Complex Hadamard matrices with a special structure

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Hadamard Matrices, Budapest, July 11, 2017

Quantum Theory: basic notation

Quantum States (in a finite dimensional Hilbert space)

a) pure states: normalized elements of the complex Hilbert space,
$|\psi\rangle \in \mathcal{H}_N$ with $||\psi||^2 = \langle\psi|\psi\rangle = 1$,
defined up to a global phase, $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$,
thus the set Ω_N of all pure states forms
a **complex projective space** $\mathbb{C}P^{N-1}$, e.g. $\Omega_2 = \mathbb{C}P^1 = S^2$

b) mixed states: convex combinations of projectors onto pure states,
$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $\sum_i p_i = 1$, $p_i \geq 0$, so that the states are:
Hermitian, $\rho = \rho^*$, positive, $\rho \geq 0$ and normalized, $\mathrm{Tr}\rho = 1$.

Unitary Quantum Dynamics

a) pure states: $|\psi'\rangle = U|\psi\rangle$,

b) mixed states: $\rho' = U\rho U^*$,

where **unitary** $U \in U(N)$ is called a **quantum gate**, (can be **complex**).
In the case $N = 2$ it is called a **single qubit quantum gate**.

Hadamard matrices $\Rightarrow$ real quantum gates

Hadamard matrices are orthogonal (up to a rescaling)

as they consist of mutually orthogonal row and columns,

$$HH^* = N\mathbb{1} \;\Rightarrow\; H' := H/\sqrt{N} \text{ is unitary}$$

$N = 2$ Hadamard matrix $\Rightarrow$ one–qubit Hadamard gate

$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so that $H_2' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is orthogonal.

The most often used gate in Quantum Information Theory,
as it forms a **quantum superposition**

$$H_2|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and

$$H_2|1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

[in quantum information one denotes H_2 (no primes) for unitary]

Quantum computing

The basic building step is based on **Hadamard matrices**:

one qubit Hadamard matrix, H_2 (of size two)

multi–qubit Hadamard matrix, $H_{2^n} = H_2^{\otimes n}$ (of size $N = 2^n$)

Examples:

a) **two qubits**, $n = 2$

Note that $H_4|0,0\rangle = (H_2 \otimes H_2)|0\rangle \otimes |0\rangle = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$

corresponds to the superposition: $0 \; + \; 1 \; + \; 2 \; + \; 3.$

b) n **qubits**: consider the n–qubit state

$$|\psi\rangle = H_2^{\otimes n}|0,\ldots 0\rangle \qquad (**)$$

which leads to the uniform superposition, $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$

and allows to process all $N = 2^n$ numbers **"in parallel"**.


beeline

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

Complex Hadamard matrices

Hadamard matrices of the Butson type

composed of q-th **roots of unity**; $H \in H(N,q)$ iff

$$HH^* = N\,\mathbb{1}\ , \qquad (H_{ij})^q = 1 \quad \text{for} \quad i,j = 1,\ldots N \tag{1}$$

Butson, 1962

special case: $q = 4$

$H \in H(N,4)$ iff $HH^* = N\,\mathbb{1}$ and $H_{ij} = \pm 1, \pm i$

(also called **complex** Hadamard matrices, **Turyn, 1970**)

Complex Hadamard matrices (*general case*)

$HH^* = N\,\mathbb{1}$ and $|H_{ij}| = 1$,

hence $H_{ij} = \exp(i\phi_{ij})$ with an **arbitrary complex** phase.

Complex Hadamard matrices do exist for any N !

example: the **Fourier matrix**

$$(F_N)_{jk} := \exp(ijk2\pi/N) \quad \text{with} \quad j,k = 0,1,\dots,N-1. \qquad (2)$$

special case : $N = 4$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \in H(4,4) \qquad (3)$$

The **Fourier matrices** are constructed of N–th root of unity, so they are of the **Butson type**,

$$F_N \in H(N,N).$$

Equivalent Hadamard matrices

$$H' \sim H$$

iff there exist permutation matrices P_1 and P_2 and diagonal unitary matrices D_1 and D_2 such that

$$H' = D_1 P_1 \; H \; P_2 D_2 \; . \tag{4}$$

Dephased form of a Hadamard matrix

$$H_{1,j} = H_{j,1} = 1 \quad \text{for} \quad j = 1, \ldots, N. \tag{5}$$

Any complex Hadamard matrix can be brought to the dephased form by an equivalence relation.

