon>0$ small enough, we arrive at the following strong invariance principles.

Corollary 1.9 *On the probability space of* Theorem 1.1, *we have almost surely, as $n\to\infty$,*

$$\sup_{t\in A}\max_{|x|\le n^{1/4-\varepsilon}} \frac{\left|\Xi((x,0),[nt]) - \frac{2}{\sigma_1^2}\eta_1\left(0,\frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2\,nt)\right)\right|}{n^{1/4}} = o(1),$$

and

$$\sup_{t\in A}\max_{0<|y|\le n^{1/4-\varepsilon}}\max_{|x|\le n^{1/4-\varepsilon}}\frac{\left|\Xi((x,y),[nt])-\frac{1}{\sigma_1^2}\eta_1\left(0,\frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2\,nt)\right)\right|}{n^{1/4}}=o(1),$$

for all compact subsets A of $[0,\infty)$, where max *is taken on the integers.*

Corollary 1.9 yields a weak convergence for $\Xi((x,y),[nt])$ with $(x,y)\in\mathbb{Z}^2$ fixed, on the function space $D([0,\infty),\mathbb{R})$ with respect to the usual sup norm

$$\Delta_2=\Delta_2(A,f(\cdot),g(\cdot)):=\sup_{t\in A}|f(t)-g(t)|$$

for real valued functions $f(\cdot)$, $g(\cdot)$ and compact subsets A of $[0,\infty)$.

In order to state our result in this regard, we define the usual measurable space $(D([0,\infty),\mathbb{R}),\mathcal{D})$, where $\mathcal{D}$ now is the σ-field generated by the collection of all Δ_2-open balls of $D([0,\infty),\mathbb{R})$, where a ball now is a subset of $D([0,\infty),\mathbb{R})$ of the form

$$\{f(\cdot):\ \Delta_2(A,f(\cdot),g(\cdot))<r\}$$

for some $g(\cdot)\in D([0,\infty),\mathbb{R})$, some $r>0$, and some compact interval A of $[0,\infty)$.

In view of these definitions, Corollary 1.9, combined with (1.19), yields weak convergence results in terms of functional convergence in distribution as follows.

Corollary 1.10 *Under the conditions of* Theorem 1.1*, with $(x,y)\in\mathbb{Z}^2$ fixed, as $n\to\infty$, we have*

$$h\left(\frac{\Xi((x,0),[nt])}{n^{1/4}}\right)\xrightarrow{d}h\left(\frac{2}{\sigma_1\sqrt{\sigma_2}}\eta_1(0,\eta_2(0,t))\right),\tag{1.28}$$

and, when $y\neq 0$,

$$h\left(\frac{\Xi((x,y),[nt])}{n^{1/4}}\right)\xrightarrow{d}h\left(\frac{1}{\sigma_1\sqrt{\sigma_2}}\eta_1(0,\eta_2(0,t))\right),\tag{1.29}$$

for all $h:D([0,\infty),\mathbb{R})\longrightarrow\mathbb{R}$ that are $(D([0,\infty),\mathbb{R}),\mathcal{D})$ measurable and Δ_2-continuous, or Δ_2-continuous except at points forming a set of measure zero on $(D([0,\infty),\mathbb{R}),\mathcal{D})$ with respect to the measure generated by $\eta_1(0,\eta_2(0,t))$, where $\eta_1(0,\cdot)$, $\eta_2(0,\cdot)$ are two independent Brownian local time processes, and $\xrightarrow{d}$ denotes convergence in distribution.

On taking h to be the identity map, and $t=1$ in (1.28) and, respectively, in (1.29), as $n\to\infty$, we obtain for $(x,y)\in\mathbb{Z}^2$

$$\frac{\Xi((x,0),n)}{n^{1/4}}\xrightarrow{d}\frac{2}{\sigma_1\sqrt{\sigma_2}}\eta_1(0,\eta_2(0,1))\stackrel{d}{=}\frac{2}{\sigma_1\sqrt{\sigma_2}}|X|\sqrt{|Y|}\tag{1.30}$$

and, when $y \neq 0$, then

$$\frac{\Xi((x,y),n)}{n^{1/4}} \xrightarrow{d} \frac{1}{\sigma_1\sqrt{\sigma_2}}\eta_1(0,\eta_2(0,1)) \stackrel{d}{=} \frac{1}{\sigma_1\sqrt{\sigma_2}}|X|\sqrt{|Y|}, \tag{1.31}$$

where X and Y are independent standard normal random variables.

We note that the statement of (1.30) can also be obtained from (1.26) in a similar way if we fix $x \in \mathbb{Z}$ in (1.26) as well. On the other hand, we emphasize that statements like (1.27) do not follow from the first statement of Corollary 1.9.

The structure of this paper from now on is as follows. In Section 2 we give preliminary facts and results, and in Sections 3-5 we prove our Theorems 1.1-1.3. In Section 6 we illustrate the general nature of our results by discussing several specific examples of simultaneous invariance principles for random walks and their local times that, in turn, yield our Theorems 1.2, 1.3 and Corollary 1.1 with explicit rates of convergence. We conclude this paper in Section 7 by making further comments, and remarks on our results, including that of Proposition 7.1 in there, and those of the examples of Section 6.

2 Preliminaries

Let

$$\rho(0) := 0, \quad \rho(N) := \min\{k > \rho(N-1) : S(k) = 0\}, \quad N = 1,2,\ldots \tag{2.1}$$

be the recurrence times of an integer valued random walk process $\{S(n);\, n = 0,1,2,\ldots\}$.

Define the inverse local time process of a standard Brownian motion $W(\cdot)$ by

$$\tau(t) := \inf\{s : \eta(0,s) \geq t\},\ t \geq 0. \tag{2.2}$$

In case of the simple symmetric random walk on $\mathbb{Z}$, denote the recurrence time $\rho(\cdot)$ by $\rho^*(\cdot)$. We quote from Révész ([37], p. 119), the following result.

