On a class of additive functionals of two-dimensional Brownian motion and random walk

Dedicated to Pál Révész on the occasion of his 65th birthday

Endre Csáki [a]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, *H*-1364, *Hungary. E-mail address*: csaki@math-inst.hu

Miklós Csörgő [b]
School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6. *E-mail address*: mcsorgo@math.carleton.ca

Antónia Földes [c]
City University of New York, 2800 *Victory Blvd., Staten Island, New York* 10314, *U.S.A. E-mail address*: afoldes@email.gc.cuny.edu

Zhan Shi [d]
Laboratoire de Probabilités UMR 7599, *Université Paris VI,* 4 *Place Jussieu, F*-75252 *Paris Cedex* 05, *France. E-mail address*: zhan@proba.jussieu.fr

Abstract: Sample path properties of a class of additive functionals of two-dimensional Brownian motion and random walk are studied.

AMS 1991 *Subject Classification*: *Primary* 60J15; *Secondary* 60F15, 60J55.

Keywords: Additive functional, Brownian motion, random walk, strong approximation.

[a]Research supported by the Hungarian National Foundation for Scientific Research, Grant No. T 019346 and T 029621, and by a Fellowship of the Paul Erdős Summer Research Center of Mathematics, Budapest.

[b]Research supported by an NSERC Canada Grant at Carleton University, Ottawa, and by a Paul Erdős Visiting Professorship of the Paul Erdős Summer Research Center of Mathematics, Budapest.

[c]Research supported by a PSC CUNY Grant, No. 6-69413, and by a Fellowship of the Paul Erdős Summer Research Center of Mathematics, Budapest.

[d]Research supported by a Fellowship of the Paul Erdős Summer Research Center of Mathematics, Budapest.

1. Introduction and main results

Let $\{W(t),\, t \geq 0\}$ be a standard Wiener process (Brownian motion) with local time $L(x,t)$ and for $\alpha < 3/2$ define

$$Y_\alpha(t) := \int_0^t \frac{ds}{W^\alpha(s)} = \int_0^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx.$$

Here $z^\alpha = |z|^\alpha \operatorname{sgn}(z)$ and the first integral is defined as Cauchy principal value for $1 \leq \alpha < 3/2$. In our previous works [CsCsFS, 99a and 99b], we were interested in the almost sure path properties of $Y_\alpha(t)$ and its corresponding counterparts for the case of simple symmetric random walk. The general case $0 < \alpha < 3/2$ was studied in [CsCsFS, 99a], and the case $\alpha = 1$ in [CsCsFS, 99b]. Moreover, in [CsCsFS, 99a], for $\alpha > 3/2$, we investigated the properties of the modified process

$$Y_\alpha^*(t) := \int_0^t \frac{1}{W^\alpha(s)} \mathbf{1}_{\{|W(s)| \geq 1\}}\, ds.$$

In this note we discuss the corresponding two-dimensional problem.

Let $\{\mathbf{W}(t) := (W_1(t), W_2(t)),\, t \geq 0\}$ be a two-dimensional Wiener process, where $W_1(t)$ and $W_2(t)$ are two independent one-dimensional Wiener processes, with $W_1(0) = W_2(0) = 0$. Put

$$R(t) := \|\mathbf{W}(t)\| = \sqrt{W_1^2(t) + W_2^2(t)}\,.$$

It is well-known that $\{R(t),\, t \geq 0\}$ is a two-dimensional Bessel process. We are interested in the additive functional

$$Z_\alpha(t) := \int_0^t \frac{ds}{R^\alpha(s)}, \tag{1.1}$$

the critical case being $\alpha = 2$ (instead of $3/2$). It can be seen that the integral in (1.1) converges for $\alpha < 2$, but diverges for $\alpha \geq 2$ almost surely. In the latter case we define the modified process

$$Z_\alpha^*(t) := \int_0^t \frac{1}{R^\alpha(s)} \mathbf{1}_{\{R(s) \geq 1\}}\, ds. \tag{1.2}$$

Considering the random walk counterpart, let $\{\mathbf{S}_n\}_{n=1}^\infty$ be a simple symmetric random walk on the integer lattice $\mathbf{Z}^2$, i.e. $\mathbf{S}_n = \sum_{k=1}^n \mathbf{X}_k$, where the random variables $\mathbf{X}_i$, $i = 1, 2, \ldots$ are i.i.d., with

$$\mathbf{P}(\mathbf{X}_1 = (0,1)) = \mathbf{P}(\mathbf{X}_1 = (0,-1)) = \mathbf{P}(\mathbf{X}_1 = (1,0)) = \mathbf{P}(\mathbf{X}_1 = (-1,0)) = \frac{1}{4}.$$

We also propose to study the discrete process

$$U_\alpha(n) := \sum_{k=1}^{n} \frac{1}{\|\mathbf{S}_k\|^\alpha} \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}\}}. \tag{1.3}$$

Define

$$\xi(\mathbf{x}, n) := \#\{k;\ 1 \le k \le n,\ \ \mathbf{S}_k = \mathbf{x}\},$$

for any lattice point $\mathbf{x}$ of $\mathbf{Z}^2$. This is the local time process of $\{\mathbf{S}_n\}_{n=1}^{\infty}$. Let furthermore $\{\rho_n,\ n \ge 0\}$ denote the consecutive return times of the walk to zero, that is

$$\rho_0 := 0, \qquad \rho_n := \min\{k > \rho_{n-1}, \quad \mathbf{S}_k = \mathbf{0}\}.$$

Our first result concerns the case $0 < \alpha < 2$, for which we show that the processes $Z_\alpha(\cdot)$ and $U_\alpha(\cdot)$ are close to each other.

Theorem 1.1. *Let $0 < \alpha < 2$. Then on a suitable probability space one can define a two-dimensional Wiener process $\mathbf{W}(\cdot)$ with the additive functional $Z_\alpha(\cdot)$ as in (1.1) and a two-dimensional simple symmetric random walk $\mathbf{S}$. with the additive functional $U_\alpha(\cdot)$ as in (1.3) such that, as $n \to \infty$, for any $\varepsilon > 0$*

$$|2^{-\alpha/2} U_\alpha(n) - Z_\alpha(n)| = \begin{cases} \mathcal{O}(n^{1-\alpha+\varepsilon}), & \text{a.s.,} \quad \text{if } 0 < \alpha < 1, \\ \mathcal{O}(\log^{4+\varepsilon} n), & \text{a.s.,} \quad \text{if } \alpha = 1, \\ \mathcal{O}(\log^{2+\varepsilon} n), & \text{a.s.,} \quad \text{if } 1 < \alpha < 2. \end{cases} \tag{1.4}$$

We note that by self-similarity, $Z_\alpha(t)$ is of order $t^{1-\alpha/2}$, so the rates in (1.4) are useful.

In the case when $\alpha > 2$, the corresponding processes, Z_α^* and U_α are not close to each other. Instead, as in the one-dimensional case, we show separately, that both processes, suitably centered, are close to certain iterated processes.

Theorem 1.2. *Let $\alpha > 2$. There exists a probability space where one can define*

(i) *a two-dimensional simple symmetric random walk $\{\mathbf{S}_n\}_0^\infty$ with its local time process $\xi(\mathbf{x}, n)$, and with the corresponding additive functional process $\{U_\alpha(n),\ n = 1, 2...\}$ as in (1.3);*

(ii) *a process $\{\xi^{(1)}(\mathbf{0}, n),\ n = 1, 2, ...\} \stackrel{\mathcal{D}}{=} \{\xi(\mathbf{0}, n),\ n = 1, 2, ...\}$;*

(iii) *a standard Wiener process $\{W(t),\ t \ge 0\}$, independent of $\{\xi^{(1)}(\mathbf{0}, n),\ n = 1, 2, ...\}$;*

such that, for some $\varepsilon > 0$, as $n \to \infty$,

$$U_\alpha(n) - \bar{f}_\alpha^D \xi(\mathbf{0}, n) = \sigma_\alpha^D W(\xi^{(1)}(\mathbf{0}, n)) + \mathcal{O}(\log^{1/2-\varepsilon} n), \qquad \text{a.s.},$$

where

$$\bar{f}_\alpha^D := \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{1}{\|x\|^\alpha}, \qquad \sigma_\alpha^D := \sqrt{\mathrm{Var}(U_\alpha(\rho_1))}\,.$$