example for $N = 3$, here $\alpha \in [0, 2\pi)$ while $w = \exp(i \cdot 2\pi/3)$, so $w^3 = 1$

$$F_3' = e^{i\alpha} \begin{bmatrix} w & 1 & w^2 \\ 1 & 1 & 1 \\ w^2 & 1 & w \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix} =: F_3 \; , \tag{6}$$

Classification of **Complex Hadamard matrices** I

$N = 2$

all complex Hadamard matrices are equivalent to the **real Hadamard (Fourier)** matrix

$$H_2 = F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \qquad (7)$$

$N = 3$

all complex Hadamard matrices are equivalent to the **Fourier matrix**

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix} , \quad w = e^{2\pi i/3}. \qquad (8)$$

U. Haagerup, Orthogonal maximal abelian $*$-subalgebras of the $N \times N$ matrices and cyclic N–rots,
in Operator Algebras and Quantum Field Theory, **1996.**

Classification of **Complex Hadamard matrices** II

$N = 4$

Lemma (Haagerup). For $N = 4$ all complex Hadamard matrices are equivalent to one of the matrices from the following 1–d orbit, $w = i$

$$F_4^{(1)}(a) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w^1 \cdot \exp(i \cdot a) & w^2 & w^3 \cdot \exp(i \cdot a) \\ 1 & w^2 & 1 & w^2 \\ 1 & w^3 \cdot \exp(i \cdot a) & w^2 & w^1 \cdot \exp(i \cdot a) \end{bmatrix}, \quad a \in [0, \pi].$$

$N = 5$

All $N = 5$ complex Hadamard matrices are equivalent to the **Fourier matrix** F_5 (**Haagerup 1996**).

$N \geq 6$

Several orbits of Complex Hadamard matrices are known, but the problem of their complete classification remains **open!**

1-d family by Beauchamp & Nicoara, April 2006

$$B_6^{(1)}(y) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1/x & -y & y & 1/x \\ 1 & -x & 1 & y & 1/z & -1/t \\ 1 & -1/y & 1/y & -1 & -1/t & 1/t \\ 1 & 1/y & z & -t & 1 & -1/x \\ 1 & x & -t & t & -x & -1 \end{bmatrix}$$

where $y = \exp(i\ s)$ is a free parameter and

$$x(y) = \frac{1 + 2y + y^2 \pm \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2}$$

$$z(y) = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}; \quad t(y) = xyz$$

W. Bruzda discovered this family independenty in May 2006

For **Complex Hadamard matrices** of size $N = 2, \ldots 16$

see online **Catalog** at

http://chaos.if.uj.edu.pl/∼karol/hadamard

(brand new 2016 engine by **Wojciech Bruzda**, some new data...)

If you know about **new complex Hadamard** matrices

(*or you found a* **misprint** *in the catalogue*)

please let know Wojtek (and me)

Wawel castle in Cracow

D.& K. Ciesielscy theorem

D.& K. Ciesielscy theorem: For any $\epsilon > 0$ there exist $\eta > 0$ such that with **probability** $1 - \epsilon$ the bench **Banach** talked to **Nikodym** in **1916** was localized in η-**neighbourhood** of the **red arrow**.

Plate commemorating the discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$ distinguished by the fact that reduced states are **maximally mixed**,

e.g. $\rho_A = \mathrm{Tr}_B|\varphi^+\rangle\langle\varphi^+| = \frac{1}{2}\mathbb{1}_2$.

Maximally entangled states of $d \times d$ system

Define bi-partite pure state by a matrix of coefficients,
$|\psi\rangle = \sum_{i,j=1}^{d} \Gamma_{ij}|i,j\rangle$.

Then reduced state $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| = \Gamma\Gamma^\dagger$.

It represents a **maximally entangled** state if $\rho_A = \Gamma\Gamma^\dagger = \mathbb{1}_d/d$, which is the case if the matrix $U = \sqrt{d}\Gamma$ of size d is **unitary**.

