On generalized Ramsey numbers

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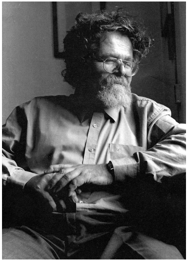


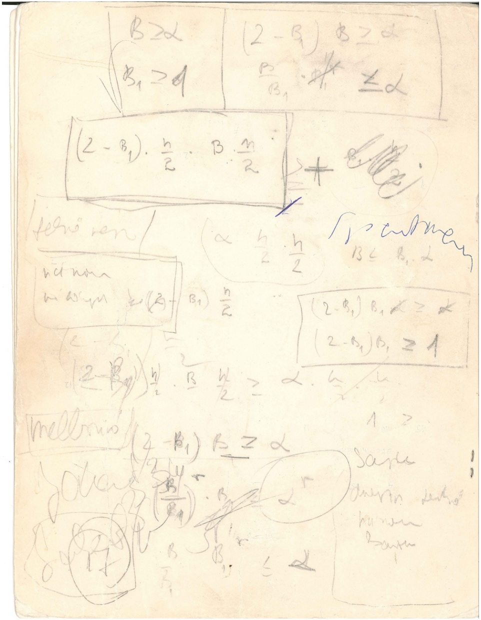
REGULAR PARTITIONS OF GRAPHS

by

E. Szemerédi

STAN-CS-75-489
APRIL 1975

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY



Ramsey theory

$$H \longrightarrow (G)_r^{\text{edge}}$$

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Every r-coloring of $E(H)$ yields a monochromatic copy of G.

$$H \longrightarrow (G)_{2}^{\text{edge}}$$

Every 2-coloring of $E(H)$ yields a monochromatic copy of G.

Ramsey theory

$$H \longrightarrow \; G$$

Goal: for given G, *minimize* H.

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Two variants:

- **Ramsey numbers**: min. # of vertices of H

$$r(G) = \min\{n \ : \ K_n \rightarrow G\}.$$

Ramsey theory

$$H \rightarrow G$$

Goal: for given G, *minimize* H.
Two variants:

- **Ramsey numbers**: min. # of vertices of H

$$r(G) = \min\{n : K_n \rightarrow G\}.$$

- **Size-Ramsey numbers**: min. # of edges of H

$$\hat{r}(G) = \min\{e(H) : H \rightarrow G\}.$$

A question of Erdős, Faudree, Rousseau and Schelp

Question (1978): For the path P_n,

$$\frac{\hat{r}(P_n)}{n} \to \infty\,? \quad \frac{\hat{r}(P_n)}{n^2} \to 0\,?$$

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COMBINATORIAL PROBLEMS 33

Bondy and I [7] conjectured

(3) $$r(C_n, C_n, C_n) \leqq 4n-3.$$

It is easy to see that if (3) is true then for odd n it is best possible. Vera Rosta [72] and independently R. Faudree and Schelp [44] determined $r(C_n, C_m)$ for every n and m.

A problem of Faudree, Rousseau, Schelp and myself: Let $\hat{r}(P_n)$ be the smallest integer for which there is a graph $\mathcal{G}$ of $\hat{r}(P_n)$ edges so that if we color the edges of $\mathcal{G}$ with two colors, there is always a monochromatic path P_n of length n. Is it true that

(4) $$\hat{r}(P_n)/n \to \infty, \quad \hat{r}(P_n)/n^2 \to 0?$$

Both of these questions seemed very interesting to us but we had no success at all with (4). It would be useful to have an asymptotic formula or at least a good inequality for $\hat{r}(P_n)$, but the first step is clearly to settle (4). I offer 100 dollars for a proof or disproof of (4).

Just one more problem of Hajnal, Rado and myself:

Is it true that

(5) $$\log\log r(K^{(3)}(n), K^{(3)}(n)) > cn?$$

Linear bounds

Theorem [Beck (1983)]: $\hat{r}(P_n) < 900n$.

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$$\hat{r}(G) = o(n^2) \text{ for every } G \in \mathcal{G}_{n,\Delta}$$

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Problem [Beck]: $\hat{r}(G) \overset{???}{<} c_\Delta n$ for every $G \in \mathcal{G}_{n,\Delta}$?