Moreover, the processes $\xi(\mathbf{0}, n)$ and $\xi^{(1)}(\mathbf{0}, n)$ can be chosen such that for some $\delta > 0$,

$$\xi(\mathbf{0}, n) = \xi^{(1)}(\mathbf{0}, n) + \mathcal{O}(\log^{1-\delta} n), \qquad \text{a.s.}$$

We would also like to prove a similar result for the corresponding problem of a two-dimensional Wiener process. Formulating a claim in this case needs some careful consideration as points are polar for the two-dimensional Wiener process, so that it has no local time. However its norm $R(t)$ does have a local time which could perfectly fit in our desired theorem. For technical reasons we introduce a slightly different definition as our modified local time for $R(t)$. This idea is not new, similar but more general concepts were already discussed e.g. in Burdzy *et al.* [BPY, 90]. Define two increasing sequences of stopping times $\{\sigma_n,\, n \geq 0\}$ and $\{\theta_n,\, n \geq 1\}$ by

$$\sigma_0 := \inf\{t > 0;\ R(t) = 1\}, \tag{1.5}$$

$$\theta_n := \inf\{t > \sigma_{n-1};\ R(t) = 2\}, \qquad n \geq 1, \tag{1.6}$$

$$\sigma_n := \inf\{t > \theta_n;\ R(t) = 1\}, \qquad n \geq 1, \tag{1.7}$$

$$\tau_i := \sigma_i - \sigma_{i-1}, \qquad i = 1, 2, \ldots \tag{1.8}$$

Let furthermore

$$\eta(t) = \max\{n;\ \sigma_n - \sigma_0 \leq t\}. \tag{1.9}$$

Theorem 1.3. *Let $\alpha > 2$. There exists a probability space where one can define*

(i) *a two-dimensional Wiener process $\{\mathbf{W}(t),\, t \geq 0\}$ with its corresponding additive functional process $\{Z_\alpha^*(t),\, t \geq 0\}$ as in (1.2), and with $\{\eta(t),\, t > 0\}$ as in (1.9);*

(ii) *a process $\{\eta^{(1)}(t),\, t > 0\} \stackrel{\mathcal{D}}{=} \{\eta(t),\, t > 0\}$;*

(iii) *a standard Wiener process $\{W(t),\, t \geq 0\}$, independent of $\{\eta^{(1)}(t),\, t \geq 0\}$;*

such that for some $\varepsilon > 0$,

$$Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t) = \sigma_\alpha^W W(\eta^{(1)}(t)) + \mathcal{O}(\log^{1/2-\varepsilon} t), \qquad \text{a.s.,} \tag{1.10}$$

where

$$\bar{f}_\alpha^W := \frac{2}{(\alpha-2)^2}, \qquad \left(\sigma_\alpha^W\right)^2 := \mathbf{E}\left(\int_1^\infty \frac{1}{x^\alpha}\ell(x,\tau_1)\,dx\right)^2 - \left(\bar{f}_\alpha^W\right)^2,$$

and $\ell(x,t)$ *is the local time at time* t *and position* x *of* $\{R(t),\, t \geq 0\}$. *Furthermore,* η *and* $\eta^{(1)}$ *can be chosen such that for some* $\delta > 0$,

$$\eta(t) = \eta^{(1)}(t) + \mathcal{O}(\log^{1-\delta} t), \qquad \text{a.s.} \tag{1.11}$$

The rest of the paper is organized as follows. Section 2 is devoted to a collection of known results which will be used later on. Theorems 1.1–1.3 are proved in Sections 3–5 respectively. In Section 6 we present some further weak and strong convergence results, as consequences of our theorems.

2. Preliminary results

In this section we list some results which will be used in the proofs of the theorems.

Let $\{\mathbf{S}_n,\, n \geq 1\}$ be a simple symmetric random walk on the plane, and let $\{\rho_n,\, n \geq 0\}$ denote, as before, the consecutive return times to zero.

For any real valued function $f(\cdot)$ on $\mathbf{Z}^2$ we define the additve functional

$$A(f,n) = \sum_{k=1}^n f(\mathbf{S}_k).$$

We assume that for some $\delta > 0$,

$$\mathbf{E}\left(\left|\sum_{k=1}^{\rho_1} f(\mathbf{S}_k)\right|^{2+\delta}\right) < \infty. \tag{2.1}$$

It is easy to compute the expectation of $\sum_{k=1}^{\rho_1} f(\mathbf{S}_k)$. Indeed, by Spitzer [S, 64], $\mathbf{E}(\xi(\mathbf{x},\rho_1)) = 1$ for any $\mathbf{x} \in \mathbf{Z}^2$, and hence we have

$$\mathbf{E}\left(\sum_{k=1}^{\rho_1} f(\mathbf{S}_k)\right) = \mathbf{E}\sum_{\mathbf{x}\in\mathbf{Z}^2} \xi(\mathbf{x},\rho_1) f(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbf{Z}^2} f(\mathbf{x}) =: \bar{f}.$$

Now we recall a few known results that will be of use later on.

Theorem A (Csáki and Földes [CsF, 99]). *Let $(X_i,\tau_i)_{i=1}^{\infty}$ be a sequence of i.i.d. vectors, such that $\tau_i \geq 0$ and*

$$\mathbf{P}(|X_i| > x) < \frac{c}{x^{\beta}}, \qquad \mathbf{P}(\tau_i > x) \leq \frac{1}{h(x)} \tag{2.2}$$

for x large enough, where $\beta > 2$, $c > 0$ and $h(x)$ is slowly varying at infinity, increasing with $\lim_{x\to\infty} h(x) = +\infty$. Then on an appropriate probability space one can construct two independent copies $(X_i^{(1)},\tau_i^{(1)})_{i=1}^{\infty}$ and $(X_i^{(2)},\tau_i^{(2)})_{i=1}^{\infty}$, together with $(X_i,\tau_i)_{i=1}^{\infty}$ such that

$$\begin{aligned}
(S_n,\rho_n)_{n=1}^{\infty} &\overset{\mathcal{D}}{=} (S_n^{(j)},\rho_n^{(j)})_{n=1}^{\infty}, \qquad j=1,2,\\
\sup_{k\leq n} |S_k - S_k^{(2)}| &= \mathcal{O}(n^{1/\beta^*}), \qquad \text{a.s.},\\
\sup_{k\leq n} |\rho_k - \rho_k^{(1)}| &= \mathcal{O}\left(h^*(n^{\gamma})\right), \qquad \text{a.s.},
\end{aligned}$$

as $n\to\infty$, where $S_k^{(j)} = \sum_{i=1}^{k} X_i^{(j)}$, $\rho_k^{(j)} = \sum_{i=1}^{k}\tau_i^{(j)}$, $S_k = \sum_{i=1}^{k} X_i$, $\rho_k = \sum_{i=1}^{k}\tau_i$, $\gamma < 1$, $\beta^* > 2$, and $h^*(\cdot)$ is the inverse of $h(\cdot)$.

The next result is an application of Theorem A for simple symmetric random walk in $\mathbf{Z}^2$.

Theorem B (Csáki and Földes [CsF, 99]). *For a simple symmetric random walk $\{\mathbf{S}_n\}_1^{\infty}$ in $\mathbf{Z}^2$ and any real valued function $f(\cdot)$ for which (2.1) holds, there exists a probability space where we can redefine $\{\mathbf{S}_n\}_1^{\infty}$ together with its local time process $\xi(\mathbf{0},n)$ and with the corresponding additive functional $A(f,n)$ in such a way, that on the same probability space there exist*

(i) *a standard Wiener process $\{W(t),\, t\geq 0\}$,*

(ii) *and a process*

$$\{\xi^{(1)}(\mathbf{0},n),\, n=0,1,2,...\} \overset{\mathcal{D}}{=} \{\xi(\mathbf{0},n),\, n=0,1,2,...\}$$

such that $\{W(t),\, t\geq 0\}$ and $\{\xi^{(1)}(\mathbf{0},n),\, n=0,1,2,...\}$ are independent and, as $n\to\infty$, we have

$$A(f,n) - \bar{f}\xi(\mathbf{0},n) = \sigma W(\xi^{(1)}(\mathbf{0},n)) + \mathcal{O}(\log^{\kappa} n), \qquad \text{a.s.},$$

and

$$|\xi^{(1)}(\mathbf{0},n) - \xi(\mathbf{0},n)| = \mathcal{O}(\log^{1-\delta} n), \qquad \text{a.s.},$$

where $\sigma = \sqrt{\mathrm{Var}(\sum_{k=1}^{\rho_1} f(\mathbf{S}_k))}$, $\kappa < 1/2$, and $\delta > 0$.