Lemma A *For any* $0 < \varepsilon < 1$, *with probability* 1 *for all large* N,

$$(1-\varepsilon)\frac{N^2}{2\log\log N} \leq \rho^*(N) \leq N^2(\log N)^{2+\varepsilon}.$$

Lemma B (cf. [14]) *On an appropriate probability space for the random walk* $\{S(n);\, n = 0,1,2,\ldots\}$ *satisfying conditions* (i)-(iii), *as* $N \to \infty$, *we have*

$$|\sigma^2\rho(N) - \tau(N\sigma^2)| = O(N^{5/3}) \qquad a.s.$$

From Lemmas A and B, we conclude the following result.

Lemma C *For any* $0 < \varepsilon < 1$, *we have with probability* 1 *for all large enough* N *that*

$$(1-\varepsilon)\frac{N^2}{2\log\log N} \leq \tau(N) \leq N^2(\log N)^{2+\varepsilon}.$$

From Lemmas B and C now, we arrive at the following conclusion for the recurrence times of our walks.

Theorem D *Suppose that the random walks* $\{S_j(n);\, n=0,1,2,\ldots\}, j=1,2$, *satisfy conditions* (i)-(iii). *Then for any* $0 < \varepsilon < 1$, *we have with probability* 1 *for all large enough* N *that*

$$(1-\varepsilon)\frac{\sigma_j^2 N^2}{2\log\log N} \leq \rho_j(N) \leq \sigma_j^2 N^2(\log N)^{2+\varepsilon},$$

where $\rho_j(\cdot)$, $j=1,2$ *are the recurrence times of* $S_j(\cdot)$ *as defined in* (2.1).

A well-known result of Lévy [32] reads as follows.

Theorem E *Let* $W(\cdot)$ *be a standard Brownian motion with local time process* $\eta(\cdot,\cdot)$. *The following equality in distribution holds:*

$$\{\eta(0,t),\, t\geq 0\} \stackrel{d}{=} \{\sup_{0\leq s\leq t} W(s),\, t\geq 0\}.$$

As to the random walk $\mathbf{C}(n)$, $n=0,1,2,\ldots$, it can be constructed as follows (cf. [16]). Consider two independent integer valued random walks $\{S_j(n);\, n=1,2,\ldots\}$, $j=1,2$, with respective one-step distributions $\mathcal{P}_j = \{p_j(k), k\in\mathbb{Z}\}$, $j=1,2$. We may assume that, on the probability space of these random walks, we have an i.i.d. sequence $G_1, G_2, \ldots$ of geometric random variables with distribution

$$\mathbf{P}(G_1 = k) = \frac{1}{2^{k+1}}, \quad k=0,1,2,\ldots,$$

independent of the random walks $S_j(\cdot)$, $j=1,2$.

On the just described probability space we may also construct the random walk $\mathbf{C}(n)$ on the 2-dimensional comb lattice $\mathbb{C}^2$ as follows. Put $T_N = G_1+G_2+\cdots+G_N$, $N=1,2,\ldots$, and let $\rho_2(N)$ denote the time of the N-th return to 0 of the random walk $S_2(\cdot)$. For $n=0,\ldots,T_1$, let $C_1(n) = S_1(n)$ and $C_2(n)=0$. For $n=T_1+1,\ldots,T_1+\rho_2(1)$, let $C_1(n)=C_1(T_1)$, $C_2(n)=S_2(n-T_1)$. In general, for $T_N+\rho_2(N) < n \leq T_{N+1}+\rho_2(N)$, let

$$C_1(n) = S_1(n-\rho_2(N)),$$

$$C_2(n) = 0,$$

and, for $T_{N+1}+\rho_2(N) < n \leq T_{N+1}+\rho_2(N+1)$, let

$$C_1(n) = C_1(T_{N+1}+\rho_2(N)) = S_1(T_{N+1}),$$

$$C_2(n) = S_2(n-T_{N+1}).$$

Then, in terms of these definitions for $C_1(n)$ and $C_2(n)$, it can be seen that $\mathbf{C}(n) = (C_1(n), C_2(n))$ is a random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ with transition probabilities as in (1.7)-(1.9).

Consider now an arbitrary random walk $\{S(n) = \sum_{l=1}^{n} X(l);\ n \geq 0\}$, on $\mathbb{Z}$. Define its potential kernel $a(\cdot)$ by

$$a(x) := \sum_{n=0}^{\infty} (\mathbf{P}(S(n) = 0) - \mathbf{P}(S(n) = -x)), \quad x \in \mathbb{Z}.$$

Introduce

$$\gamma(x) := \frac{1}{a(x) + a(-x)}. \tag{2.3}$$

Then, for every one-dimensional aperiodic recurrent random walk $S(n)$, for $x = \pm 1, \pm 2, \ldots$, on simply writing ρ for $\rho(1)$, we have (cf. Kesten and Spitzer, [31])

$$\begin{gathered}\mathbf{P}(\xi(x,\rho) = 0) = 1 - \gamma(x), \\ \mathbf{P}(\xi(x,\rho) = k) = \gamma^2(x)(1-\gamma(x))^{k-1}, \quad k = 1, 2, \ldots \\ \mathbf{E}\xi(x,\rho) = 1, \quad \mathbf{Var}\xi(x,\rho) = 2(a(x) + a(-x) - 1),\end{gathered} \tag{2.4}$$

$$\lim_{x \to \infty} \frac{a(x) + a(-x)}{x} = \frac{2}{\sigma^2}, \tag{2.5}$$

where $\sigma^2 = \mathbf{Var}(X(1))$.

Lemma 2.1 *Let $\theta(x,1)$ be the time between the first visit and the first return of $\mathbf{C}(\cdot)$ to $(x,0)$. Then for $x = \pm 1, \pm 2, \ldots, \quad y = \pm 1, \pm 2, \ldots,$ we have*

$$\mathbf{P}(\Xi((x,y), \theta(x,1)) = 0) = 1 - \frac{\gamma_2(y)}{2}, \tag{2.6}$$

$$\mathbf{P}(\Xi((x,y), \theta(x,1)) = k) = \frac{\gamma_2^2(y)}{2}(1 - \gamma_2(y))^{k-1}, \quad k = 1, 2 ..., \tag{2.7}$$

$$\mathbf{E}(\Xi((x,y), \theta(x,1)) = \frac{1}{2}. \tag{2.8}$$

Furthermore, for

$$g(\lambda) := \mathbf{E}\exp(\lambda \Xi((x,y), \theta(x,1))) = 1 + \frac{1}{2} \frac{1 - e^{-\lambda}}{1 - \dfrac{1}{\gamma_2(y)}(1 - e^{-\lambda})}, \tag{2.9}$$

we have

$$g(\lambda) \leq \exp\left(\frac{\lambda}{2}\left(1 + \frac{2\lambda}{\gamma_2(y)}\right)\right) \tag{2.10}$$

if $2\lambda < \gamma_2(y)$, where $\gamma_2(\cdot)$ is defined à la $\gamma(\cdot)$ in (2.3) associated with the random walk $S_2(\cdot)$.