$k-$ uniform state of n subsystems

Consider a state of n subsystems with d levels each, $|\psi\rangle \in \mathcal{H}_d^{\otimes n}$.
Such a state is called k-**uniform** if for any choice of part X consisting of k subsystems out of n the partial trace over the part $\bar{X}$ consisting of remaining $n-k$ subsystems is maximally mixed,

$$\mathrm{Tr}_{\bar{X}} |\psi\rangle\langle\psi| = \frac{1}{d^k} \mathbb{1}_{d^k}. \tag{9}$$

Examples

a) 2–qubit state $|00\rangle + |01\rangle + |10\rangle - |11\rangle$ is **1–uniform** (Bell-like)

(as the coefficient matrix $\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is Hadamard !)

b) 3–qubit state $|GHZ\rangle = (|000\rangle + |111\rangle)$ is **1–uniform**

c) there are no 2–**uniform** states of 4 qubits,

but they exist for larger systems...

Hadamard matrices & quantum states

A Hadamard matrix $H_8 = H_2^{\otimes 3}$ of order $N = 8$ implies

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

This '**orthogonal array**'

allows us to construct a **2–uniform state** of 7 qubits:

$$|\Phi_7\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle.$$

a **simplex** state $|\Phi_7\rangle$

Examples of 2–uniform states obtained form H_{12}

8 qubits

$$|\Phi_8\rangle = |00000000\rangle + |00011101\rangle + |10001110\rangle + |01000111\rangle + |10100011\rangle + |11010001\rangle + |01101000\rangle + |10110100\rangle + |11011010\rangle + |11101101\rangle + |01110110\rangle + |00111011\rangle.$$

9 qubits

$$|\Phi_9\rangle = |000000000\rangle + |100011101\rangle + |010001110\rangle + |101000111\rangle + |110100011\rangle + |011010001\rangle + |101101000\rangle + |110110100\rangle + |111011010\rangle + |011101101\rangle + |001110110\rangle + |000111011\rangle.$$

10 qubits

$$|\Phi_{10}\rangle = |0000000000\rangle + |0100011101\rangle + |1010001110\rangle + |1101000111\rangle + |0110100011\rangle + |1011010001\rangle + |1101101000\rangle + |1110110100\rangle + |0111011010\rangle + |0011101101\rangle + |0001110110\rangle + |1000111011\rangle,$$

Higher dimensions: uniform states of *qutrits*, $d = 3$, and *ququarts*, $d = 4$

A pair of **orthogonal Latin squares** of size 3,

$$\begin{array}{|c|c|c|} \hline 0\alpha & 1\beta & 2\gamma \\ \hline 1\gamma & 2\alpha & 0\beta \\ \hline 2\beta & 0\gamma & 1\alpha \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline A\spadesuit & K\clubsuit & Q\diamondsuit \\ \hline K\diamondsuit & Q\spadesuit & A\clubsuit \\ \hline Q\clubsuit & A\diamondsuit & K\spadesuit \\ \hline \end{array} .$$

yields a 2–**uniform** state of **4 qutrits**:

$$\begin{aligned} |\Psi_3^4\rangle \quad = \quad & |0000\rangle + |0112\rangle + |0221\rangle + \\ & |1011\rangle + |1120\rangle + |1202\rangle + \\ & |2022\rangle + |2101\rangle + |2210\rangle. \end{aligned}$$

Corresponding **Quantum Code**:
$$\begin{aligned} |0\rangle &\rightarrow |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\ |1\rangle &\rightarrow |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\ |2\rangle &\rightarrow |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle \end{aligned}$$

Combinatorial designes

$\Longrightarrow$ An introduction to "*Quantum Combinatorics*"

A classical example:

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

Combinatorial designes

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and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

$$\begin{array}{cccc} A\spadesuit & K\diamondsuit & Q\heartsuit & J\clubsuit \\ K\heartsuit & A\clubsuit & J\spadesuit & Q\diamondsuit \\ Q\clubsuit & J\heartsuit & A\diamondsuit & K\spadesuit \\ J\diamondsuit & Q\spadesuit & K\clubsuit & A\heartsuit \end{array}$$

Two **mutually orthogonal Latin squares** of size $N = 4$
2 **MOLS(4)** = **Graeco–Latin** square !