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Answer [R-Szemerédi]: No

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Problem [Beck]: $\hat{r}(G) \overset{???}{<} c_\Delta n$ for every $G \in \mathcal{G}_{n,\Delta}$?

Answer [R-Szemerédi]: No, i.e., $\hat{r}(G) \gg n$ for some $G \in \mathcal{G}_{n,3}$.

Construction of the graph G

Goal: construct G on n vertices with $\Delta(G) = 3$ such that

$$H \not\to G \text{ if } e(H) \leq n(\log n)^{1/30}.$$

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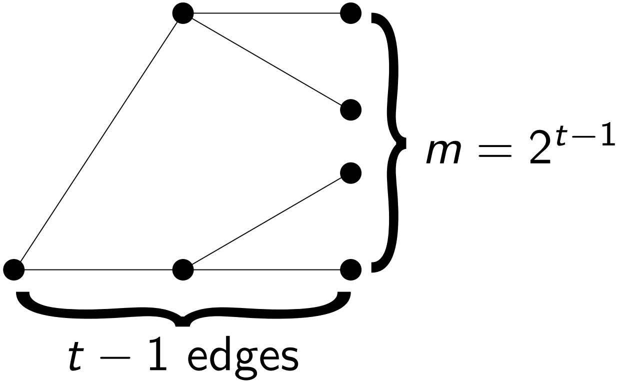
Proof: Take t and $m = 2^{t-1}$ satisfying

$$\frac{2\log n}{\log\log n} \leq m \leq \frac{4\log n}{\log\log n}.$$

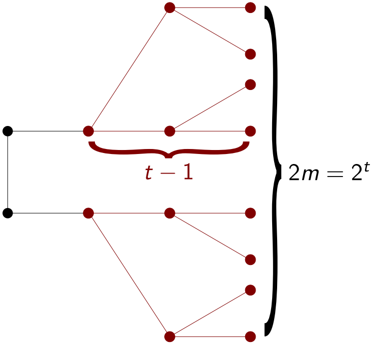
Construction of the graph G

Building block: binary tree of height $t-1$.



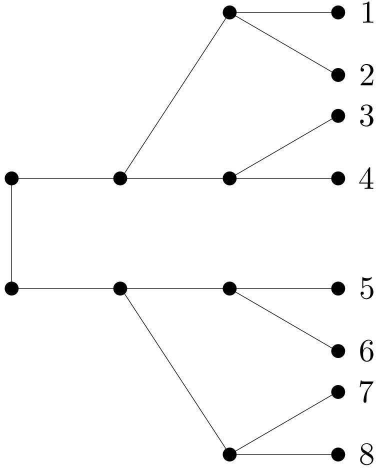
Construction of the graph G

Tree T



$$\# \text{ of vertices} = 2 + \sum_{i=1}^{t} 2^i = 2^{t+1} = 4m$$

Automorphisms of T



Construction of the graph G

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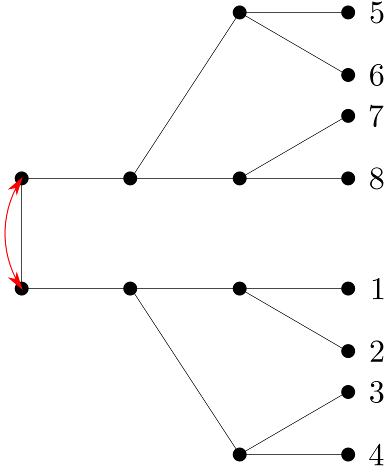

Number of choices $= 2^{2^0}$

Construction of the graph G

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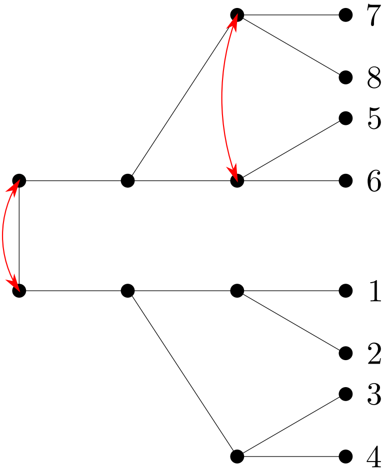
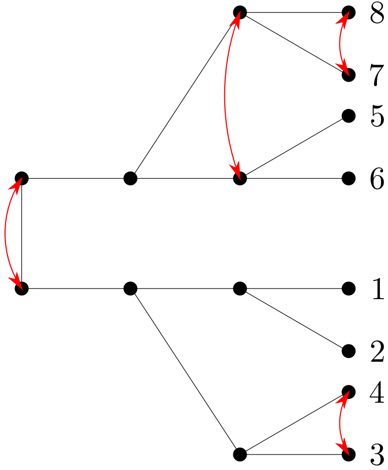