Proof. Define the indicator variable $\mathbf{I}$ as $\mathbf{P}(\mathbf{I}=0)=\mathbf{P}(\mathbf{I}=1)=\frac{1}{2}$, that is independent from the sequence $X_2(k), k=1,2,\ldots$ Observe that, with this notation,

$$\Xi((x,y),\theta(x,1))=\mathbf{I}\xi_2(y,\rho_2),\quad |y|>0,$$

where $\mathbf{I}=1$ if the first step from $(x,0)$ is vertical, and 0 if it is horizontal, and ρ_2 is the time of the first return to 0 of the random walk $S_2(\cdot)$. Using now (2.4), we get (2.6) and (2.7). As regards (2.9), it follows by straightforward calculations, from which we can conclude (2.10) as well, via some more calculations.

We make use of the following almost sure properties of the increments for a Brownian motion (Csörgő and Révész [22]), Brownian local time (Csáki *et al.* [13]), and random walk local time (Csáki and Földes [18], Jain and Pruitt [30]).

Theorem F *Let $0<a_T\le T$ be a non-decreasing function of T. Then, as $T\to\infty$, we have almost surely*

$$\sup_{0\le t\le T-a_T}\sup_{s\le a_T}|W(t+s)-W(t)|=O(a_T^{1/2}(\log(T/a_T)+\log\log T)^{1/2}),\tag{2.11}$$

$$\sup_{x\in\mathbb{R}}\sup_{0\le t\le T-a_T}(\eta(x,t+a_T)-\eta(x,t))=O(a_T^{1/2}(\log(T/a_T)+\log\log T)^{1/2}),\tag{2.12}$$

and, under the conditions (i)-(iii) *for a random walk local time $\xi(0,\cdot)$, as $N\to\infty$, we have almost surely*

$$\max_{0\le n\le N-a_N}(\xi(0,n+a_N)-\xi(0,n))=O(a_N^{1/2}(\log(N/a_N)+\log\log N)^{1/2}).\tag{2.13}$$

Remark 3 *We note that for* (2.13) *of* Theorem F *to hold, instead of condition* (ii), *we only need the existence of two moments.*

In the proofs we also need increment results for $\xi(x,\cdot)$, uniformly in x. Such results are not found in the cited papers, but combining (2.12) and (2.13) with the assumed rate (1.11), we can obtain the following result.

Corollary A *Under the conditions of* Theorem 1.1, *for any $\varepsilon>0$, we have almost surely, as $N\to\infty$,*

$$\sup_{x\in\mathbb{Z}}\sup_{0\le n\le N-a_N}(\xi_j(x,n+a_N)-\xi_j(x,n))=O(a_N^{1/2+\varepsilon})+O(N^{\beta+\varepsilon}),\quad j=1,2.\tag{2.14}$$

The following theorem is a version of Hoeffding's inequality, which is explicitly stated in [40].

Theorem G *Let G_i be i.i.d.random variables with the common geometric distribution $\mathbf{P}(G_i=k)=2^{-k-1},\quad k=0,1,2...$ Then*

$$\mathbf{P}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(G_i-1)\right|>\lambda\right)\le 2\exp(-\lambda^2/8n)$$

for $0<\lambda<na$ with some $a>0$.

3 Proof of Theorem 1.1

The proof is based on the following result.

Lemma 3.1 *Suppose that conditions* (i)-(iii) *are met. If* $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, *then, as* $n \to \infty$, *we have for any* $\varepsilon > 0$

$$N = O(n^{1/2+\varepsilon}) \quad a.s.$$

and

$$\xi_2(0,n) = N + O(n^{1/4+\varepsilon}) \quad a.s.$$

On using Theorem D and Theorem F, the proof of this lemma goes exactly the same way as that of the corresponding Lemma 2.1 in [16].

As to the proof of our Theorem 1.1, it goes along the lines of Theorem 1.1 in [16], but we present it for the sake of completeness. In what follows we use several times the assumptions (1.10) and (1.11), as well as increment results for the Brownian motion (see Theorem F).

If $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N)$, then

$$C_1(n) = S_1(n - \rho_2(N)) = \sigma_1 W_1(n - \rho_2(N)) + O(T_N^{\alpha+\varepsilon}) = \sigma_1 W_1(T_N) + O(N^{\alpha+\varepsilon})$$

$$= \sigma_1 W_1(N) + O(N^{\alpha^*+\varepsilon}) = \sigma_1 W_1(\xi_2(0,n)) + O(n^{\alpha^*/2+\varepsilon}) = \sigma_1 W_1\left(\frac{1}{\sigma_2^2}\eta_2(0, n\sigma_2^2)\right) + O(n^{\vartheta/2+\varepsilon}) \quad \text{a.s.}$$

Since $C_2(n) = 0$ if $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N)$, we only have to estimate $W_2(n)$. We have

$$\begin{aligned}|W_2(n)| &\le |W_2(\rho_2(N))| + \sup_{0 \le t \le T_{N+1}} |W_2(\rho_2(N) + t) - W_2(\rho_2(N))| \\ &= |W_2(\rho_2(N))| + O(N^{1/2+\varepsilon}) = \frac{1}{\sigma_2} S_2(\rho_2(N)) + O(n^{\alpha^*+\varepsilon}) = O(n^{\alpha^*+\varepsilon}),\end{aligned}$$

as $S_2(\rho_2(N)) = 0$, i.e.,

$$0 = C_2(n) = \sigma_2 W_2(n) + O(n^{\alpha^*+\varepsilon}).$$

In the case when $T_{N+1} + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, then, for any $\varepsilon > 0$, we have almost surely

$$\begin{aligned}C_1(n) &= S_1(T_{N+1}) = \sigma_1 W_1(N) + O(N^{\alpha^*+\varepsilon}) = \sigma_1 W_1(\xi_2(0,n)) + O(n^{\alpha^*/2+\varepsilon}) \\ &= \sigma_1 W_1\left(\frac{1}{\sigma_2^2}\eta_2(0, n\sigma_2^2)\right) + O(n^{\vartheta/2+\varepsilon}),\end{aligned}$$

and

$$C_2(n) = S_2(n - T_{N+1}) = \sigma_2 W_2(n - T_{N+1}) + O(n^{\alpha+\varepsilon}) = \sigma_2 W_2(n) + O(n^{\alpha^*+\varepsilon}).$$