Concerning the moments of the local time of $\xi(\mathbf{x}, \rho_1)$ we quote the following result.

Theorem C (Révész [R, 90]). *For any integer* $m \geq 1$,

$$|\mathbf{E}\,(\xi(\mathbf{x}, \rho_1) - 1)^m\,| \leq q + m!p^{1-m}q,$$

where

$$p := \mathbf{P}(\mathbf{0} \rightsquigarrow \mathbf{x}) = \mathbf{P}(\{\mathbf{S}_n\} \text{ hits } \mathbf{x} \text{ before returning to } \mathbf{0}),$$

and $q = 1 - p$.

The order of magnitude of p is well-known, namely we have

Theorem D (Spitzer [S, 64], pp. 117, 124, 125, and Révész, [R, 90] p. 219). *As* $\|\mathbf{x}\|$ *goes to infinity,*

$$\mathbf{P}(\mathbf{0} \rightsquigarrow \mathbf{x}) = \frac{\pi + o(1)}{4 \log \|\mathbf{x}\|}.$$

The next theorem concerns the asymptotics of $\xi(\mathbf{0}, n)$.

Theorem E (Dvoretzky and Erdős [DE, 51], Erdős and Taylor [ET, 60]).

$$\lim_{n\to\infty} \mathbf{P}(\xi(\mathbf{0}, n) < x \log n) = 1 - e^{-\pi x}.$$

We also have

$$\limsup_{n\to\infty} \frac{\xi(\mathbf{0}, n)}{\log n \log_3 n} = \frac{1}{\pi}, \qquad \text{a.s.}$$

$$\limsup_{n\to\infty} \frac{\sup_{\mathbf{x}} \xi(\mathbf{x}, n)}{(\log n)^2} \leq \frac{1}{\pi}, \qquad \text{a.s.}$$

Furthermore, as n *goes to infinity,*

$$\mathbf{P}(\rho_1 > n) = \mathbf{P}(\xi(\mathbf{0}, n) = 0) = \frac{\pi}{\log n} + \mathcal{O}((\log n)^{-2}).$$

Now we collect certain results for the two-dimensional Bessel process $\{R(t),\, t \geq 0\}$ (with $R(0) = 0$). Let $\{\sigma_n,\, n \geq 0\}$ and $\{\theta_n,\, n \geq 1\}$ be as in (1.5)–(1.7). It is a consequence of the strong Markov property that $\{\sigma_i - \sigma_{i-1}\}_1^\infty$ is an i.i.d. sequence of random variables, for which we quote the following result.

Theorem F (Csáki et al. [CsFRS, 98]). *We have,*

$$\mathbf{P}(\sigma_1 - \sigma_0 > x) = \frac{2\log 2}{\log x} + \mathcal{O}(\log^{-2} x),$$

as $x \to \infty$.

We also need the following iterated logarithm law.

Theorem G (Csáki et al. [CsFRS, 98]). *Let* $\{\eta(t),\, t > 0\}$ *be as in* (1.9). *Then*

$$\limsup_{t\to\infty} \frac{\eta(t)}{(\log t)\log_3 t} = \frac{1}{2\log 2}, \qquad \text{a.s.} \tag{2.3}$$

Let us recall now the following theorem concerning the increments of $n \mapsto \xi(\mathbf{0}, n)$.

Theorem H (Csáki, Földes and Révész [CsFR, 98]). *Let* $a_n = \exp((\log n)^K)$ *and* $b_n = \exp((\log n)^b)$ *for some constants* $K > 0$ *and* $b > 0$. *Then for any* $\varepsilon > 0$,

$$\sup_{a\le a_n} (\xi(\mathbf{0}, a + b_n) - \xi(\mathbf{0}, a)) = \mathcal{O}(\log^{b+\varepsilon} n), \qquad \text{a.s.}$$

The next is a strong approximation theorem for the two-dimensional random walk and the Wiener process.

Theorem J (Révész [R, 90]). *On a rich enough probability space one can define a Wiener process* $\mathbf{W}(\cdot) \in \mathbf{R}^2$ *and a simple symmetric random walk* $\mathbf{S}_n \in \mathbf{Z}^2$ *such that*

$$\|\sqrt{2}\,\mathbf{S}_n - \mathbf{W}(n)\| = \mathcal{O}(\log n), \qquad \text{a.s.}$$

Finally, we will need the following two theorems concerning the two-dimensional Bessel process $\{R(t),\, t \ge 0\}$. They are both borrowed from Bertoin and Werner [BW, 94] (see their Proposition 4 and Theorem 3, respectively).

Theorem K (Bertoin and Werner [BW, 94]). *We have*

$$\limsup_{t\to\infty} \frac{1}{(\log t)\log_3 t} \int_0^t \mathbf{1}_{\{R(s)<1\}} ds = \frac{1}{2}, \qquad \text{a.s.}$$

Theorem L (Bertoin and Werner [BW, 94]). *Let* Z_2^* *be as in* (1.2). *For any* $\varepsilon > 0$, *when* $t \to \infty$,

$$Z_2^*(t) = o\left((\log t)^2 (\log_2 t)^{2+\varepsilon}\right), \qquad \text{a.s.}$$

3. Proof of Theorem 1.1

We begin this section with a few lemmas and facts which will be needed in the proof.

Fact 1. *For* $x \geq 1,\ y \geq 1$

$$|x^\alpha - y^\alpha| \leq \alpha |x-y|(x^{\alpha-1} + y^{\alpha-1}) \qquad \text{if } \alpha > 1 \tag{3.1}$$

and

$$|x^\alpha - y^\alpha| \leq |x-y| \qquad \text{if } 0 \leq \alpha \leq 1. \tag{3.2}$$

Fact 1 is a simple consequence of the mean value theorem.

Fact 2. *For* $x \geq 1,\ y \geq 1,$ *and* $\alpha \geq 0$ *we have*

$$\frac{1}{xy^\alpha} + \frac{1}{yx^\alpha} \leq 2\left(\frac{1}{x^{\alpha+1}} + \frac{1}{y^{\alpha+1}}\right). \tag{3.3}$$

Fact 2 can be checked by elementary computations.

Recall that $\{R(t),\, t \geq 0\}$ is a two-dimensional Bessel process starting from 0, and $\{\ell(x,t),\, x \geq 0,\, t \geq 0\}$ is its local time. First we give a rough estimate for $Z_\alpha(t)$ for large t.

Lemma 3.1. *For* $0 \leq \alpha < 2$*, and any* $\varepsilon > 0$*, as* $t \to \infty$*,*

$$Z_\alpha(t) := \int_0^t \frac{ds}{R^\alpha(s)} = \mathcal{O}\left(t^{1-\alpha/2+\varepsilon}\right), \qquad \text{a.s.}$$

Proof: By the scaling property, for any fixed $t > 0$,

$$\{R(ut),\, u \geq 0\} \stackrel{\mathcal{D}}{=} \{\sqrt{t}\, R(u),\, u \geq 0\},$$

which implies that

$$Z_\alpha(t) \stackrel{\mathcal{D}}{=} t^{1-\alpha/2} \int_0^1 \frac{du}{R^\alpha(u)}. \tag{3.4}$$

Observe furthermore that $R^2(u)/u$ has the exponential distribution with mean 2, thus we have

$$\mathbf{E}\left(\int_0^1 \frac{du}{R^\alpha(u)}\right) = c_1 \int_0^1 \frac{du}{u^{\alpha/2}} = c_2,$$

where c_1 and c_2 are finite constants. Hence by Markov's inequality, we get for any $A > 0$ that

$$\mathbf{P}\left(Z_\alpha(t) > t^{1-\alpha/2}\, A\right) \le \frac{c_2}{A}. \tag{3.5}$$

For any given $\varepsilon > 0$, let $\beta > 1/\varepsilon$ and $t_k = k^\beta$. From (3.5) we get that

$$\mathbf{P}\left(Z_\alpha(t_{k+1}) > t_k^{1-\alpha/2+\varepsilon}\right) \le \frac{2c_2}{k^{\beta\varepsilon}},$$

for k large enough, as $t_{k+1}/t_k \to 1$. Using the Borel–Cantelli lemma and the monotonicity of $t \mapsto Z_\alpha(t)$, we obtain that

$$Z_\alpha(t) = \mathcal{O}\left(t^{1-\alpha/2+\varepsilon}\right), \qquad \text{a.s.},$$

completing the proof of Lemma 3.1. □

Lemma 3.1 implies

Lemma 3.2. *For $0 \le \alpha < 2$, and any $\varepsilon > 0$*

$$Z_\alpha^*(t) := \int_0^t \frac{\mathbf{1}_{\{R(s)\ge 1\}}}{R^\alpha(s)}\, ds = \mathcal{O}\left(t^{1-\alpha/2+\varepsilon}\right), \qquad \text{a.s.}$$