Number of choices $= 2^{2^0} \times 2^{2^1}$

Construction of the graph G

Automorphisms of T



Number of choices $= 2^{2^0} \times 2^{2^1} \times 2^{2^2}$

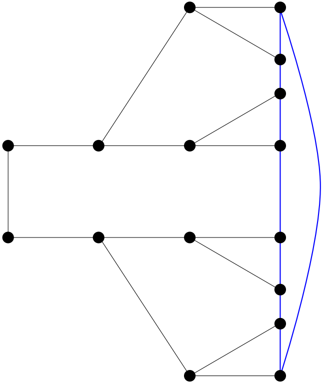
Construction of the graph G

Automorphisms of T

$$|\operatorname{Aut}(T)| = 2^{1+2+\cdots+2^{t-1}} = 2^{2^t-1}.$$

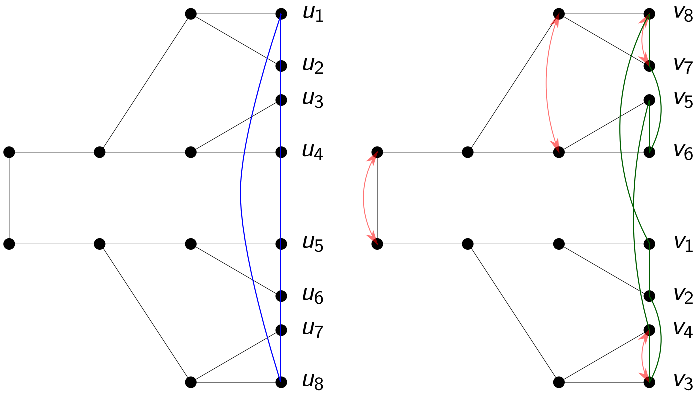
Construction of the graph G

Basic component G^i of G:



$G^i = T + C^i$, where C^i is a Hamiltonian cycle on the $2m$ leaves of T.

Two isomorphic components



Construction of the graph G

$\#$ of possible (labeled) Hamiltonian cycles on $2m$ leaves

$$= \frac{(2m)!}{2m}$$

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$\#$ of non-isomorphic basic components

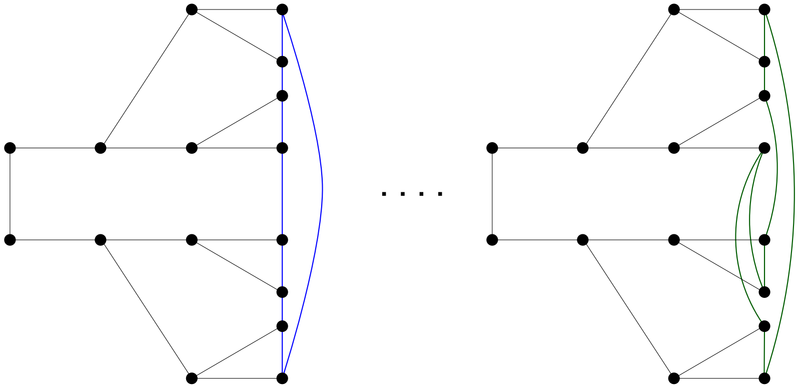
$$\frac{(2m)!}{2m\,|\,\mathrm{Aut}(T)|} > \frac{(2m-1)!}{2^{2^t}} = \frac{(2m-1)!}{2^{2m}} > \cdots > m^m > n > \frac{n}{4m} =: q$$

since $m \geq \frac{2\log n}{\log\log n}$.

Construction of the graph G

$G =$ disjoint union of q pairwise non-isomorphic $G^i = T + C^i$.