This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Recall the definitions and constructions in Section 2. For $T_N + \rho_2(N) < n \le T_{N+1} + \rho_2(N)$ the number of horizontal steps, out of the first n steps, is equal to $n - \rho_2(N)$, and for $T_{N+1} + \rho_2(N) < n \le T_{N+1} + \rho_2(N+1)$ it is equal to T_{N+1}. So we may define the number of horizontal visits to $(x,0) \in \mathbb{Z}^2$ up to time n by

$$H((x,0),n) := \begin{cases} \xi_1(x, n-\rho_2(N)) & \text{if} \quad T_N + \rho_2(N) < n \le T_{N+1} + \rho_2(N), \\ \xi_1(x, T_{N+1}) & \text{if} \quad T_{N+1} + \rho_2(N) < n \le T_{N+1} + \rho_2(N+1), \end{cases} \tag{4.1}$$

and the number of vertical visits to $(x,0)$ up to time n is defined by

$$V((x,0),n) := \Xi((x,0),n) - H((x,0),n). \tag{4.2}$$

For T_N, as a sum of i.i.d. geometric random variables, we have

$$T_N = N + O((N \log\log N)^{1/2}) \qquad a.s.$$

as $N \to \infty$. Therefore, using Corollary A, we easily obtain for any $\delta > 0$ that, as $N \to \infty$, we have almost surely

$$\xi_1(x, T_N) = \xi_1(x,N) + O(N^{\beta^*+\delta}), \tag{4.3}$$

and

$$\xi_1(x, T_{N+1}) = \xi_1(x, N+1) + O(N^{\beta^*+\delta}) = \xi_1(x,N) + O(N^{\beta^*+\delta}),$$

where β^* is as in Theorem 1.2.

If $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, then

$$\xi_1(x, T_N) \le \xi_1(x, n-\rho_2(N)) \le \xi_1(x, T_{N+1}). \tag{4.4}$$

Hence, if $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, then, by (4.3) and (4.4), we have almost surely for any $\delta > 0$, as $n \to \infty$,

$$\begin{aligned} H((x,0),n) &= \xi_1(x,N) + O(N^{\beta^*+\delta}) \\ &= \xi_1(x, \xi_2(0,n)) + O(n^{\beta^*/2+\delta}) \\ &= \frac{1}{\sigma_1^2}\eta_1\left(x, \sigma_1^2 \xi_2(0,n)\right) + O(n^{\beta^*/2+\delta}) \\ &= \frac{1}{\sigma_1^2}\eta_1\left(x, \frac{\sigma_1^2}{\sigma_2^2}\eta_2(0,\sigma_2^2 n)\right) + O(n^{\beta^*/2+\delta}), \end{aligned} \tag{4.5}$$

where we used the assumed approximation rates, Lemma 3.1 and Theorem F.

In the following lemma we show that the number of horizontal and vertical visits are very close to each other.

Lemma 4.1 *For any $\delta > 0$, as $n \to \infty$, we have*

$$\sup_{x\in\mathbb{Z}} |H((x,0),n) - V((x,0),n)| = O(n^{1/8+\delta}) \quad a.s. \tag{4.6}$$

Proof. It follows from Theorem 1.1 that $C_1(n) \le n^{1/4+\delta}$ almost surely for any $\delta > 0$ and sufficiently large n. Hence it suffices to show that

$$\max_{|x|\le n^{1/4+\delta}} |H((x,0),n) - V((x,0),n)| = O(n^{1/8+\delta}) \quad \text{a.s.} \tag{4.7}$$

as $n \to \infty$. Here and throughout the proof max is taken on the integers.

Let $\kappa(x,0)$ be the time of the first visit of $\mathbf{C}(\cdot)$ to $(x,0)$, and for $\ell \ge 1$ let $\kappa(x,\ell)$ be the time of the ℓ-th horizontal visit to $(x,0)$. Then

$$V((x,0),\kappa(x,\ell)) = \sum_{j=1}^{\ell} \left(V((x,0),\kappa(x,j)) - V((x,0),\kappa(x,j-1))\right),$$

which is a sum of i.i.d. random variables with geometric distribution, with parameter 1/2. Then we have by Theorem G that

$$\mathbf{P}(\max_{|x|\le m}\max_{\ell\le m} |V((x,0),\kappa(x,\ell) - \ell| > u) \le m \exp\left(-\frac{u^2}{8m}\right).$$

Putting $u = m^{1/2+\delta}$, Borel-Cantelli lemma implies

$$\max_{|x|\le m}\max_{\ell\le m} |V((x,0),\kappa(x,\ell)) - \ell| = O(m^{1/2+\delta}) \quad \text{a.s.}$$

as $m \to \infty$.

It follows from (4.5) that

$$\sup_{x\in\mathbb{Z}} H((x,0),n) \le n^{1/4+\delta}$$

almost surely for any $\delta > 0$ and large n. Hence putting $m = n^{1/4+\delta}$, we obtain

$$\max_{|x|\le n^{1/4+\delta}} |V((x,0),n) - H((x,0),n)|$$

$$\le \max_{|x|\le n^{1/4+\delta}}\ \max_{\ell\le n^{1/4+\delta}} |V((x,0),\kappa(x,\ell)) - \ell| = O(n^{1/8+\delta}) \quad \text{a.s.}$$

as $n \to \infty$. This verifies the lemma and completes the proof of Theorem 1.2. □

5 Proof of Theorem 1.3

The proof of this theorem consists of establishing the next two lemmas. Note that as before, throughout this proof max is taken on the integers, even for Brownian local time $\eta(x,\cdot)$ as well.