Mutually ortogonal Latin Squares (MOLS)

♣) $N = 2$. There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)
♡) $N = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(N-1)$ MOLS.
♠) $N = 6$. Only a **single** Latin Square exists (No OLS!).

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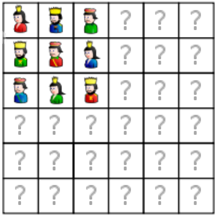
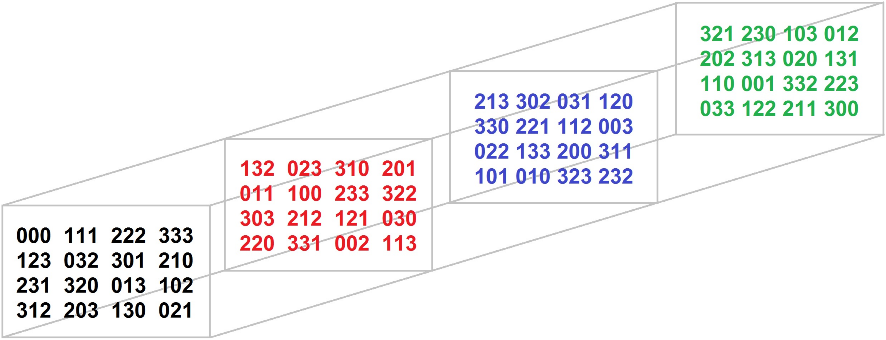
Euler's problem: **36** officers of six different ranks from six different units come for a **military parade** Arrange them in a square such that: in each row / each column all uniforms are different.



No solution exists ! (conjectured by **Euler**), proof by:
Gaston Terry "Le Probléme de 36 Officiers". *Compte Rendu* **(1901)**.

Mutually ortogonal Latin Squares (MOLS)

An apparent solution of the $N = 6$ **Euler's** problem of **36 officers**.



State $|\Psi_4^6\rangle$ of **six** **ququarts** can be generated by three mutually orthogonal **Latin cubes of order four!**

(three quarts + three address quarts = 6 quarts in $4^3 = 64$ terms)

Six ququarts

The 3–**uniform** state of **6 ququarts**:

read from three **mutually orthogonal Latin cubes**

$$|\Psi_4^6\rangle =$$

$$|000000\rangle + |001111\rangle + |002222\rangle + |003333\rangle + |010123\rangle + |011032\rangle +$$
$$|012301\rangle + |013210\rangle + |020231\rangle + |021320\rangle + |022013\rangle + |023102\rangle +$$
$$|030312\rangle + |031203\rangle + |032130\rangle + |033021\rangle + |100132\rangle + |101023\rangle +$$
$$|102310\rangle + |103201\rangle + |110011\rangle + |111100\rangle + |112233\rangle + |113322\rangle +$$
$$|120303\rangle + |121212\rangle + |122121\rangle + |123030\rangle + |130220\rangle + |131331\rangle +$$
$$|132002\rangle + |133113\rangle + |200213\rangle + |201302\rangle + |202031\rangle + |203120\rangle +$$
$$|210330\rangle + |211221\rangle + |212112\rangle + |213003\rangle + |220022\rangle + |221133\rangle +$$
$$|222200\rangle + |223311\rangle + |230101\rangle + |231010\rangle + |232323\rangle + |233232\rangle +$$
$$|300321\rangle + |301230\rangle + |302103\rangle + |303012\rangle + |310202\rangle + |311313\rangle +$$
$$|312020\rangle + |313131\rangle + |320110\rangle + |321001\rangle + |322332\rangle + |323223\rangle +$$
$$|330033\rangle + |331122\rangle + |332211\rangle + |333300\rangle.$$

k–uniform states and k–unitary matrices

Consider a **2–uniform** state of four parties A, B, C, D with d levels each,

$$|\psi\rangle = \sum_{i,j,l,m=1}^{d} \Gamma_{ijlm} |i,j,l,m\rangle$$

It is **maximally entangled** with respect to all **three** partitions:

$$AB|CD \text{ and } AC|BD \text{ and } AD|BC.$$

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions are **maximally mixed**,
$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD} = \rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD} = \rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD} = \mathbb{1}_{d^2}/d^2$

Thus matrices $U_{\mu,\nu}$ of order d^2 obtained by reshaping the tensor $d\Gamma_{ijkl}$ are **unitary** for three reorderings:

a) $\mu, \nu = ij, lm$, b) $\mu, \nu = im, jl$, c) $\mu, \nu = il, jm$.