Lemma 3.3. *For any $b > 2$, and any $0 \le \alpha < 2$, as $t \to \infty$,*

$$\int_0^t \frac{\mathbf{1}_{\{R(s)<1\}}}{R^\alpha(s)}\, ds = \mathcal{O}(\log^b t), \quad \text{a.s.}$$

Proof: Using the notation introduced in (1.5)–(1.9) we can decompose the local time $\ell(x,t)$ of $R(t)$ as follows: for any $\delta > 0$ and all large t,

$$\begin{aligned}
\ell(x,t) &= \ell(x,\sigma_0) + \sum_{i=1}^{\eta(t)} (\ell(x,\sigma_i) - \ell(x,\sigma_{i-1})) + \ell(x,t) - \ell(x,\sigma_{\eta(t)}) \\
&\le \ell(x,\sigma_0) + \sum_{i=1}^{\eta(t)+1} (\ell(x,\sigma_i) - \ell(x,\sigma_{i-1})) \\
&\le \ell(x,\sigma_0) + \sum_{i=1}^{\log^{1+\delta} t} (\ell(x,\sigma_i) - \ell(x,\sigma_{i-1})), \qquad (3.6)
\end{aligned}$$

the last inequality following from Theorem G. Since $b > 2$, we can choose $\delta > 0$ such that $2 + \delta < b$.

By writing $\int_0^t \frac{\mathbf{1}_{\{R(s)<1\}}}{R^\alpha(s)}\,ds = \int_0^1 \frac{\ell(x,t)}{x^\alpha}\,dx$, and in view of (3.6) (noting that $\int_0^1 \frac{\ell(x,\sigma_0)}{x^\alpha}\,dx = Z^*_\alpha(\sigma_0) < \infty$ almost surely), it only remains to check that

$$Q(t) := \sum_{i=1}^{\log^{1+\delta} t} \int_0^1 \frac{\ell(x,\sigma_i) - \ell(x,\sigma_{i-1})}{x^\alpha}\,dx = \mathcal{O}(\log^b t), \qquad \text{a.s.} \tag{3.7}$$

To this end, we look at the Laplace transforms of $\ell(x,\theta_1) - \ell(x,\sigma_0)$ and $\ell(x,\sigma_1) - \ell(x,\theta_1)$ which can be found in Borodin and Salminen ([BS, 96] p. 297, Formula 2.3.1): for any $\gamma > 0$,

$$\mathbf{E}\left(e^{-\gamma(\ell(x,\theta_1)-\ell(x,\sigma_0))}\right) = \begin{cases} \frac{1+2\gamma x|\log x|}{1+2\gamma x|\log(x/2)|} & \text{if } 0 < x \le 1 \\ \frac{1}{1+2\gamma x|\log(x/2)|} & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x \ge 2 \end{cases},$$

$$\mathbf{E}\left(e^{-\gamma(\ell(x,\sigma_1)-\ell(x,\theta_1))}\right) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ \frac{1}{1+2\gamma x|\log x|} & \text{if } 1 \le x \le 2 \\ \frac{1+2\gamma x|\log(x/2)|}{1+2\gamma x|\log x|} & \text{if } x \ge 2 \end{cases}.$$

Since $\ell(x,\theta_1) - \ell(x,\sigma_0)$ and $\ell(x,\sigma_1) - \ell(x,\theta_1)$ are independent, we get the Laplace transform of $\ell(x,\sigma_1) - \ell(x,\sigma_0)$:

$$\mathbf{E}\left(e^{-\gamma(\ell(x,\sigma_1)-\ell(x,\sigma_0))}\right) = \begin{cases} \frac{1+2\gamma x|\log x|}{1+2\gamma x|\log(x/2)|} & \text{if } 0 < x \le 1 \\ \frac{1}{(1+2\gamma x|\log x|)(1+2\gamma x|\log(x/2)|)} & \text{if } 1 \le x \le 2 \\ \frac{1+2\gamma x|\log(x/2)|}{1+2\gamma x|\log x|} & \text{if } x \ge 2. \end{cases} \tag{3.8}$$

Hence for $0 < x \le 1$,

$$\mathbf{E}\left(\ell(x,\sigma_i) - \ell(x,\sigma_{i-1})\right) = \mathbf{E}\left(\ell(x,\sigma_1) - \ell(x,\sigma_0)\right) = 2x\log 2.$$

As a consequence, when t goes to ∞,

$$\mathbf{E}(Q(t)) \le (\log^{1+\delta} t) \int_0^1 \frac{x\log 2}{x^\alpha}\,dx = \mathcal{O}(\log^{1+\delta} t) \tag{3.9}$$

for all $0 \le \alpha < 2$. Based on (3.9), we can finish the proof in the same way as that of Lemma 3.1, using the monotonicity of $t \mapsto Q(t)$, the subsequence $t_k = e^k$ and the Borel–Cantelli lemma. Indeed, by (3.9) and Markov's inequality,

$$\mathbf{P}\left(Q(t_{k+1}) > (\log t_k)^b\right) \le c_3 \frac{(\log t_{k+1})^{1+\delta}}{(\log t_k)^b} \le c_4 \frac{(k+1)^{1+\delta}}{k^b},$$

which sums for k (recalling that $b > 2+\delta$). This yields (3.7), and completes the proof of Lemma 3.3. □

Lemma 3.4. *Let $\alpha \geq 0$. As $n \to \infty$, we have almost surely for any $\varepsilon > 0$,*

$$U_{2\alpha}(n) := \sum_{k=1}^{n} \frac{\mathbf{1}_{\{\|\mathbf{S}_k\| \neq \mathbf{0}\}}}{\|\mathbf{S}_k\|^{2\alpha}} = \begin{cases} \mathcal{O}\left(n^{1-\alpha+\varepsilon}\right) & \text{if} \quad \alpha < 1 \\ \mathcal{O}(\log^{3+\varepsilon} n) & \text{if} \quad \alpha = 1 \\ \mathcal{O}(\log^2 n) & \text{if} \quad \alpha > 1. \end{cases}$$

Proof: Let $M_n = \max_{k \leq n} \|\mathbf{S}_k\|$. Then by the LIL,

$$M_n = \mathcal{O}\left((n \log\log n)^{1/2}\right), \qquad \text{a.s.} \tag{3.10}$$

By Theorem E,

$$\sum_{k=1}^{n} \frac{\mathbf{1}_{\{\|\mathbf{S}_k\| \neq \mathbf{0}\}}}{\|\mathbf{S}_k\|^{2\alpha}} = \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{\xi(\mathbf{x}, n)}{\|\mathbf{x}\|^{2\alpha}} = \mathcal{O}(\log^2 n) \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}, \|\mathbf{x}\| \leq M_n} \frac{1}{\|\mathbf{x}\|^{2\alpha}}. \tag{3.11}$$

Due to the following equiconvergence relation

$$\sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}, \|\mathbf{x}\| \leq N} \frac{1}{\|\mathbf{x}\|^{2\alpha}} \sim \int_1^N r^{1-2\alpha}\, dr, \tag{3.12}$$

and the fact that

$$\int_1^N r^{1-2\alpha}\, dr = \begin{cases} \mathcal{O}\left(N^{2-2\alpha}\right) & \text{if} \quad \alpha < 1 \\ \mathcal{O}(\log N) & \text{if} \quad \alpha = 1 \\ \mathcal{O}(1) & \text{if} \quad \alpha > 1. \end{cases} \tag{3.13}$$

we have our statement combining (3.10)–(3.13). □

Proof of Theorem 1.1: In view of Lemma 3.3, we only have to check that almost surely,

$$\left|2^{-\alpha/2} U_\alpha(n) - Z^*_\alpha(n)\right| = \begin{cases} \mathcal{O}(n^{1-\alpha+\varepsilon}), & \text{if} \;\; 0 < \alpha < 1, \\ \mathcal{O}(\log^{4+\varepsilon} n), & \text{if} \;\; \alpha = 1, \\ \mathcal{O}(\log^{2+\varepsilon} n), & \text{if} \;\; 1 < \alpha < 2. \end{cases} \tag{3.14}$$