Lemma 5.1 *On the probability space of* Theorem 1.1, *for any* $0<\varepsilon<1/4$ *and sufficiently small* $0<\delta<\varepsilon/2$, *as* $n\to\infty$, *we have*

$$\max_{|x|\le n^{1/4-\varepsilon}} |H((x,0),n)-H((0,0),n)| = O(n^{1/4-\delta}) \quad a.s. \tag{5.1}$$

Proof. First we prove for a Brownian local time $\eta(\cdot,\cdot)$ that, as $t\to\infty$,

$$\max_{|x|\le t^{1/2-\varepsilon}} |\eta(x,t)-\eta(0,t)| = O(t^{1/2-\delta}) \quad \text{a.s.} \tag{5.2}$$

Recall that $\tau(\cdot)$ stands for the inverse local time of $W(\cdot)$. Then (cf. Perkins [34], Bass and Griffin [4])

$$\mathbf{E}\left(e^{\lambda\eta(x,\tau(r))}\right) = \exp\left(\frac{\lambda r}{1-2\lambda|x|}\right), \quad \lambda<1/(2|x|).$$

Hence, with $\lambda=u/(4r|x|)$ and some $c>0$,

$$\mathbf{P}(\eta(x,\tau(r))-r>u) \le \exp\left(\frac{2\lambda^2 r|x|}{1-2\lambda|x|}-u\lambda\right) \le \exp\left(-c\frac{u^2}{r|x|}\right),$$

as long as $u\le r/2$. Similarly,

$$\mathbf{P}(r-\eta(x,\tau(r))>u) \le \exp\left(\frac{2\lambda^2 r|x|}{1+2\lambda|x|}-u\lambda\right) \le \exp\left(-c\frac{u^2}{r|x|}\right).$$

Consequently,

$$\mathbf{P}(\max_{|x|\le r^{1-\varepsilon}} |\eta(x,\tau(r))-r| > r^{1-\delta}) \le c_1 r^{1-\varepsilon}\exp\left(-cr^{\varepsilon-2\delta}\right)$$

for some $c_1>0$. Hence, if $\varepsilon>2\delta$, then by Borel-Cantelli lemma

$$\max_{|x|\le r^{1-\varepsilon}} |\eta(x,\tau(r))-r| = O(r^{1-\delta}) \quad \text{a.s.,} \quad r\to\infty. \tag{5.3}$$

Putting $r=\eta(0,t)$, we obtain

$$\max_{|x|\le(\eta(0,t))^{1-\varepsilon}} |\eta(x,\tau(\eta(0,t)))-\eta(0,t)| = O((\eta(0,t))^{1-\delta}) \quad \text{a.s.,} \quad t\to\infty. \tag{5.4}$$

Consequently, we have also

$$\max_{|x|\le t^{1/2-\varepsilon}} |\eta(x,\tau(\eta(0,t)))-\eta(0,t)| = O(t^{1/2-\delta}) \quad \text{a.s.} \tag{5.5}$$

as $t\to\infty$. Observe that

$$\eta(x,t)-\eta(0,t) = (\eta(x,t)-\eta(x,\tau(\eta(0,t))))$$
$$+(\eta(x,\tau(\eta(0,t)))-\eta(0,\tau(\eta(0,t)))) + (\eta(0,\tau(\eta(0,t)))-\eta(0,t)). \tag{5.6}$$

The first term in (5.6) being non-negative, and the last one being zero, we can conclude that

$$\eta(x,t)-\eta(0,t) \ge \eta(x,\tau(\eta(0,t)))-\eta(0,\tau(\eta(0,t))). \tag{5.7}$$

Similarly,

$$\eta(x,t)-\eta(0,t) = (\eta(x,t)-\eta(x,\tau(\eta(0,t)+1)))$$
$$+(\eta(x,\tau(\eta(0,t)+1))-\eta(0,\tau(\eta(0,t)+1))) + (\eta(0,\tau(\eta(0,t)+1))-\eta(0,t)). \tag{5.8}$$

Here the first term being non-positive, and the last term being 1, we arrive at

$$\eta(x,t)-\eta(0,t) \le \eta(x,\tau(\eta(0,t)+1))-\eta(0,\tau(\eta(0,t)+1))+1. \tag{5.9}$$

Taking maximums in (5.7) and (5.9), we obtain (5.2).

It follows from the assumed nearness (1.11) and applying the increment result (5.2) for $\eta_1(x,t)$, that for any $0<\varepsilon<1/4$ and sufficiently small $0<\delta<\varepsilon/2$, we have also

$$\max_{|x|\le n^{1/2-\varepsilon}} |\xi_1(x,n)-\xi_1(0,n)| \le \max_{|x|\le n^{1/2-\varepsilon}} \left|\xi_1(x,n)-\frac{1}{\sigma_1^2}\eta_1(x,n\sigma_1^2)\right|$$
$$+\max_{|x|\le n^{1/2-\varepsilon}} \left|\frac{1}{\sigma_1^2}\eta_1(x,n\sigma_1^2)-\frac{1}{\sigma_1^2}\eta_1(0,n\sigma_1^2)\right| + \left|\frac{1}{\sigma_1^2}\eta_1(0,n\sigma_1^2)-\xi_1(0,n)\right| = O(n^{1/2-\delta}) \quad \text{a.s.}$$

as $n\to\infty$.

