The ideas of step 1 help to show other results.

Conjecture: $\forall$ even d: $p_{n+1} - p_n = d$ i.o. (inf. often)
(de Polignac) [1849]

<u>Def</u> $\mathcal{D}_W$ = de Polignac numbers (1849) in the weaker sense $p, p+d \in \mathcal{P}$ i.o.

<u>Def</u> $\mathcal{D}_S$ = strong de Polignac numbers: $p_{n+1} - p_n = d$ i.o.

(weak de Polignac numbers could be called Kronecker numbers (1901))

Thm G. (GPY) $\mathcal{D}_W$ has a pos. lower density if $\vartheta > 1/2$

Thm H. (GPY). $|\mathcal{D}_S| \geq 1$ if $\vartheta > 1/2$

Thm 3. $\mathcal{D}_S$ has a pos. lower density if $\vartheta > 1/2$.

Remark: $\mathcal{D}_W \neq \{0\} \Leftrightarrow \underline{\lim}(p_{n+1} - p_n) < \infty$

<u>Problem</u>: Thm C asserts that

$\forall c > 0 \quad p_{n+1} - p_n < c \log p_n$ i.o.

Howewer, is this true for $\forall c > 0$ for a set of primes with relative pos. lower density?

<u>Thm I</u> (GY, 200?) If $c > 1/4$ the answer is yes.

<u>Thm 4</u> (GPY, 201?) The answer is yes for $\forall c > 0$

<u>Remark</u>. The answer is no, if the fixed $c > 0$ is substitufed by any $g(n) \to 0$ as $n \to \infty$.

The strongest hypothesis about the distribution level of primes, $\vartheta=1$, the Elliott-Halberstam conjecture implies

(*) $p_{n+1}-p_n \le 16$ inf. often (GPY) (E)

and $\exists$ m-term AP's with (*) $\forall m$ (Thm 1)

Problem: does there exists a plausible hypothesis $\Rightarrow$ $\forall m$ $\exists m$-term AP of twin primes [Not. $\theta(n)=\begin{cases}\log p & n=p\\ 0 & \text{else}\end{cases}$

Thm 5. Suppose $\vartheta>0.724$ is a distribution level for primes AND for $f(n)=\lambda(n),\ \lambda(n)\lambda(n+h),\ \log p\,\lambda(p+h)$ [$\theta(n)\lambda(n+h)$] and $\lambda(p-h)\log p$, i.e. $\forall \varepsilon, A>0$:

$$\Big[\sum_{q\le N^{\vartheta-\varepsilon}} \max_a \Big|\sum_{n\equiv a\,(q)} f(n)\Big| \ll_{\varepsilon,A} \frac{N}{\log^A N}\Big]$$

Then $\forall m$ $\exists$ m-term AP of primes p such that $p+h$ is prime too

<u>**Def**</u> Let $Q=\{q_n\}_{n=1}^{\infty}$ be the set of q's which are the products of two diffent primes, called also semiprimes or E_2-numbers

<u>**Contrast**</u>.

Thm I' (Chen 1966/73): $\exists$ inf. many primes p with $p+2 \in P_2$

$p+2 \in \mathcal{P}$ or $p+2 = p'p'' \in Q$

In 2005 it was still open

$$\liminf_{n\to\infty} \frac{q_{n+1}-q_n}{\log q_n/\log\log q_n} \overset{?}{\leq} 0$$

Thm J (GGPY = S.W. Graham + GPY)

$q_{n+1}-q_n \leq 6$ inf. often

(2008)

PROBLEM: PARITY PHENOMENON

Thm 6 $\exists\, d = 2, 4$ or 6 s.t. $\forall m$ $\exists$ m-term AP of semiprimes q s.t. $q + d \in Q$ too

Thm 7. For at least one third of all even numbers: $\mathcal{D} = \{d_\nu\}$ **dens** $\mathcal{D} \geq 1/6$,
$\forall d_\nu$ $\forall m$ $\exists$ m-term AP of q's s.t. $q, q + d_\nu \in Q$

Thm 8. $\forall k$ $\exists$ $\mathcal{H} = \{h_i\}_{i=1}^{k}$ s.t. $\forall m$ $\exists$ m-term AP of q's s.t.
(Hardy-Littlewood semiprime k-tuple conj)
$q, q + h_1, \ldots q + h_k \in Q$

Thm 9. If $|\mathcal{H}| = 3$, $\mathcal{H} = \{h_k\}_{k=1}^{3}$ is admissible, then $\exists\, i, j \in \{1, 2, 3\}$, $i \neq j$ s.t. $\exists$ m-term AP of n's $n + h_i, n + h_j \in Q$

Problems of Erdős

C1: Erdős–Mirsky: Is $d(n) = d(n+1)$ inf. often?

