Commutators of projectors, mutually unbiased bases and projective geometry.

A.Kocherova[1] and I.Zhdanovskiy[2]

[1]Department of Higher Mathematics
Moscow Institute of Physics and Technologies (MIPT)

[2]Department of Higher Mathematics
Moscow Institute of Physics and Technologies (MIPT),
Lab. of algebraic geometry
Higher School of Economics (HSE)

5^{th} workshop on real and complex Hadamard matrices, 2017

Algebra $A(t), t \in \mathbb{P}^3$.

Fix point $t = (t_{11} : t_{12} : t_{21} : t_{22}) \in \mathbb{P}^3$. Consider unital associative algebra $A(t)$ generated by pairs of orthogonal projectors π_1, π_2 and ρ_1, ρ_2:

$$\pi_1\pi_2 = \pi_2\pi_1 = \rho_1\rho_2 = \rho_2\rho_1 = 0.$$

Also, generators $\pi_i, \rho_j, i, j = 1, 2$ satisfy to relation:

$$\sum_{i,j=1}^{2} t_{ij}[\pi_i, \rho_j] = 0. \qquad (1)$$

Properties of $A(t)$.

Theorem

$\dim_{\mathbb{C}} A(t) = 18$ *and* $A(t) \cong \mathbb{C}^{\oplus 9} \oplus Mat_3(\mathbb{C})$ *for general* $t \in \mathbb{P}^3$.

There are exceptional cases: for some points $t \in \mathbb{P}^3$ algebra $A(t)$ is infinite-dimensional. For example, algebra $A(t_0)$ for $t_0 = (1 : 0 : 0 : -1)$ is infinite-dimensional. There is a connection of algebra $A(t_0)$ and Petrescu's construction of mutually unbiased bases in dimension 7.

Projectors and MUBs.

Consider two sets of orthogonal Hermitian projectors $p_1, ..., p_n$ and $q_1, ..., q_n$ of rank 1, acting in n-dimensional vector space V. Assume that

$$p_i q_j p_i = \frac{1}{n} p_i, q_i p_j q_i = \frac{1}{n} q_i \quad (2)$$

for any i, j. These relations mean that $\mathrm{Tr} p_i q_j = \frac{1}{n}$.
Choose $e_i \in \mathrm{Im} p_i, i = 1, ..., n$ and $f_j \in \mathrm{Im} q_j, j = 1, ..., n$ such that $|e_i| = |f_j| = 1$. Orthogonality of p_i's and q_j's mean that $e_i, i = 1, ..., n$ and $f_j, j = 1, ..., n$ are orthonormal bases in V

Fact

Two bases e_i and f_j are mutually unbiased.

Note that if e_i and f_j are mutually unbiased bases then we can construct two sets of Hermitian projectors p_i, q_j satisfying to (2).

MUB in dimension 7

Suppose that $n = 7$. It is well-known that there is one-dimensional family of MUBs (up to natural equivalence). This family is given by transition matrix $P(x)$:

$$P(x) = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w \cdot x & w^4 \cdot x & w^5 & w^3 & w^3 & w \\ 1 & w^4 \cdot x & w \cdot x & w^3 & w^5 & w^3 & w \\ 1 & w^5 & w^3 & \frac{w}{x} & \frac{w^4}{x} & w & w^3 \\ 1 & w^3 & w^5 & \frac{w^4}{x} & \frac{w}{x} & w & w^3 \\ 1 & w^3 & w^3 & w & w & w^4 & w^5 \\ 1 & w & w & w^3 & w^3 & w^5 & w^4 \end{pmatrix}, \qquad (3)$$

where $x \in \mathbb{C}^*, |x| = 1$ and w is a primitive root of unity of degree 6. If $x \in \mathbb{C}^*$ then $P(x)$ is a one-dimensional family of generalized Hadamard matrices of order 7. This family of complex Hadamard matrices was discovered by Petrescu.