Now if $T_N+\rho_2(N)\le n<T_N+\rho_2(N+1)$, then

$$H((x,0),n)=\xi_1(x,T_N).$$

Hence, we have almost surely, as $n\to\infty$,

$$\max_{|x|\le n^{1/4-\varepsilon}} |H((x,0),n)-H((0,0),n)| \le \max_{|x|\le n^{1/4-\varepsilon}} |\xi_1(x,T_N)-\xi_1(0,T_N)| = O(T_N^{1/2-\delta}) = O(n^{1/4-\delta}). \tag{5.10}$$

Since $T_{N+1}-T_N=O(\log N)$ a.s. for large N, we conclude

$$\sup_{x\in\mathbb{Z}} |\xi_1(x,T_{N+1})-\xi_1(x,T_N)| = O(\log N) \quad \text{a.s.,} \quad N\to\infty.$$

Consequently, we have (5.10) for $T_N+\rho_2(N+1)\le n<T_{N+1}+\rho_2(N+1)$ as well. This also proves Lemma 5.1. □

Lemma 5.2 *On the probability space of* Theorem 1.1, *for any* $0 < \varepsilon < 1/4$ *and sufficiently small* $0 < \delta < \varepsilon/2$, *as* $n \to \infty$, *we have*

$$\max_{|x|\le n^{1/4-\varepsilon}} \max_{0<|y|\le n^{1/4-\varepsilon}} |\Xi((x,y),n) - V((x,0),n)| = O(n^{1/4-\delta}) \quad \text{a.s.} \tag{5.11}$$

Proof. Let $\theta(x,0)$ be the time of the first visit of $\mathbf{C}(\cdot)$ to $(x,0)$, and for $\ell \ge 1$ let $\theta(x,\ell)$ be the time of the ℓ-th return of $\mathbf{C}(\cdot)$ to $(x,0)$. Then

$$\Xi((x,y),\theta(x,\ell)) = \sum_{i=1}^{\ell} (\Xi((x,y),\theta(x,i)) - \Xi((x,y),\theta(x,i-1))),$$

a sum of i.i.d. random variables with distribution given in Lemma 2.1, with expectation 1/2. Estimating the common moment generating function, we get by exponential Markov inequality

$$\mathbf{P}(\max_{\ell\le L} |\Xi((x,y),\theta(x,\ell)) - \ell/2| \ge u) \le L \exp\left(\frac{L\lambda^2}{\gamma_2(y)} - \lambda u\right).$$

By selecting $\lambda = \dfrac{u\gamma_2(y)}{2L}$ and applying (2.5), for $u < L$ and some $c > 0$, we get

$$\mathbf{P}(\max_{\ell\le L} |\Xi((x,y),\theta(x,\ell)) - \ell/2| \ge u) \le L \exp\left(-c\frac{u^2}{|y|L}\right).$$

Putting $u = L^{1-\delta}$, we obtain

$$\mathbf{P}(\max_{|x|\le L^{1-\varepsilon}} \max_{0<|y|\le L^{1-\varepsilon}} \max_{\ell\le L} (\Xi((x,y),\theta(x,\ell)) - \ell/2) \ge L^{1-\delta}) \le c_1 L^3 \exp\left(-cL^{\varepsilon-2\delta}\right),$$

with some $c_1 > 0$. Hence, selecting $\delta < \varepsilon/2$, by Borel-Cantelli lemma we arrive at

$$\max_{|x|\le L^{1-\varepsilon}} \max_{0<|y|\le L^{1-\varepsilon}} \max_{\ell\le L} |\Xi((x,y),\theta(x,\ell)) - \ell/2|$$

$$= \max_{|x|\le L^{1-\varepsilon}} \max_{0<|y|\le L^{1-\varepsilon}} \max_{\ell\le L} |\Xi((x,y),\theta(x,\ell)) - \frac{1}{2}\Xi((x,0),\theta(x,\ell))| = O(L^{1-\delta}) \quad \text{a.s.} \tag{5.12}$$

as $L \to \infty$.

We will now use (5.12) via letting

$$L = \sup_{x\in\mathbb{Z}} \Xi((x,0),n).$$

By Theorem 1.2 and Theorem C we have that for any $\varepsilon_1 > 0$, as $n \to \infty$,

$$\sup_{x\in\mathbb{Z}} \Xi((x,0),n) = O(n^{1/4+\varepsilon_1}) \quad \text{a.s.}$$

On choosing δ and ε_1 small enough, we conclude

$$\max_{|x|\le n^{1/4-\varepsilon}} \max_{0<|y|\le n^{1/4-\varepsilon}} \max_{\ell\le n^{1/4+\varepsilon_1}} |\Xi((x,y),\theta(x,\ell)) - \frac{1}{2}\Xi((x,0),\theta(x,\ell))| = O(n^{1/4-\varepsilon_2}) \quad \text{a.s.}$$

as $n\to\infty$. Consequently, by Lemma 4.1, we also have

$$\max_{|x|\le n^{1/4-\varepsilon}} \max_{0<|y|\le n^{1/4-\varepsilon}} \max_{\ell\le n^{1/4+\varepsilon_1}} |\Xi((x,y),\theta(x,\ell)) - V((x,0),\theta(x,\ell))| = O(n^{1/4-\varepsilon_2}) \quad \text{a.s.} \tag{5.13}$$

as $n\to\infty$.

For each $n\ge 1$, let $\theta_n\le n$ be the last visit of $\mathbf{C}(n)$ on the $x-$axis before time n, and let $\theta_n^*>n$ be its first visit on the $x-$axis after time n.

Observe that if $\mathbf{C}(n)=(x,y)$ with $y\ne 0$, then $C_1(\theta_n)=C_1(n)=C_1(\theta_n^*)=x$, thus for any $x'\ne x$ the local times $\Xi((x',y),\cdot)$ and $V((x',0),\cdot)$ do not change in the interval $[\theta_n,\theta_n^*)$. Furthermore, if $\mathbf{C}(n)=(x,0)$, then $\theta_n=n$. Consequently, we only have to deal with the case of $x=C_1(n)$ when $y=C_2(n)\ne 0$. We have

$$\begin{aligned} &V((x,0),n)-\Xi((x,y),n)= \\ &(V((x,0),\theta_n)-\Xi((x,y),\theta_n))+(V((x,0),n)-V((x,0),\theta_n))+(\Xi((x,y),\theta_n)-\Xi((x,y),n)) \\ &\le V((x,0),\theta_n)-\Xi((x,y),\theta_n), \end{aligned} \tag{5.14}$$

as the second term of the three summands in (5.14) is zero and the last one is non-positive.