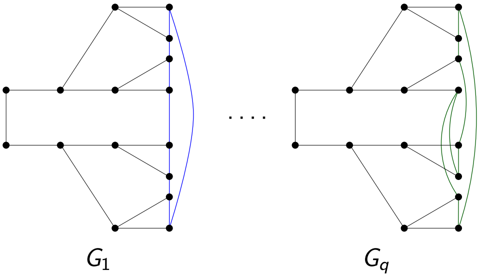
Construction of the graph G



G_1 G_q

Some facts about G

- every vertex of G has degree either 2 or 3;



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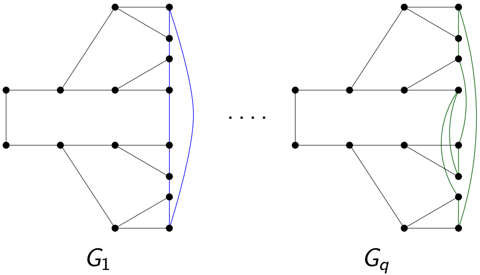
- every vertex of G has degree either 2 or 3;
- $|V(G)| = n$ and $|E(G)| \leq \dfrac{3n}{2}$;



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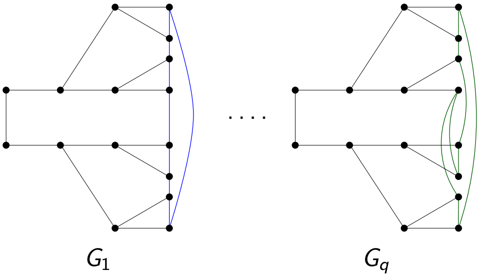

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Let I be a maximum independent set of G.

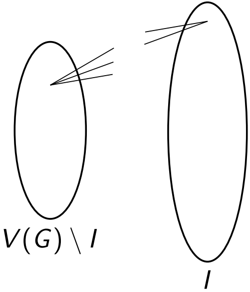
$$e(I, V(G) \setminus I) \geq 2\,|I| = 2\alpha$$

$$e(I, V(G) \setminus I) \leq 3\,|V(G) \setminus I| = 3(n - \alpha).$$

Therefore

$$2\alpha \leq 3(n - \alpha)$$

and $\alpha \leq \frac{3n}{5}$.



Lower bound on $\hat{r}(G)$

Let $\ell = (\log n)^{1/30}$ and suppose H has $n\ell$ edges. Then $H \not\to G$.

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Define $k = 10\ell$ and

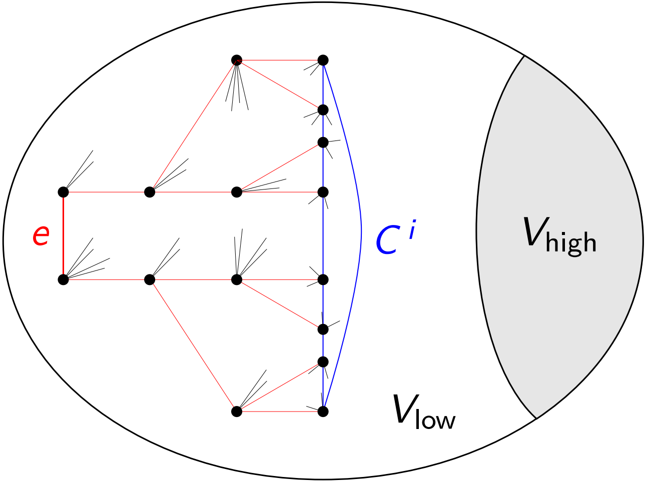
$$V_{\text{high}} = \{x \in V(H) \,:\, \deg_H(x) > k\}$$

$$V_{\text{low}} = \{x \in V(H) \,:\, \deg_H(x) \leq k\}.$$

Note: $|V_{\text{high}}| < \frac{n}{5}$.

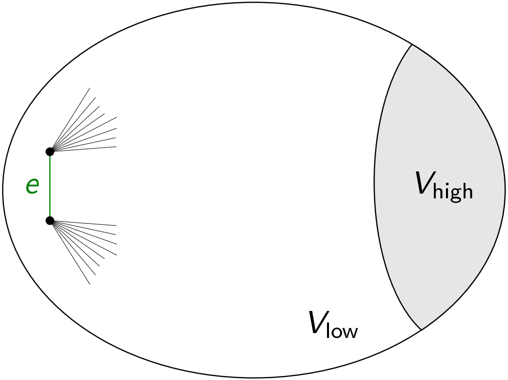
Lower bound on $\hat{r}(G)$

An edge $e \in H[V_{\text{low}}]$ *can see* the cycle C^i, if there is a copy of G^i in $H[V_{\text{low}}]$ in which the edge e is the "root edge" of T.