We start with the following decomposition:

$$\begin{aligned} \left|2^{-\alpha/2} U_\alpha(n) - Z^*_\alpha(n)\right| &= \left| \sum_{k=1}^{n} \int_{k-1}^{k} \left(\frac{\mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}\}}}{\|\mathbf{S}^*_k\|^\alpha} - \frac{\mathbf{1}_{\{R(s) \geq 1\}}}{R^\alpha(s)} \right) ds \right| \\ &\leq \left| \sum_{k=1}^{n} \int_{k-1}^{k} \frac{R^\alpha(s) - \|\mathbf{S}^*_k\|^\alpha}{\|\mathbf{S}^*_k\|^\alpha R^\alpha(s)} \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}, R(s) \geq 1\}}\, ds \right| \\ &+ \left| \sum_{k=1}^{n} \int_{k-1}^{k} \frac{\mathbf{1}_{\{\mathbf{S}_k = \mathbf{0}, R(s) \geq 1\}}}{R^\alpha(s)}\, ds \right| + \left| \sum_{k=1}^{n} \int_{k-1}^{k} \frac{\mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}, R(s) < 1\}}}{\|\mathbf{S}^*_k\|^\alpha}\, ds \right| \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.15}$$

where $\mathbf{S}_k^* = \sqrt{2}\,\mathbf{S}_k$.

First observe that

$$I_2 \le \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{S}_k=\mathbf{0}\}} = \xi(\mathbf{0},n) = \mathcal{O}(\log^{1+\varepsilon} n), \qquad \text{a.s.} \tag{3.16}$$

for any $\varepsilon > 0$, by Theorem E. Similarly, by Theorem K, we have that for any $\varepsilon > 0$

$$I_3 \le \int_0^n \mathbf{1}_{\{R(s)<1\}}\, ds = \mathcal{O}(\log^{1+\varepsilon} n), \qquad \text{a.s.} \tag{3.17}$$

The estimation of I_1 is more delicate and we have to consider two cases. First we consider the case $0 < \alpha \le 1$. According to (3.2) we have

$$\begin{aligned} I_1 &\le \sum_{k=1}^{n} \int_{k-1}^{k} \frac{|R^\alpha(s) - \|\mathbf{S}_k^*\|^\alpha|}{\|\mathbf{S}_k^*\|^\alpha R^\alpha(s)}\, \mathbf{1}_{\{\mathbf{S}_k\neq\mathbf{0},R(s)\ge 1\}}\, ds \\ &\le \sum_{k=1}^{n} \int_{k-1}^{k} \frac{|R(s) - \|\mathbf{S}_k^*\|\,|}{\|\mathbf{S}_k^*\|^\alpha R^\alpha(s)} \mathbf{1}_{\{\mathbf{S}_k\neq\mathbf{0},R(s)\ge 1\}}\, ds \\ &\le \mathcal{O}(\log n) \sum_{k=1}^{n} \int_{k-1}^{k} \frac{\mathbf{1}_{\{\mathbf{S}_k\neq\mathbf{0},R(s)\ge 1\}}}{\|\mathbf{S}_k^*\|^\alpha R^\alpha(s)}\, ds, \end{aligned}$$

where in the last line we used Theorem J. Since $xy \le (x^2+y^2)/2$, this yields

$$\begin{aligned} I_1 &\le \mathcal{O}(\log n) \sum_{k=1}^{n} \int_{k-1}^{k} \mathbf{1}_{\{\mathbf{S}_k\neq\mathbf{0},R(s)\ge 1\}} \left(\frac{1}{\|\mathbf{S}_k^*\|^{2\alpha}} + \frac{1}{R^{2\alpha}(s)} \right) ds \\ &\le \mathcal{O}(\log n) \left(\sum_{k=1}^{n} \frac{\mathbf{1}_{\{\mathbf{S}_k\neq\mathbf{0}\}}}{\|\mathbf{S}_k^*\|^{2\alpha}} + \int_0^n \frac{\mathbf{1}_{\{R(s)\ge 1\}}}{R^{2\alpha}(s)}\, ds \right) \\ &\le \mathcal{O}(\log n)\left(U_{2\alpha}(n) + Z_{2\alpha}^*(n)\right). \end{aligned} \tag{3.18}$$

Applying Lemmas 3.2 and 3.4 yields that for $0 < \alpha < 1$,

$$I_1 = \mathcal{O}\left(n^{1-\alpha+\varepsilon}\right), \qquad \text{a.s.}$$

For $\alpha = 1$, we can use (3.18), Lemma 3.4 and Theorem L to see that

$$I_1 = \mathcal{O}\left(\log^{4+\varepsilon} n\right), \qquad \text{a.s.}$$

These estimates, together with (3.15)–(3.17), yield (3.14) when $0 < \alpha \le 1$.

Consider now the case $1 < \alpha < 2$. We estimate I_1 differently. By (3.1) and Theorem J,

$$\begin{aligned} I_1 &\leq \sum_{k=1}^{n} \int_{k-1}^{k} \frac{|R^\alpha(s) - \|\mathbf{S}_k^*\|^\alpha|}{\|\mathbf{S}_k^*\|^\alpha R^\alpha(s)} \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}, R(s) \geq 1\}}\, ds \\ &\leq \sum_{k=1}^{n} \int_{k-1}^{k} \frac{2|R(s) - \|\mathbf{S}_k^*\|\,|\,(R^{\alpha-1}(s) + \|\mathbf{S}_k^*\|^{\alpha-1})}{\|\mathbf{S}_k^*\|^\alpha R^\alpha(s)} \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}, R(s) \geq 1\}}\, ds \\ &\leq \mathcal{O}(\log n) \sum_{k=1}^{n} \int_{k-1}^{k} \left(\frac{1}{\|\mathbf{S}_k^*\|^\alpha R(s)} + \frac{1}{\|\mathbf{S}_k^*\| R^\alpha(s)} \right) \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}, R(s) \geq 1\}}\, ds. \end{aligned}$$

Using (3.3) gives

$$\begin{aligned} I_1 &\leq \mathcal{O}(\log n) \sum_{k=1}^{n} \frac{1}{\|\mathbf{S}_k^*\|^{\alpha+1}} \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}\}} + \mathcal{O}(\log n) \int_0^n \frac{1}{R^{\alpha+1}(s)} \mathbf{1}_{\{R(s) \geq 1\}}\, ds \\ &\leq \mathcal{O}(\log n) \left(U_{\alpha+1}(n) + Z_{\alpha+1}^*(n) \right). \end{aligned}$$

When $\alpha > 1$, we have $\alpha + 1 > 2$, so that we can use Theorems 1.2 and 1.3 (or rather their consequences as stated in (6.4) and (6.5) in Section 6) to see that $U_{\alpha+1}(n) = \mathcal{O}(\log^{1+\varepsilon} n)$ and $Z_{\alpha+1}^*(n) = \mathcal{O}(\log^{1+\varepsilon} n)$ almost surely. This yields that for $1 < \alpha < 2$,

$$I_1 = \mathcal{O}\left(\log^{2+\varepsilon} n\right), \qquad \text{a.s.}$$

Combining this with (3.15)–(3.17) yields (3.14) when $1 < \alpha < 2$. Consequently, Theorem 1.1 is now proved. □

4. Proof of Theorem 1.2

In view of Theorem B in Section 2, to prove Theorem 1.2, it suffices to show that, when $\alpha > 2$, the function $\mathbf{x} \mapsto f(\mathbf{x}) = \mathbf{1}_{\{\mathbf{x} \neq \mathbf{0}\}} / \|\mathbf{x}\|^\alpha$ satisfies (2.1) with an appropriately chosen $\delta > 0$.

Write $\|Y\|_p := (\mathbf{E}(|Y|^p))^{1/p}$ for any random variable Y and $p \geq 1$. Fix an integer $m \geq 1$. It follows from Theorems C and D that for any $\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}$,

$$\begin{aligned} \|\xi(\mathbf{x}, \rho_1)\|_m &\leq 1 + \|\xi(\mathbf{x}, \rho_1) - 1\|_m \\ &\leq 1 + C_m p^{(1-m)/m} \\ &\leq C_m (\log\|\mathbf{x}\| + 1)^{(m-1)/m} \\ &\leq C_m (\log\|\mathbf{x}\| + 1), \end{aligned} \tag{4.1}$$

where C_m denotes a (finite and positive) constant depending only on m, whose value varies from line to line.