We have also

$$\begin{aligned} &V((x,0),n)-\Xi((x,y),n)= \\ &(V((x,0),\theta_n^*)-\Xi((x,y),\theta_n^*))+(V((x,0),n)-V((x,0),\theta_n^*))+(\Xi((x,y),\theta_n^*)-\Xi((x,y),n)) \\ &\ge (V((x,0),\theta_n^*)-\Xi((x,y),\theta_n^*))-1, \end{aligned} \tag{5.15}$$

as the second term of the three summands in (5.15) is equal to -1, and the last one is non-negative. Combining (5.13)-(5.15), we get Lemma 5.2. □

This also completes the proof of Theorem 1.3. □

6 Examples

In this section we discuss a number of works, as examples, that deal with various joint strong invariance principles for integer valued random walks and their local times. Naturally, our specific set of examples may not be exhaustive. Also, the original conditions of these invariance principles are kept unchanged or, on occasions, are replaced by equivalent ones. However, we have not made any attempt to improve them.

Example 1. In 1981 Révész in [36] proved that for simple symmetric walk (which clearly satisfies conditions (i)-(iii)), (1.10) and (1.11) hold simultaneously with $\alpha = \beta = 1/4$. Thus, for simple symmetric random walk, our Theorems 1.1 and 1.2 and Corollary 1.1 hold with $\alpha = \alpha^* = \beta = \beta^* = \vartheta = 1/4$.

Example 2. In 1983 Csáki and Révész [20] proved that under conditions (i) and (iii), if we have $m + 1$ moments with $m > 6$, then (1.10) holds with $\alpha = 1/4$, simultaneously with (1.11) with $\beta = \beta^* = 1/4 + 3/(2m)$. Thus, under these conditions, our Theorems 1.1 and 1.2 and Corollary 1.1 hold with $\alpha^* = 1/4$, $\vartheta = 1/4 + 3/(2m)$.

Example 3. In 1989 Borodin [10] proved that under condition (i) with eight moments, and with

- (iii)* $\quad |\psi(\theta)| = |\sum_{k=-\infty}^{\infty} e^{i\theta k} p_j(k)| = 1$ if and only if θ is an integer multiple of 2π,

instead of (iii), (1.10) and (1.11) hold simultaneously with $\alpha = \beta = 1/4$. Thus, under these conditions, our Theorems 1.1 and 1.2 and Corollary 1.1 hold with $\alpha^* = \beta^* = \vartheta = \frac{1}{4}$. We note in passing that condition (iii)* is equivalent to saying that the random walk in hand is strongly aperiodic (cf. Spitzer [39], p.75).

Example 4. In 1993 Bass and Khoshnevisan [5] proved that under conditions (i) and (iii)*, and assuming more than five moments in case of $\sigma_1 = \sigma_2 = 1$, (1.10) and (1.11) hold simultaneously, respectively with $\alpha = 1/4$ and $\beta = 1/4$. Thus under these conditions our Theorems 1.1 and 1.2 and Corollary 1.1 hold with $\alpha = \alpha^* = \beta = \beta^* = \vartheta = 1/4$.

Example 5. A further result of Bass and Khoshnevisan in 1993, namely Theorem 3.2 in [6], implies that, under the conditions (i)-(iii)* with $\sigma_1 = \sigma_2 = 1$, and $m \geq 3$ moments, (1.10) and (1.11) hold simultaneously, respectively with $\alpha = 1/m$ and $\beta = \beta^* = 3/10$. Thus, under these conditions, our Theorems 1.1 and 1.2 and Corollary 1.1 hold with $\alpha = 1/m$, $\alpha^* = \max(1/m, 1/4)$, and $\vartheta = \max(1/m, 3/10)$.

7 Further comments, results and remarks

First we note that, in the case of Example 1 that is based on the simultaneous strong approximation result of Révész [36] for a simple symmetric random walk and that of its local time, the obtained rates are nearly best possible (cf. Csörgő and Horváth [21]). As of the other examples, their assumptions may very well be improvable for obtaining their strong approximations. This however remains an open problem.

The weak convergence conclusions that are spelled out in Section 1 are based on the strong approximation results of Theorems 1.1, 1.2 and Corollary 1.1. We note however that in probability nearness versions of these approximations would suffice for our approach to proving functional limit theorems, i.e., weak convergence, for the various processes in hand. Moreover, these in probability nearnesses in various sup norm metrics may very well be provable under weaker conditions than those used for their present strong versions. This again remains an open problem in general, and also in the case of Examples 2–5 in particular, for dealing with weak convergence in their context.

A few more remarks in view of Theorem 1.2. It follows from (1.13) that our random walk $\mathbf{C}(\cdot)$ on the 2-dimensional comb lattice $\mathbb{C}^2$ spends about $n^{1/2}$ portion of its time up to n on the x-axis. The rest of its time is spent away from this axis. It is of interest to explore how far away it may go vertically from any particular value of x, as well as from a collection of x values, on the x-axis. More precisely, we are interested in establishing lower and upper bounds for

$$\max_{k\le n:\, C_1(k)=x} |C_2(k)| \qquad \text{and} \qquad \max_{k\le n:\, |C_1(k)|\le x_n} |C_2(k)|. \tag{7.1}$$

In the latter of these two quantities, the magnitude of the size x_n is of special interest on its own, and also in terms of the size of its possible contribution to the desired second set of upper and lower bounds, as compared to those of the first set.

First we note that, in view of the approximation of Theorem 1.1 for $C_2(n)$ by a standard Brownian motion, for an unrestricted maximal behaviour of $C_2(n)$, as compared to the restricted ones in (7.1), with any $\varepsilon > 0$, we have the following immediate almost sure upper and lower bounds for large n.

$$n^{1/2-\varepsilon} \le \max_{0\le k\le n} |C_2(k)| \le n^{1/2+\varepsilon}. \tag{7.2}$$

On the other hand, for the restricted maximal quantities in (7.1), we are now to establish the following bounds.

Proposition 7.1 *Under the conditions of* Theorem 1.1*, with any $\varepsilon > 0$, we have almost surely for large n*

$$n^{1/4-\varepsilon} \le \max_{k\le n:\, C_1(k)=x} |C_2(k)| \le n^{1/4+\varepsilon} \tag{7.3}$$

with any fixed $x \in \mathbb{Z}$, and

$$x_n n^{1/4-\varepsilon} \le \max_{k\le n:\, |C_1(k)|\le x_n} |C_2(k)| \le x_n n^{1/4+\varepsilon}, \tag{7.4}$$

where $x_n \le n^{1/4-\delta}$ with some $\delta \ge 0$.