Lower bound on $\hat{r}(G)$

Question: How many cycles can an edge e see?



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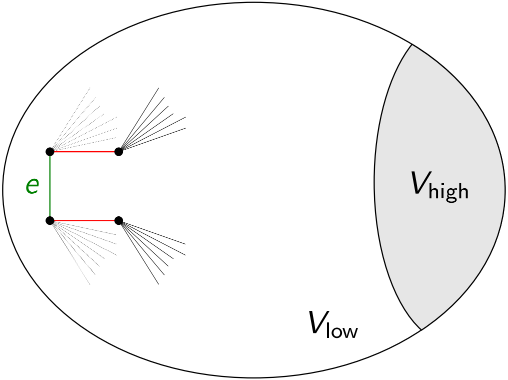

$k^2 \ldots$

Lower bound on $\hat{r}(G)$

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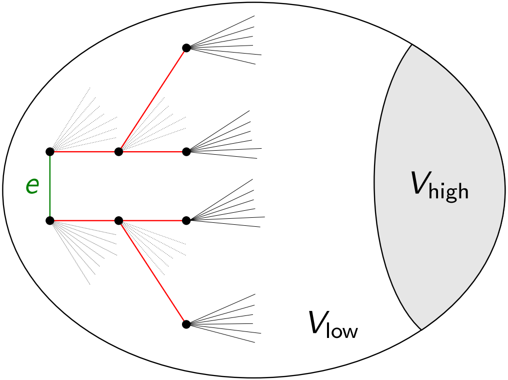

$$k^2 \binom{k}{2}^{(2+\cdots \qquad)} \cdots$$

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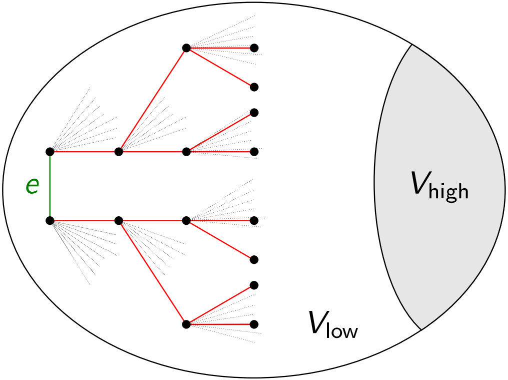

$$k^2 \binom{k}{2}^{(2+\cdots+2^{t-1})} \cdots$$

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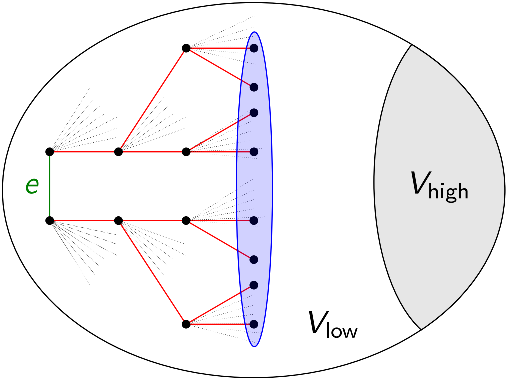

$$k^2 \binom{k}{2}^{(2+\cdots+2^{t-1})} \times \underbrace{k^{2m-1}}_{\#\text{ of cycles spanned by } 2m \text{ vertices}}$$

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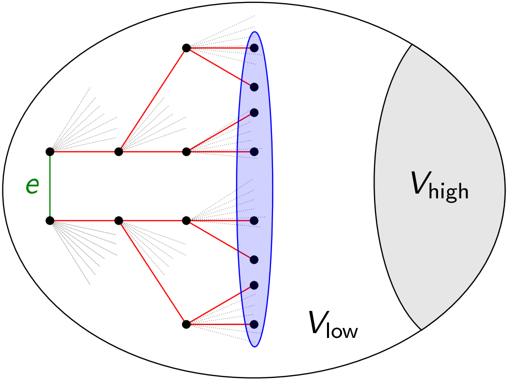

$$k^2 \binom{k}{2}^{(2+\cdots+2^{t-1})} \times \underbrace{k^{2m-1}}_{\#\text{ of cycles spanned by } 2m \text{ vertices}} < k^{6m}.$$

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$$k^{6m} \leq n^{4/5+o(1)}.$$

Recall

$$k = 10(\log n)^{1/30}, \quad m \leq \frac{4 \log n}{\log \log n}$$

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Claim: $\exists\ i_0 \in [q]$ for which C^{i_0} can be seen by at most $n^{5/6}$ edges.