Nicoara relation and algebra $A(t_0)$

Nicoara proved that if $\{e_i\}_{i=1}^7, \{f_j\}_{j=1}^7$ are mutually unbiased bases with transition matrix $P(x)$, then Hermitian projectors $p_i, q_j, i, j = 1, ..., 7$ satisfy to relation:

$$[p_1 + p_2, q_1 + q_2] = [p_3 + p_4, q_3 + q_4]. \tag{4}$$

Denote by $\mathcal{N}$ the algebra generated by p_i and q_j satisfying to relations (2) and (4). It is easy that pairs of projectors $p_1 + p_2$, $p_3 + p_4$ and $q_1 + q_2$,$q_3 + q_4$ are orthogonal.
Denote by $A(t_0)$ the algebra $A(t)$ for $t_0 = (1 : 0 : 0 : -1)$. It is easy that there is a morphism: $A(t_0) \to \mathcal{N}$ given by formulas:
$\pi_1 \mapsto p_1 + p_2, \pi_2 \mapsto p_3 + p_4$ and $\rho_1 \mapsto q_1 + q_2, \rho_2 \mapsto q_3 + q_4$.

Algebra $A(t_0)$

Let us note the following properties of $A(t_0)$:

Properties of $A(t_0)$

- Algebra $A(t_0)$ is infinite-dimensional
- Algebra $A(t_0)$ has 9 one-dimensional modules
- Dimension of irreducible representations of $A(t_0)$ is less or equal 2
- Algebra $A(t_0)$ has infinite-dimensional center Z, $SpecZ$ is a union of three lines in 3-dimensional affine space
- Irreducible $A(t_0)$-modules are parameterized by points of $SpecZ$.

Using representation theory of $A(t_0)$, one can construct Petrescu's family

Geometric description of orthogonal projectors of rank 1.

Consider orthogonal projectors π_1, π_2 of rank 1 acting in 3-dimensional space V. It is easy that $\pi_1, \pi_2, \pi_3 = 1 - \pi_1 - \pi_2$ are orthogonal projectors of rank 1.

As we know, projector is defined by its image and kernel. Images of π_1, π_2, π_3 are 1-dimensional subspaces of V, and hence, these images define points S_1, S_2, S_3 of $\mathbb{P}V$ respectively.

Description of orthogonal projectors

Three orthogonal projectors of rank 1 acting in 3-dimensional space V are defined by 3 independent points (don't lie on the same line) in $\mathbb{P}V$

Actually, image of π_i is defined by point S_i and kernel is defined by subspace corresponding to line passing through $S_k, S_l, k, l \neq i$.

Geometric description of two sets of orthogonal projectors

Analogous to the case of π_1, π_2, π_3, we get that projectors ρ_1, ρ_2 and $\rho_3 = 1 - \rho_1 - \rho_2$ are defined by 3 independent points T_1, T_2, $T_3 \in \mathbb{P}V$. Therefore,

Description of two sets of projectors

Two sets of orthogonal projectors: π_1, π_2, π_3 and ρ_1, ρ_2, ρ_3 are defined by 6 points S_1, S_2, S_3; T_1, T_2, $T_3 \in \mathbb{P}V$

Of course, action of $\mathrm{GL}(V)$ by conjugation on 6 projectors is compatible with action of $\mathrm{PGL}(V)$ on 6 points of $\mathbb{P}V$.

Commutators of projectors

Consider space generated by commutators of two sets of three 1-dimensional orthogonal projectors π_1, π_2, π_3 and ρ_1, ρ_2, ρ_3. Using formulas $\pi_3 = 1 - \pi_1 - \pi_2$ and $\rho_3 = 1 - \rho_1 - \rho_2$, we get that this space is generated by $[\pi_i, \rho_j], i, j = 1, 2$, and hence, dimension of this space is not more than 4.