Remark 4 *First we note that the upper bound in (7.4) is valid without any restriction on x_n. The assumption that $x_n \le n^{1/4-\delta}$, with $\delta \ge 0$, is to have a correct lower bound as well. In particular, with $x_n = n^{1/4-\delta}$, $\delta > 0$, (7.4) reads as follows,*

$$n^{1/2-\delta-\varepsilon} \le \max_{k\le n:\, |C_1(k)|\le n^{1/4-\delta}} |C_2(k)| \le n^{1/2-\delta+\varepsilon}. \tag{7.5}$$

Thus, on taking $\varepsilon > 0$ small enough, both bounds in (7.5) are seen to fluctuate around the value $n^{1/2-\delta}$ for any $\delta > 0$, i.e., unlike in the unrestricted maximal path behaviour of $C_2(\cdot)$ as in (7.2), *with $\delta > 0$, the bound $n^{1/2-\varepsilon}$ cannot be reached in* (7.5) *on taking $\varepsilon > 0$ small enough. In the same vein, we have also*

$$\liminf_{n\to\infty} \frac{\max_{\{k\le n:\, |C_1(k)|\le n^{1/4-\delta}\}} |C_2(k)|}{n^{1/2}} = 0 \qquad a.s.,$$

and

$$\limsup_{n\to\infty} \frac{\max_{\{k\le n:\, |C_1(k)|\le n^{1/4-\delta}\}} |C_2(k)|}{n^{1/2}} = 0 \qquad a.s.$$

On the other hand, the assertion in (7.5) *continues to hold true with $\delta = 0$ as well, i.e., in this case, the bounds in* (7.2) *and* (7.5) *coincide. Moreover, in this case,*

$$\liminf_{n\to\infty} \frac{\max_{\{k\le n:\, |C_1(k)|\le n^{1/4}\}} |C_2(k)|}{n^{1/2}} = 0 \qquad a.s.,$$

just like before, however, we now have that

$$\limsup_{n\to\infty} \frac{\max_{\{k\le n:\, |C_1(k)|\le n^{1/4}\}} |C_2(k)|}{n^{1/2}} = \infty \qquad a.s.$$

Remark 5 *We are to compare now the two assertions of* Proposition 7.1*. First, for each fixed x as in* (7.3)*, like for example on the y-axis, $C_2(\cdot)$ does almost surely exceed the bound $n^{1/4-\varepsilon}$, however the bound $n^{1/4+\varepsilon}$ cannot be reached. In view of this,* (7.4) *via* (7.5) *tells us that for a large enough collection of x values on the x-axis, $C_2(\cdot)$ does get away more and more from this axis as the distance x_n of $C_1(\cdot)$ from zero increases, so that, eventually, for any $\delta \ge 0$, it exceeds the bound $n^{1/2-\delta-\varepsilon}$ with any $\varepsilon > 0$.*

Proof of Proposition 7.1 It follows from Theorems 1 and 3 of Földes [28] that, for a standard Brownian motion $W(\cdot)$ and large T, we have almost surely

$$T^{1-\varepsilon} \le \sup_{0\le s\le \tau(T)} |W(s)| \le T^{1+\varepsilon}$$

with any $0 < \varepsilon < 1$, where $\tau(\cdot)$ is the inverse local time process as in (2.2). Using now the assumption (1.10) in combination with Lemma B and Theorem F, we obtain the almost sure bounds with any $0 < \varepsilon < 1$

$$N^{1-\varepsilon} \le \max_{i\le \rho(N)} |S(i)| \le N^{1+\varepsilon} \tag{7.6}$$

for large N.

Now recall that $V((x,0),n) =: V(x)$ as in (4.2) is the number of vertical returns of $\mathbf{C}(\cdot)$ to $(x,0)$ up to time n which, in turn, equals the number of excursions of $S_2(\cdot)$, corresponding to these vertical returns, up to time n, Then, with any $0 < \varepsilon < 1$, we can also conclude from (7.6) that

$$V(x)^{1-\varepsilon} \leq \max_{k \leq n: C_1(k)=x} |C_2(k)| \leq V(x)^{1+\varepsilon}. \tag{7.7}$$

To estimate $V(x)$ now, we combine Lemma 4.1 with the law of the iterated logarithm as stated in (1.21) and, on using also Corollary 1.5, we get

$$n^{1/4-\varepsilon_1} \leq V((x,0),n) \leq n^{1/4+\varepsilon_1} \tag{7.8}$$

with any $0 < \varepsilon_1 < 1/4$, almost surely for large n. Now, the statements of (7.7) and (7.8) together result in (7.3).

In order to prove (7.4), we apply (7.6) with

$$N = \sum_{|x| \leq x_n} V((x,0),n),$$

that is the total number of vertical returns to x in the interval $-x_n \leq x \leq x_n$, which is also the number N of corresponding excursions of $S_2(\cdot)$. Consequently, with any $0 < \varepsilon < 1$, we can also conclude

$$\left(\sum_{|x| \leq x_n} V((x,0),n) \right)^{1-\varepsilon} \leq \max_{k \leq n: |C_1(k)| \leq x_n} |C_2(k)| \leq \left(\sum_{|x| \leq x_n} V((x,0),n) \right)^{1+\varepsilon}, \tag{7.9}$$

and, clearly,

$$x_n \min_{|x| \leq x_n} V((x,0),n) \leq \sum_{|x| \leq x_n} V((x,0),n) \leq (2x_n+1) \max_{|x| \leq x_n} V((x,0),n). \tag{7.10}$$

We also note that the estimate of $V((x,0),n)$ as in (7.8) also holds true uniformly in x over the interval $(-x_n, x_n)$, on account of the very same information that was already used in arguing (7.8) itself. Consequently, with the latter in mind, in view of (7.10) and (7.9), we arrive at (7.4) as well. This also completes the proof of Proposition 7.1. □

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- the factor 1/2 of the right hand side of (1.8) by χ $(0 < \chi < 1)$
- the factor 1/2 of the right hand side of (1.9) by $1 - \chi$.

We fully agree with him/her saying that this replacement would not make too much difficulties, and that the same results would continue to hold with appropriate changes in their corresponding constants. Indeed, this is so. For example, the factor 1/2 on the left hand side of (1.15) would be changed to χ, as well as some other constants along these lines.

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