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Proof: On average a cycle can be seen by at most

$$\frac{1}{q}|E(H[V_{\text{low}}])| \cdot k^{6m}$$

edges.

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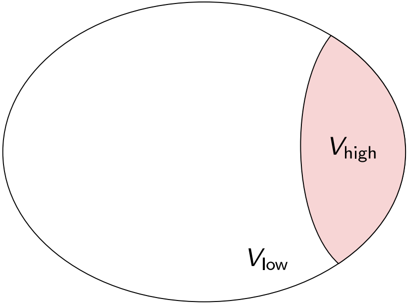
$$\frac{1}{q}|E(H[V_{\text{low}}])| \cdot k^{6m} \leq \frac{n\ell \cdot n^{4/5+o(1)}}{q} \underbrace{=}_{q=\frac{n}{4m}} \overbrace{(4m) \cdot \ell}^{\text{logarithms}} \cdot n^{4/5+o(1)} < n^{5/6}$$

edges.

The coloring:



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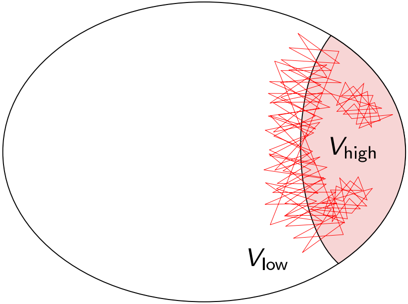

Red edges:

- edges incident to V_{high}

Lower bound on $\hat{r}(G)$

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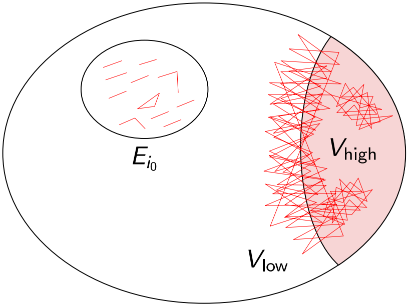

Red edges:

- edges incident to V_{high}
- $E_{i_0} = \{e \in H[V_{\text{low}}] \,:\, e \text{ can see } C^{i_0}\}, \quad |E_{i_0}| \leq n^{5/6}.$

Lower bound on $\hat{r}(G)$

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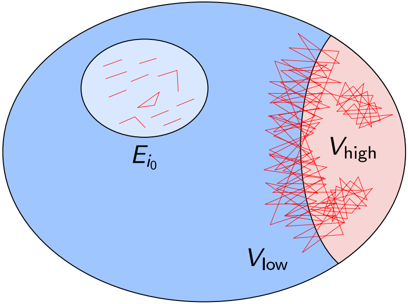

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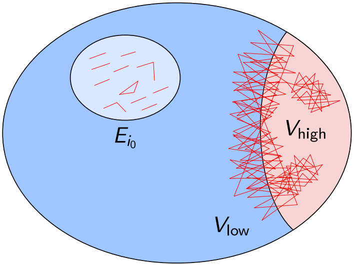
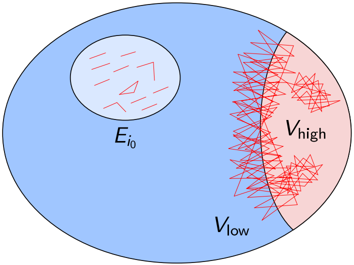
All other edges are blue.

Lower bound on $\hat{r}(G)$



No BLUE copy of G since no blue edge can see the cycle C^{i_0}.