Such a tensor Γ is called **perfect**.

Corresponding **unitary matrix** U of order d^2 is called **two–unitary**
if reordered matrices U^{R_1} and U^{R_2} remain **unitary.**

Unitary matrix U of order d^k with analogous property is called **k–unitary**

Exemplary multiunitary matrices

Two–unitary permutation matrix of size $9 = 3^2$
associated to 2 **MOLS(3)** and **2–uniform** state $|\Psi_3^4\rangle$ of 4 qutrits

$$U = U_{\substack{ij\\ml}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \in U(9)$$

Furthermore, also two reordered matrices
(by partial transposition and reshuffling) remain **unitary**:

$$U^{T_1} = U_{\substack{il \\ mj}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in U(9)$$

$$U^{R} = U_{\substack{im \\ jl}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \in U(9)$$

Are there multiunitary Hadamard matrices?

no for $d = 2$ and $N = d^2 = 4$ (*to many constraints!*)
Yes for $d = 3$ and $N = d^2 = 9$ and for $d = 2$ and $N = d^3 = 8$
Example: 3–unitary real Hadamard matrix of size $N = 2^3 = 8$
associated to the **3–uniform** state $|\Psi_2^6\rangle$ of 6 qubits

$$H_{\substack{ijk\\lmn}} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix} \in H_8$$

This unitary matrix remains **unitary** after any of $\frac{1}{2}\binom{6}{3} = 10$ reorderings related to different decomposition of the hypercube with $8^2 = 2^6 = 64$ entries.

$$H_{\substack{ijk\\lnm}} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix} \in H_8$$

$$H_{\substack{ijk\\nlm}} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix} \in H_8$$

Open issue I

Multi–unitary Hadamard (and other unitary) matrices

Let H be a Hadamard matrix of size $N = d^k$

It is called **multi–unitary**

if the corresponding tensor (of size d with $2k$ indices) is **perfect**,

which means that all its $\frac{1}{2}\binom{2k}{k}$ **reorderings** also form a **Hadamard matrix**

To be done: Identify and classify

a) multi–unitary **real Hadamard matrices**

b) multi–unitary **complex Hadamard matrices**

c) all **multi–unitary matrices**

$N = 36$ Euler–like conjecture

Euler 36–**officers** problem: no 2 **MOLS(6)** $\Leftrightarrow$
there are no 2–unitary **permutation** matrices of order $N = 6^2 = 36$.

Is there at all a 2–**unitary** matrix of order $N = 36$?
(= a set of 36 **"entangled officers"** of **Euler**) ??

$N = 36$ Euler–like conjecture

Euler 36–**officers** problem: no 2 **MOLS(6)** $\Leftrightarrow$
there are no 2–unitary **permutation** matrices of order $N = 6^2 = 36$.

Is there at all a **2–unitary** matrix of order $N = 36$?
(= a set of 36 **"entangled officers"** of **Euler**) ??

basing on numerical results by **Z. Puchała** and **W. Bruzda**
we advance the following

Conjecture

There are no **two–unitary** matrices of order $N = 6^2 = 36$.

A proof of this conjecture would imply
the $N = 6$ non-existence theorem of **Euler – Terry**.

Cracow

and Tatra mountains in the background

Open issue II

H_2 reducible Hadamard matrices, (Bengt R. Karlsson, 2011)

A Hadamard matrix of size $N = 2m$ is called H_2 **reducible**,
if each of its 2×2 blocks forms a **Hadamard** matrix.

Example: **complex Hadamard** matrix of size $N = 6$ defined by the tensor product $F_3 \otimes H_2$ is H_2 **reducible** as it consists of 9 blocks of size two, each of them forming a (complex) Hadamard matrix.

Task II: Identify and classify

a) H_2 reducible **complex Hadamard matrices**
(done by **Karlsson, 2011** for $N = 6 = 2 \times 3$.)

b) H_3 reducible **complex Hadamard matrices**
(done by **Karlsson, 2016** for $N = 9 = 3 \times 3$.)