Let $f(\mathbf{x}) = \mathbf{1}_{\{\mathbf{x}\neq\mathbf{0}\}}/\|\mathbf{x}\|^\alpha$. To check (2.1), observe that

$$\sum_{k=1}^{\rho_1} f(\mathbf{S}_k) = \sum_{x\in\mathbf{Z}^2-\{\mathbf{0}\}} \frac{\xi(\mathbf{x},\rho_1)}{\|\mathbf{x}\|^\alpha}.$$

Hence for any integer $m \geq 1$, by the triangle inequality and (4.1),

$$\|\sum_{k=1}^{\rho_1} f(\mathbf{S}_k)\|_m \leq \sum_{x\in\mathbf{Z}^2-\{\mathbf{0}\}} \frac{\|\xi(\mathbf{x},\rho_1)\|_m}{\|\mathbf{x}\|^\alpha} \leq C_m \sum_{\mathbf{x}\in\mathbf{Z}^2-\{\mathbf{0}\}} \frac{1+\log\|\mathbf{x}\|}{\|\mathbf{x}\|^\alpha},$$

which is finite whenever $\alpha > 2$. Thus any moments of order $m \geq 1$ of $\sum_{k=1}^{\rho_1} f(S_k)$ exist, proving Theorem 1.2. □

5. Proof of Theorem 1.3

The proof of this theorem is based on Theorem A. Denote by $\ell(x,t)$ the local time (at time t and position x) of the Bessel process $\{R(t),\, t\geq 0\}$. Let $\{\sigma_n,\, n\geq 0\}$ and $\{\theta_n,\, n\geq 1\}$ be as in (1.5)–(1.7). Both $\{\theta_i-\sigma_{i-1}\}_1^\infty$ and $\{\sigma_i-\theta_i\}_1^\infty$ are i.i.d. sequences. The Laplace transform of $\ell(x,\sigma_1)-\ell(x,\sigma_0)$ is given in (3.8), from which we can obtain: for $x\geq 1$,

$$\mathbf{E}(\ell(x,\sigma_1)-\ell(x,\sigma_0)) = 2x\log x, \tag{5.1}$$

$$\mathbf{E}\left((\ell(x,\sigma_1)-\ell(x,\sigma_0))^3\right) \leq Cx^3(\log x)^2, \tag{5.2}$$

where $C>0$ is a constant.

Define, for $i\geq 1$,

$$\tau_i := \sigma_i - \sigma_{i-1}, \tag{5.3}$$

$$X_i := \int_{\sigma_{i-1}}^{\sigma_i} \frac{1}{R^\alpha(s)} \mathbf{1}_{\{R(s)\geq 1\}}\, ds. \tag{5.4}$$

We show that the sequence $(X_i,\tau_i)_{i=1}^\infty$ of i.i.d. vectors satisfies the conditions of Theorem A. Indeed, by (5.2),

$$\|X_i\|_3 = \left(\mathbf{E}_1\left(\int_{\sigma_0}^{\sigma_1} \frac{1}{R^\alpha(s)}\mathbf{1}_{\{R(s)\geq 1\}}\, ds\right)^3\right)^{1/3}$$

$$
\begin{aligned}
&= \left(\mathbf{E}\left(\int_1^\infty \frac{\ell(x,\tau_1)}{x^\alpha}\,dx\right)^3\right)^{1/3} \\
&\le \int_1^\infty \frac{\|\ell(x,\tau_1)\|_3}{x^\alpha}\,dx \\
&\le C\int_1^\infty \frac{(x^3(\log x)^2)^{1/3}}{x^\alpha}\,dx,
\end{aligned}
$$

which is finite, whenever $\alpha > 2$. Using Chebyshev's inequality, this leads to

$$
\mathbf{P}\left(|X_i| > x\right) \le \frac{c}{x^3}, \tag{5.5}
$$

for some constant $c > 0$ and all $x > 0$. Thus, the first condition of (2.2) is satisfied by X_i with $\beta = 3$. It is clear that the second condition in (2.2) is satisfied by τ_i with $h(x) = C\log x$, since by Theorem F, $\mathbf{P}(\tau_i > x) \le C/\log x$.

Consequently, the sequence of vectors $(X_i, \tau_i)_{i=1}^\infty$ defined in (5.3)–(5.4) satisfies the conditions of Theorem A. Let

$$
\begin{aligned}
S_i &:= \int_{\sigma_0}^{\sigma_i} \frac{1}{R^\alpha(s)}\mathbf{1}_{\{R(s)\ge 1\}}\,ds = Z_\alpha^*(\sigma_i), \qquad i \ge 1, \\
\rho_n &:= \sigma_n - \sigma_0, \qquad n \ge 0.
\end{aligned}
$$

According to Theorem A, on an appropriate probability space one can construct two independent copies, say $\left(X_i^{(1)}, \tau_i^{(1)}\right)_{i=1}^\infty$ and $\left(X_i^{(2)}, \tau_i^{(2)}\right)_{i=1}^\infty$, of $(X_i, \tau_i)_{i=1}^\infty$, such that

$$
(S_n, \rho_n)_{n=1}^\infty \overset{\mathcal{D}}{=} (S_n^{(1)}, \rho_n^{(j)})_{n=1}^\infty, \qquad j = 1, 2, \tag{5.6}
$$

$$
\sup_{k\le n} |S_k - S_k^{(2)}| = \mathcal{O}\left(n^{1/\beta^*}\right), \qquad \text{a.s.}, \tag{5.7}
$$

$$
\sup_{k\le n} |\rho_k - \rho_k^{(1)}| = \mathcal{O}\left(e^{cn^\gamma}\right), \qquad \text{a.s.} \tag{5.8}
$$

where $S_k^{(j)} = \sum_{i=1}^k X_i^{(j)}$, $\rho_k^{(j)} = \sum_{i=1}^k \tau_i^{(j)}$, $\gamma < 1$ and $2 < \beta^* < 3$,

Apply now the Komlós–Major–Tusnády theorem [KMT, 75] (see also Csörgő and Révész [CsR, 81], p. 108, Theorem 2.6.6) to get that, there exists a Wiener process $\{W(t),\, t \ge 0\}$ such that for any $2 < \beta^* < 3$, when N goes to infinity,

$$
S_N^{(2)} - N\mathbf{E}(X_1) - \sigma_\alpha^W W(N) = O(N^{1/\beta^*}), \qquad \text{a.s.}, \tag{5.9}
$$

where $\sigma_\alpha^W = \sqrt{\mathrm{Var}(X_1)}$. Since W is constructed from $\{S_n^{(2)},\, n \ge 1\}$, it can be chosen to be independent of $(X_i^{(1)}, \tau_i^{(1)})_{i=1}^\infty$.

By (5.1),

$$\mathbf{E}(X_1) = \mathbf{E}\int_1^\infty \frac{\ell(x,\sigma_1-\sigma_0)}{x^\alpha}\,dx = \int_1^\infty \frac{2\log x}{x^{\alpha-1}}\,dx = \frac{2}{(\alpha-2)^2} =: \bar{f}_\alpha^W.$$

It follows from (5.9) and (5.7) that

$$S_N - N\bar{f}_\alpha^W - \sigma_\alpha^W W(N) = O(N^{1/\beta^*}), \qquad \text{a.s.}$$

Now let $\eta(t)$ be as in (1.9). We get

$$S_{\eta(t)} - \bar{f}_\alpha^W \eta(t) - \sigma_\alpha^W W(\eta(t)) = \mathcal{O}\left((\eta(t))^{1/\beta^*}\right), \qquad \text{a.s.},$$

which, in view of (2.3), yields that, for any small enough $\varepsilon > 0$,

$$S_{\eta(t)} - \bar{f}_\alpha^W \eta(t) - \sigma_\alpha^W W(\eta(t)) = \mathcal{O}(\log^{1/2-\varepsilon} t), \qquad \text{a.s.} \tag{5.10}$$

Now we want to get an almost sure upper bound for $|Z_\alpha^*(t) - S_{\eta(t)}|$. Since $\eta(t) < t \le \eta(t)+1$, we have

$$|Z_\alpha^*(t) - S_{\eta(t)}| \le X_{\eta(t)+1}.$$

On the other hand, a routine Borel–Cantelli argument, using (5.5), yields that for a small enough $\delta > 0$, $X_k = \mathcal{O}(k^{1/2-\delta})$ almost surely (when k goes to infinity). Therefore, by (2.3), $X_{\eta(t)+1} = \mathcal{O}(\log^{1/2-\delta^*} t)$, for any $0 < \delta^* < \delta$. Accordingly,

$$Z_\alpha^*(t) - S_{\eta(t)} = \mathcal{O}(\log^{1/2-\delta^*} t), \qquad \text{a.s.}$$

Going back to (5.10), we arrive at

$$Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t) = \sigma_\alpha^W W(\eta(t)) + \mathcal{O}(\log^{1/2-\gamma^*} t), \qquad \text{a.s.}, \tag{5.11}$$

with a small enough $\gamma^* > 0$.