Thm K (Heath-Brown 1984): Yes

also for $\Omega(n)$ instead of $d(n)$

But, due to the parity problem the common value or even the parity of the common value could not be given in advance

Thm L (Schlage-Puchta, 2003/5)

$\omega(n) = \omega(n+1)$ inf. often

In joint work with GGPY we could prove this where the common value could be given in advance almost arbitrarily

<u>**Def**</u>: the exponent pattern of $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the multiset $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$

Question 1: Do we have (a-priori) any given exponent pattern $\mathcal{A}$ such that n and $n+1$ have e.p. $\mathcal{A}$ infinitely often?

Question 2: Do we have any à priori given e.p. $\mathcal{A}$ such that there exist arbitrarily long AP's of n such that n and $n+1$ have e.p. $\mathcal{A}$ for elements "n" of the AP's?

Theorem 9A. The answer for Q2 is yes for any $\mathcal{A}$ if $(1,1,1,2) \subset \mathcal{A}$

Thm 10. $\forall m\ \exists m$-term AP of n-values such that simultaneously

(i) $\omega(n) = 4 + B$, $\Omega(n) = 5 + B$,

$d(n) = 24 \cdot 2^B$ and

$\omega(n) = \omega(n+1)$, $\Omega(n) = \Omega(n+1)$, $d(n) = d(n+1)$

with any $B \geq 0$ or

(ii) $\omega(n) = 4$, $\Omega(n) = 5 + B$, $d(n) = 24(B+1)$

$\omega(n) = \omega(n+1)$, $\Omega(n) = \Omega(n+1)$, $d(n) = d(n+1)$

Thm 11 $\forall m\ \exists m$-term AP of n-values with $\omega(n) = \omega(n+1) = 3$

Thm 12 $\forall m\ \exists m$-term AP of n-values with $\Omega(n) = \Omega(n+1) = 4$

Thm 13: The number of n's below N satisfying the above conditions is $\gg c N / (\log N)^3$ [Exp. $c \log^2 N (\log_2 N)^c$)

<u>Proof of Thm 11 in the weaker form</u> that $\omega(n)=\omega(n+1)=3$ inf. o. or in the original form using Thm

Let $L_1(n) = 6m+1$, $L_2(n) = 8m+1$

and $L_3(m) = 9m+1$. Then

$3L_1 = 2L_3 + 1$, $4L_1 = 3L_2 + 1$, $9L_2 = 8L_3 + 1$

If for example the are arbitrarily long AP's of ~~m's~~ semiprimes ~~with~~ $L_1(m)$ and $L_2(m)$ (i.e. arb. long AP's of m_ν values s.t. $L_1(m_\nu), L_2(m_\nu) \in Q$) then

$\omega(3L_1) = 3$, $\omega(2L_3) = 3$, $3L_1 = 2L_3 + 1$

<u>Proof of Thm 11 in the weaker form</u> that $\omega(n)=\omega(n+1)=3$ inf. o.

or in the original form using Thm 9

Let $L_1(n) = 6m+1$, $L_2(n) = 8m+1$

and $L_3(m) = 9m+1$. Then

$3L_1 = 2L_3 + 1$, $4L_1 = 3L_2 + 1$, $9L_2 = 8L_3 + 1$

If for example the are arbitrarily long AP's of semiprimes ~~with~~ $L_1(m)$ and $L_2(m)$ (i.e. arb. long AP's of m_ν values s.t. $L_1(m_\nu), L_2(m_\nu) \in Q$) then

$\omega(3L_1) = 3$, $\omega(2L_3) = 3$, $3L_1 = 2L_3 + 1$,