This issue is helpful in classyfying all
complex Hadamard matrices of order N

Robust Hadamard matrices I

Definition

A Hadamard matrix H of size N will be called **robust** if any of its projection onto 2-dimensional subspace forms a **Hadamard** matrix.

Equivalently, if

a) for **any** choice of indices i, j the truncated matrix $H_2 = \begin{bmatrix} H_{ii} & H_{ij} \\ H_{ji} & H_{jj} \end{bmatrix}$ is **Hadamard**,

b) any principal minor of H is extremal, $|\det(H_2)| = 2$

$$\textbf{Example } N = 4: \quad H_4^R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

A problem of **robust matrices** with $N(N-1)/2$ constraints, differs from Karlsson's problem of **H_2-reducible matrices** with $N^2/4$ constraints.

Robust Hadamard matrices II

a) doubly even dimension, $N = 4m$

Skew Hadamard matrix satisfies: $H_S + H_S^T = 2\mathbb{1}$.

Proposition: A **real Hadamard** matrix H is **robust** if it is sign–equivalent to a **skew Hadamard** matrix, $H_R = DH_S D'$ with D, D' diagonal sign matrices.

Existence: For $m < 69$ there exists a **skew Hadamard** matrix of size $4m$.

b) even dimension, $N = 4m + 2$

Conference matrix of size N satisfies: $CC^T = (N-1)\mathbb{1}$ with $C_{ij} = \pm 1$.

Construction with symmetric conference matrix $C = C^T$. The matrix $H_R = C + iI$ is **robust complex Hadamard**

as its main minors read $$\det\left(\begin{bmatrix} i & \pm 1 \\ \pm 1 & i \end{bmatrix}\right) = -2$$

Robust Hadamard matrices III

Example. **Robust Complex Hadamard** for $N = 6$

Using a **symmetric conference matrix** C_6 we obtain

$$H_6^R = C_6 + iI = \begin{bmatrix} i & 1 & 1 & 1 & 1 & 1 \\ 1 & i & 1 & -1 & -1 & 1 \\ 1 & 1 & i & 1 & -1 & -1 \\ 1 & -1 & 1 & i & 1 & -1 \\ 1 & -1 & -1 & 1 & i & 1 \\ 1 & 1 & -1 & -1 & 1 & i \end{bmatrix}.$$

Hence **robust complex Hadamard** matrices exist for $N = 6, 10, 14, 18, 26...,$ for which **symmetric** conference matrices exist.

Question: Is there a **complex robust Hadamard** for $N = 22$?

A more general set-up to Hadamard matrices

Birkhoff Polytope & Unistochastic matrices

Let B be a **bistochastic matrix** of order n, so that $\sum_i B_{ij} = \sum_j B_{ij} = 1$ and $B_{ij} \geq 0$ (also called *doubly stochastic*).
B **is called unistochastic** if there exist a unitary U such that $B_{ij} = |U_{ij}|^2$ what implies $B = f(U)$

Existence problem: **Which B is unistochastic?**

Every B of size $N = 2$ **is unistochastic**, for $N = 3$ it is not the case **(Schur)**. Constructive conditions for unistochasticity are known for $N = 3$ **Au-Yeung and Poon 1979**, but for $N = 4$ this problem remains open.

Classification problem: **Assume B is unistochastic**

Find all preimages U such that $f^{-1}(B) = U$.
Special case: Flat matrix of **van der Waerden** of size N, so $W_{ij} = 1/N$.
Then the problem of classification of all preimages of W reduces to the search for all **complex Hadamard matrices** of size N.

Robust Hadamard matrices & unistochasticity

Consider the **Birkhoff polytope** $\mathcal{P}_N$ containing **bistochastic** matrices of size N with the flat matrix W at its center.

Its **ray** r is formed by convex combinations of a given permutation matrix P and the center, $B = aP + (1-a)W \in r$

Unistochastic and orthostochastic rays

Proposition i) If there exists a **robust real Hadamard** matrix of size N any ray r of $\mathcal{P}_N$ is **orthostochastic**.

ii) If there exists a **robust complex Hadamard** matrix of size N any ray of $\mathcal{P}_N$ is **unistochastic**.