What remains to show is that on the right hand side of (5.11) we can replace $W(\eta(t))$ by $W(\eta^{(1)}(t))$, at the price of a possible small increase in the order of the error term in the approximation. For this purpose we need two preliminary estimates. The first one (Lemma 5.1) is analogous to Theorem H. Its proof, which goes along the same lines as that of Theorem H, is omitted.

Lemma 5.1. *Let $a_t = \exp((\log t)^K)$ and $b_t = \exp((\log t)^b)$, for $K > 0$ and $b > 0$. Then for any $\varepsilon > 0$,*

$$\sup_{a \le a_t}(\eta(a+b_t) - \eta(a)) = \mathcal{O}(\log^{b+\varepsilon} t), \qquad \text{a.s.}$$

Lemma 5.2. *There exists $\nu < 1$ such that, as $t \to \infty$,*

$$|\eta^{(1)}(t) - \eta(t)| = \mathcal{O}(\log^{\nu} t), \qquad \text{a.s.}$$

Proof: Since $\eta(t) = \eta^{(1)}(\rho^{(1)}_{\eta(t)})$, $\eta^{(1)}(t) = \eta(\rho_{\eta^{(1)}(t)})$, $t \geq \rho_{\eta(t)}$ and $t \geq \rho^{(1)}_{\eta^{(1)}(t)}$, we have

$$\eta^{(1)}(\rho_{\eta(t)}) - \eta^{(1)}(\rho^{(1)}_{\eta(t)}) \leq \eta^{(1)}(t) - \eta(t) \leq \eta(\rho_{\eta^{(1)}(t)}) - \eta(\rho^{(1)}_{\eta^{(1)}(t)}). \tag{5.12}$$

By (2.3), for any $\varepsilon > 0$ and all large t, $\eta(t) < (\log t)^{1+\varepsilon}$. It follows from (5.8) that for some $\gamma < 1$ and all large t,

$$|\rho_{\eta(t)} - \rho^{(1)}_{\eta(t)}| \leq \sup_{i \leq (\log t)^{1+\varepsilon}} |\rho_i - \rho^{(1)}_i| \leq \exp\left(c\,(\log t)^{(1+\varepsilon)\gamma}\right), \qquad \text{a.s.}$$

Now apply Lemma 5.1 to $b_t = \exp(c\,(\log t)^{(1+\varepsilon)\gamma})$ and $a_t = t$ (noting that $\rho^{(1)}_{\eta^{(1)}(t)} \leq t$) to see that

$$\eta^{(1)}(\rho_{\eta(t)}) - \eta^{(1)}(\rho^{(1)}_{\eta(t)}) = \mathcal{O}(\log^{(1+2\varepsilon)\gamma} t), \qquad \text{a.s.}$$

A similar argument shows that

$$\eta(\rho_{\eta^{(1)}(t)}) - \eta(\rho^{(1)}_{\eta^{(1)}(t)}) = \mathcal{O}(\log^{(1+2\varepsilon)\gamma} t), \qquad \text{a.s.}$$

In view of (5.12), we have

$$\eta^{(1)}(t) - \eta(t) = \mathcal{O}(\log^{(1+2\varepsilon)\gamma} t), \qquad \text{a.s.}$$

Choosing $\varepsilon > 0$ so small that $(1 + 2\varepsilon)\gamma < 1$, gives the lemma. □

We now go back to the proof of Theorem 1.3. Note that Lemma 5.2 yields (1.11). To check (1.10), we can control the the increments of W by using Theorem 1.2.1 of Csörgő and Révész [CsR, 81]: for any $\varepsilon > 0$ and any $\nu > 0$,

$$\sup_{0 \leq u,v \leq (\log t)^{1+\varepsilon},\ |u-v| \leq (\log t)^{\nu}} |W(u) - W(v)| = \mathcal{O}\left((\log t)^{\nu/2} (\log\log t)^{1/2}\right), \qquad \text{a.s.}$$

Since $\eta(t) = \mathcal{O}(\log^{1+\varepsilon} t)$ (cf. Theorem G), it follows from Lemma 5.2 and (5.11) that for any $1 > \nu^* > \nu$,

$$Z^*_\alpha(t) - \bar{f}^W_\alpha \eta(t) = \sigma^W_\alpha W(\eta^{(1)}(t)) + \mathcal{O}(\log^{\nu^*/2} t) + \mathcal{O}(\log^{1/2-\gamma^*} t), \qquad \text{a.s.}$$

Take

$$\kappa := \max\left(\frac{1}{2} - \gamma^*, \frac{\nu^*}{2}\right) < \frac{1}{2}$$

to get

$$Z_\alpha(t) - \bar{f}^W_\alpha \eta(t) = \sigma^W_\alpha W(\eta^{(1)}(t)) + \mathcal{O}(\log^{\kappa} t), \qquad \text{a.s.},$$

as $t \to \infty$. Since W was chosen to be independent of $(X^{(1)}_i, \tau^{(1)}_i)_{i=1}^{\infty}$, it is a fortiori independent of $\{\eta^{(1)}(t),\, t > 0\}$. This completes the proof of Theorem 1.2. □

6. Further results

Our theorems can be used to deduce weak and strong laws for the additive functionals studied in this paper. Considering $Z_\alpha(t)$, defined by (1.1), by scaling we have (cf. (3.4)), for any $t > 0$,

$$Z_\alpha(t) \stackrel{\mathcal{D}}{=} t^{1-\alpha/2} Z_\alpha(1). \tag{6.1}$$

Hence, Theorem 1.1 implies for $0 < \alpha < 2$,

$$2^{-\alpha/2} n^{\alpha/2-1} U_\alpha(n) \xrightarrow{\mathcal{D}} Z_\alpha(1), \quad n \to \infty,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. It follows also that the LIL for the process $Z_\alpha(\cdot)$ is inherited by $U_\alpha(\cdot)$.

Now we study limsup and liminf properties of Z_α. It is obvious from (6.1) that first we have to study the upper and lower tail behaviour of $Z_\alpha(1)$. A direct approach however does not seem available, instead, we apply Corollary XI.1.12 of Revuz and Yor [RY, 99] telling us that

$$Z_\alpha(1) := \int_0^1 \frac{ds}{R^\alpha(s)} \stackrel{\mathcal{D}}{=} \left(\frac{2}{2-\alpha}\right)^\alpha \left(\int_0^1 R^\beta(t)\, dt\right)^{-(1-\alpha/2)}, \tag{6.2}$$

where $\beta = 2\alpha/(2-\alpha)$. For $\beta \geq 1$, Azencott [A, 80] shows that there exists a finite constant $c(\beta)$ such that

$$\log \mathbf{P}\left(\int_0^1 R^\beta(t)\, dt > u\right) \sim -c(\beta) u^{2/\beta}, \quad u \to \infty,$$

which combined with (6.2) yields

$$\log \mathbf{P}(Z_\alpha(1) < x) \sim -c_1(\alpha) x^{-2/\alpha}, \quad x \to 0,$$

where

$$c_1(\alpha) = \frac{4c(\beta)}{(2-\alpha)^2}.$$

Now the usual argument gives the following liminf result for $2/3 \leq \alpha < 2$:

$$\liminf_{t\to\infty} t^{-(1-\alpha/2)} (\log\log t)^{\alpha/2} Z_\alpha(t) = K(\alpha), \qquad \text{a.s.},$$

where

$$K(\alpha) = (c_1(\alpha))^{\alpha/2}.$$

Hence, by Theorem 1.1, we have also for $2/3 \leq \alpha < 2$

$$\liminf_{n\to\infty} n^{-(1-\alpha/2)} (\log\log n)^{\alpha/2} 2^{-\alpha/2} U_\alpha(n) = K(\alpha), \qquad \text{a.s.}$$

Concerning the limsup results, we can use the following small deviation theorem of Borovkov and Mogulskii [BM, 91]: for $\beta \geq 1$

$$\log \mathbf{P}\left(\int_0^1 R^{\beta}(t)\,dt < u\right) \sim -\tilde{c}(\beta)u^{-2/\beta}, \quad u \to 0, \tag{6.3}$$

where $\tilde{c}(\beta)$ is a finite positive constant. (We note that in [BM, 91] the analogue result is proved for one-dimensional Wiener process, but as they remark, their result holds also true for higher dimensional case.)