No RED copy of G: any subgraph of H_{red} of order n has an independent set of size

$$\geq n - |V_{\text{high}}| - \left| \bigcup_{e \in E_{i_0}} e \right| \geq n - \frac{n}{5} - 2n^{5/6} > \frac{3n}{5}.$$

On the other hand $\alpha(G) \leq \frac{3n}{5}$. □

Summary

$$\mathcal{G}_{n,\Delta} = \{G \ : \ |V(G)| = n, \Delta(G) \leq \Delta\}$$

$$\hat{r}_{n,\Delta} = \max\{\hat{r}(G) \ : \ G \in \mathcal{G}_{n,\Delta}\}.$$

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- Conjecture [R-Szemerédi (2000)]:

$$\forall\, \Delta \geq 3\ \exists\, \varepsilon > 0, \quad n^{1+\varepsilon} \leq \hat{r}_{n,\Delta} \leq n^{2-\varepsilon}.$$

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$$\forall\, \Delta \geq 3\; \exists\, \varepsilon > 0, \quad n^{1+\varepsilon} \leq \hat{r}_{n,\Delta} \leq n^{2-\varepsilon}.$$

- Theorem [Kohayakawa-R-Schacht-Szemerédi (2010)]:

$$\hat{r}_{n,\Delta} \leq n^{2-1/\Delta+o(1)} \quad \text{for all} \quad \Delta \geq 3$$

Universal Ramsey graphs

$$H \longrightarrow \mathcal{G}_{n,\Delta} \quad \Leftrightarrow \quad H \longrightarrow G \text{ for all } G \in \mathcal{G}_{n,\Delta}.$$

Universal Ramsey graphs

$$H \to \mathcal{G}_{n,\Delta} \quad \Leftrightarrow \quad H \to G \text{ for all } G \in \mathcal{G}_{n,\Delta}.$$

Theorem [KRSS]: For all $\Delta \geq 3$ there exist an H such that

$$H \to \mathcal{G}_{n,\Delta}$$

and

$$|E(H)| \leq n^{2-1/\Delta+o(1)}.$$

Universal Ramsey graphs

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and

$$|E(H)| \leq n^{2-1/\Delta+o(1)}.$$

Fact: If $H \supseteq G$ for every $G \in \mathcal{G}_{n,\Delta}$ then

$$|E(H)| \geq n^{2-2/\Delta}.$$

Universal Ramsey graphs

$$H \to \mathcal{G}_{n,\Delta} \quad \Leftrightarrow \quad H \to G \text{ for all } G \in \mathcal{G}_{n,\Delta}.$$

Theorem [KRSS]: For all $\Delta \geq 3$ there exist an H such that

$$H \to \mathcal{G}_{n,\Delta}$$

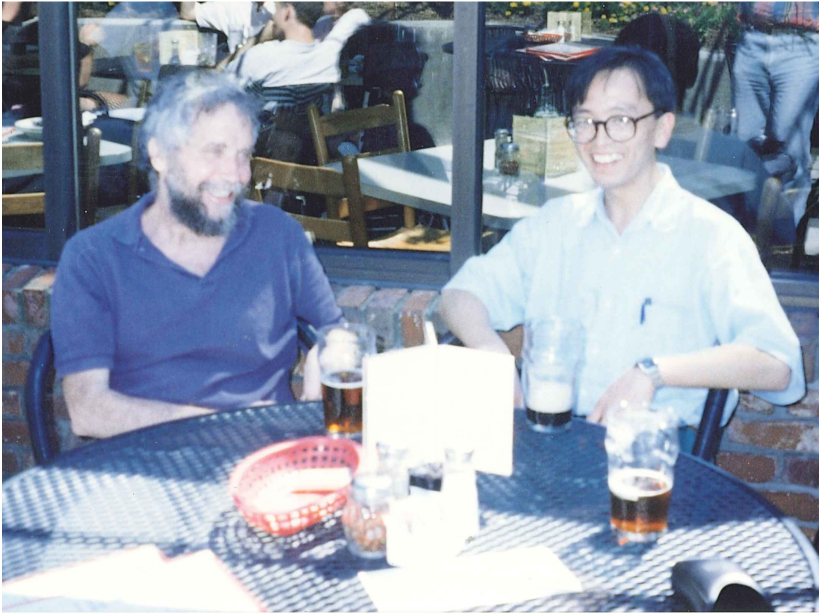
and

$$|E(H)| \leq n^{2-1/\Delta+o(1)}.$$

Fact: If $H \supseteq G$ for every $G \in \mathcal{G}_{n,\Delta}$ then

$$|E(H)| \geq n^{2-2/\Delta}.$$

Question: Is $\hat{r}_{n,\Delta} < n^{1.99}$ for every Δ?

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Happy Birthday, Endre!!!