Thus for all even cases $N = 2, 4, 6, \ldots, 20$ the rays are **unistochastic**.

Open questions: a) What about $N = 22$?

b) Is the set $\mathcal{U}_N$ of **unistochastic** matrices **star shaped** ?

c) For which N there exists a **unistochastic ball** around the center W_N ?

Open issue III

Real and complex robust Hadamard matrices

A Hadamard matrix H of size N is called **robust**
if any of its projection onto 2-dimensional subspace forms a **Hadamard** matrix.

Task III: Identify and classify

a) robust **real Hadamard matrices**
(exist e.g. for N for which a **skew Hadamard** matrix exists)

b) robust **complex Hadamard matrices**
(exist e.g. for N for which **complex skew Hadamard** matrix exists)

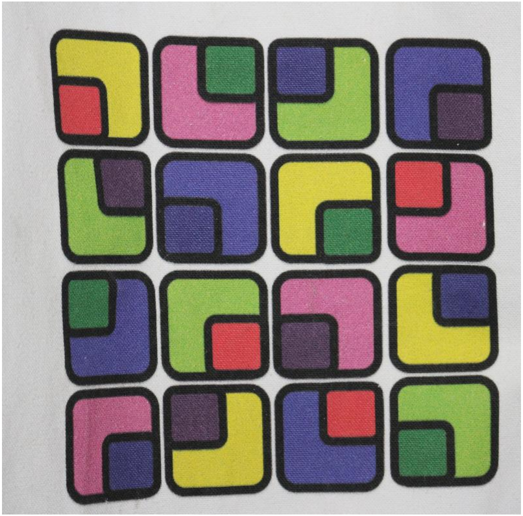
This issue is helpful in solving the
unistochasticity problem:
What bistochastic matrix B of order N is uni–(orto–)stochastic.

Bench commemorating the discussion between
Otton Nikodym and **Stefan Banach** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa   Fot. Andrzej Kobos

opened in Planty Garden, Cracow, Oct. 14, 2016

A quick quiz



What **quantum state** can be associated with this design ?

Hints

$A\spadesuit$	$K\diamondsuit$	$Q\heartsuit$	$J\clubsuit$
$K\heartsuit$	$A\clubsuit$	$J\spadesuit$	$Q\diamondsuit$
$Q\clubsuit$	$J\heartsuit$	$A\diamondsuit$	$K\spadesuit$
$J\diamondsuit$	$Q\spadesuit$	$K\clubsuit$	$A\heartsuit$

Two **mutually orthogonal** **Latin squares** of size $N = 4$

Hints

A♠	K♢	Q♡	J♣
K♡	A♣	J♠	Q♢
Q♣	J♡	A♢	K♠
J♢	Q♠	K♣	A♡

Two **mutually orthogonal** **Latin squares** of size $N = 4$

A♠	K♢	Q♡	J♣
K♡	A♣	J♠	Q♢
Q♣	J♡	A♢	K♠
J♢	Q♠	K♣	A♡

Three **mutually orthogonal** **Latin squares** of size $N = 4$

The answer

Bag shows **three mutually orthogonal Latin squares** of size $N = 4$ with three attributes A, B, C of each of $4^2 = 16$ squares.
Appending two indices, $i, j = 0, 1, 2, 3$ we obtain a 16×5 table,
$A_{00}, B_{00}, C_{00}, 0, 0$
$A_{01}, B_{01}, C_{01}, 0, 1$
....
$A_{33}, B_{33}, C_{33}, 3, 3.$
It forms an **orthogonal array OA(16,5,4,2)**
leading to the 2–**uniform** state of **5 ququarts**,

$$
\begin{aligned}
|\Psi_4^5\rangle = \quad & |00000\rangle + |12301\rangle + |23102\rangle + |31203\rangle \\
& |13210\rangle + |01111\rangle + |30312\rangle + |22013\rangle + \\
& |21320\rangle + |33021\rangle + |02222\rangle + |10123\rangle + \\
& |32130\rangle + |20231\rangle + |11032\rangle + |03333\rangle
\end{aligned}
$$

related to the **Reed–Solomon code** of length 5.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Banach tells his side of the story