Theorem

Commutators of projectors $\pi_1, \pi_2, \rho_1, \rho_2$ are linear dependent (i.e.satisfy to relation:

$$\sum_{i,j=1}^{2} t_{ij}[\pi_i, \rho_j] = 0$$

for some $t = (t_{11} : t_{12} : t_{21} : t_{22}) \in \mathbb{P}^3$) if and only if there is a conic passing through 6 points $S_1, S_2, S_3; T_1, T_2, T_3$.

One-dimensional family of Cartan subalgebras

Consider Cartan subalgebra of $gl(V), \dim V = 3$. It is well-known that there is a basis in which Cartan subalgebra is a subalgebra of diagonal matrices. This basis defines 3 points of $\mathbb{P}V$. Thus, Cartan subalgebras are parameterized by 3 non-ordered points of $\mathbb{P}V$.

Consider one-dimensional family of Cartan subalgebras $\mathcal{H}_x, x \in \mathbb{P}^1$. Taking all 3 points of $\mathbb{P}V$ corresponding to bases in which Cartan subalgebras $\mathcal{H}_x$ are diagonal subalgebra, we get the curve $C \subset \mathbb{P}V$. Note that this curve is non-degenerated (i.e. is not a line). Also, we have map: $C \to \mathbb{P}^1$ of degree 3.

We can rewrite the relation (1) in the following manner:

$$[x_0\pi_1 + x_1(-t_{21}\rho_1 - t_{22}\rho_2), x_0\pi_2 + x_1(t_{11}\rho_1 + t_{12}\rho_2)] = 0 \qquad (5)$$

for any $x = (x_0 : x_1) \in \mathbb{P}^1$. Consider Cartan subalgebras $\mathcal{H}_0, \mathcal{H}_1$ of $gl(V)$ generated by π_1, π_2 and ρ_1, ρ_2 respectively. Relation (5) means that there is one-dimensional family of Cartan subalgebras $\mathcal{H}_x, x \in \mathbb{P}^1$ in subspace $\mathcal{H}_0 + \mathcal{H}_1$.

Theorem

Consider 1-dimensional family of Cartan subalgebras $\mathcal{H}_x, x \in \mathbb{P}^1$. Denote by C the curve corresponding to bases in which $\mathcal{H}_x$ are diagonal. Denote by $< \mathcal{H}_x > \subset gl(V)$ the space generated by all elements of $\mathcal{H}_x$. Thus, $\dim < \mathcal{H}_x > \leq 5$ iff curve C is a conic.

Generalization

This statement has the following generalization.
Consider one-dimensional family of Cartan subalgebras $\mathcal{H}_x, x \in \mathbb{P}^1$ of gl_n. C is a curve corresponding to bases of $\mathcal{H}_x$. In this case we have morphism: $C \to \mathbb{P}^1$ of degree n. Also, $C \subset \mathbb{P}^{n-1}$ is non-degenerated curve (i.e. is not contained in any hyperplane).

Theorem

$\dim < \mathcal{H}_x > \leq 2n-1$ *iff curve* $C \subset \mathbb{P}^{n-1}$ *has degree* $n-1$.

Let us formulate the following general idea:

Idea

Any commutativity (or "almost" commutativity) has geometrical nature, i.e it can be formulated in geometrical terms.

Application to pair of operators and quantum mechanics

Consider pair of operators A, B acting in finite-dimensional vector space V. Assume that each of these operators has n different eigenvalues. Thus, operators A and B define $2n$ points $a_1, ..., a_n; b_1, ..., b_n$ of $\mathbb{P}^{n-1} = \mathbb{P}V$ corresponding to eigenspaces of A and B respectively.

- There are $n-1$ pair of independent polynomials $f_i, g_i, i = 1, ..., n-1$ such that $[f_i(A) + g_i(B), f_j(A) + g_j(B)] = 0$ for any i, j if and only if there is a curve of degree $n-1$ passing through $a_1, ..., a_n; b_1, ..., b_n$.
- It can be formulated in terms of quantum mechanics

THANK YOU!