Based on (6.3) and (6.2), similarly to the liminf case, one can easily obtain

$$\limsup_{t\to\infty} \frac{Z_\alpha(t)}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}} = \tilde{K}(\alpha), \qquad \text{a.s.},$$

with

$$\tilde{K}(\alpha) = (\tilde{c}(\beta))^{-\alpha/2} \frac{2^\alpha}{(2-\alpha)^\alpha}$$

and consequently

$$\limsup_{n\to\infty} \frac{2^{-\alpha/2} U_\alpha(n)}{n^{1-\alpha/2}(\log\log n)^{\alpha/2}} = \tilde{K}(\alpha), \qquad \text{a.s.}$$

Turning now to the case $\alpha > 2$, we first consider some limiting distribution results. Let E be a random variable with density function $e^{-|x|}/2$, $x \in \mathbf{R}$. Then E is a bilateral exponential random variable and $|E|$ has the exponential distribution of parameter 1, i.e., it has the density function e^{-x}, $x \in [0,\infty)$. By Theorem E,

$$\frac{\pi\xi(\mathbf{0},n)}{\log n} \xrightarrow{\mathcal{D}} |E|, \quad n \to \infty,$$

and similarly, from Theorem F one can see that

$$\frac{(2\log 2)\eta(t)}{\log t} \xrightarrow{\mathcal{D}} |E|, \quad t \to \infty.$$

Moreover, as easily seen,

$$\frac{\sqrt{2\pi}\, W(\xi^{(1)}(\mathbf{0},n))}{\sqrt{\log n}} \xrightarrow{\mathcal{D}} E, \quad n \to \infty$$

and

$$\frac{2\sqrt{\log 2}\, W(\eta^{(1)}(t))}{\sqrt{\log t}} \xrightarrow{\mathcal{D}} E, \quad t \to \infty,$$

where $W(\xi^{(1)}(\mathbf{0},\cdot))$ and $W(\eta^{(1)}(\cdot))$, resp. are the processes in Theorems 1.2 and 1.3, resp.

Now our Theorems 1.2 and 1.3 imply the following weak convergence results:

$$\begin{aligned} \frac{\pi U_\alpha(n)}{\bar{f}^D_\alpha \log n} &\xrightarrow{\mathcal{D}} |E|, \quad n\to\infty, \\ \frac{U_\alpha(n)-\bar{f}^D_\alpha \xi(\mathbf{0},n)}{\sigma^D_\alpha\sqrt{\log n}}\sqrt{2\pi} &\xrightarrow{\mathcal{D}} E, \quad n\to\infty, \\ \frac{(2\log 2)Z^*_\alpha(t)}{\bar{f}^W_\alpha \log t} &\xrightarrow{\mathcal{D}} |E|, \quad t\to\infty, \\ \frac{Z^*_\alpha(t)-\bar{f}^W_\alpha\eta(t)}{\sigma^W_\alpha\sqrt{\log t}}2\sqrt{\log 2} &\xrightarrow{\mathcal{D}} E, \quad t\to\infty. \end{aligned}$$

Concerning strong limit results, our Theorems 1.2 and 1.3 combined with Theorems E and G yield

$$\limsup_{n\to\infty}\frac{U_\alpha(n)}{\log n\log_3 n}=\frac{\bar{f}^D_\alpha}{\pi}, \qquad \text{a.s.,} \tag{6.4}$$

and

$$\limsup_{t\to\infty}\frac{Z^*_\alpha(t)}{\log t\log_3 t}=\frac{\bar{f}^W_\alpha}{2\log 2}, \qquad \text{a.s.} \tag{6.5}$$

Moreover, the following LIL is true (cf. [MR, 94], [CsFR, 98], [CsF, 99])

$$\limsup_{n\to\infty}\frac{W(\xi^{(1)}(\mathbf{0},n))}{\sqrt{\log n}\log_3 n}=\frac{1}{\sqrt{2\pi}}, \qquad \text{a.s.}$$

Similarly, one can see that

$$\limsup_{t\to\infty}\frac{W(\eta^{(1)}(t))}{\sqrt{\log t}\log_3 t}=\frac{1}{2\sqrt{\log 2}}, \qquad \text{a.s.}$$

Consequently, we have the following LIL's:

$$\limsup_{n\to\infty}\frac{U_\alpha(n)-\bar{f}^D_\alpha\xi(\mathbf{0},n)}{\sqrt{\log n}\log_3 n}=\frac{\sigma^D_\alpha}{\sqrt{2\pi}}, \qquad \text{a.s.}$$

and

$$\limsup_{t\to\infty}\frac{Z^*_\alpha(t)-\bar{f}^W_\alpha\eta(t)}{\sqrt{\log t}\log_3 t}=\frac{\sigma^W_\alpha}{2\sqrt{\log 2}}, \qquad \text{a.s.}$$

Acknowledgements

Cooperation between the authors was supported by the joint French-Hungarian Intergovernmental Grant “Balaton” (grant no. F25/97).

References

[A, 80] Azencott, R. (1980). Grandes déviations et applications. In: *Ecole d'Eté St.-Flour VIII*, Lecture Notes in Mathematics **774**, pp. 1–176, Springer, Berlin.

[BW, 94] Bertoin, J. and Werner, W. (1994). Comportement asymptotique du nombre de tours effectués par la trajectoire brownienne plane. In: *Séminaire de Probabilités XXVIII*, Lecture Notes in Mathematics **1583**, pp. 164–171, Springer, Berlin.

[BS, 96] Borodin, A.N. and Salminen, P. (1996). *Handbook of Brownian motion – Facts and Formulae*, Birkhäuser, Basel.

[BM, 91] Borovkov, A.A. and Mogulskii, A.A. (1991). On probabilities of small deviations for stochastic processes. *Siberian Adv. Math.* **1**, 39–63.

[BPY, 90] Burdzy, K., Pitman, J.W. and Yor, M. (1990). Brownian crossings between spheres. *J. Math. Anal. Appl.* **148**, 101–120.

[CsCsFS, 99a] Csáki, E., Csörgő, M., Földes, A. and Shi, Z. (1999). Path properties of Cauchy's principal values related to local time. (Preprint)

[CsCsFS, 99b] Csáki, E., Csörgő, M., Földes, A. and Shi, Z. (1999). Increment sizes of the principal value of Brownian local time. (Preprint)

[CsF, 99] Csáki, E. and Földes A. (1999). Asymptotic independence and additive functionals. (Preprint)

[CsFR, 98] Csáki, E., Földes, A. and Révész, P. (1998). A strong invariance principle for the local time difference of a simple symmetric planar random walk. *Studia Sci. Math. Hung.* **34**, 25–39.

[CsFRS, 98] Csáki, E., Földes, A., Révész, P. and Shi, Z: (1998). On the excursions of two-dimensional random walk and Wiener process. In: *Proceedings of the Paul Erdős Summer School '98.* (To appear)

[CsR, 81] Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*, Academic Press, New York.

[DE, 51] Dvoretzky, A. and Erdős, P. (1951). Some problems on random walk in space, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pp. 353–367, University of California Press, Berkeley.

[ET, 60] Erdős, P. and Taylor, S.J. (1960). Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hung.* **11**, 137–162.

[KMT, 75] Komlós, J., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent r.v.'s and the sample df. I. *Z. Wahrsch. verw. Gebiete* **32**, 111–131.

[MR, 94] Marcus, M.B. and Rosen, J. (1994). Laws of the iterated logarithm for the local times of recurrent random walks on Z^2 and of Lévy processes and random walks in the domain of attraction of Cauchy random variables. *Ann. Inst. H. Poincaré* **30**, 467–499.

[R, 90] Révész, P. (1990). *Random Walk in Random and Non-Random Environments*, World Scientific, Singapore.

[RY, 99] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, Third Edition, Springer, Berlin.

[S, 64] Spitzer, F. (1964). *Principles of Random Walk*, Van Nostrand, Princeton.