On Near Butson-Hadamard Matrices

Oğuz Yayla

joint work with Sibel Kurt

Hacettepe University
Department of Mathematics
Ankara, Turkey

July, 2017

The author is supported by TÜBİTAK Project No: 116R026.

Outline

- Introduction
- Ideal Factorization Method
- Application to matrices and sequences

Near Butson-Hadamard Matrices

- $\mathcal{E}_m = \{1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1}\}$
- A *Butson-Hadamard matrix* is a square matrix H of order v with entries in $\mathcal{E}_m$ such that $H\overline{H}^T = vI$.
- denoted by $\mathrm{BH}(v, m)$.
- $\mathrm{BH}(v, 2)$ is so called *Hadamard matrix* of order v
- *Near Butson-Hadamard matrix* $\mathrm{BH}_\gamma(v, m)$ of type γ:

Near Butson-Hadamard Matrices

- $\mathcal{E}_m = \{1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1}\}$
- A *Butson-Hadamard matrix* is a square matrix H of order v with entries in $\mathcal{E}_m$ such that $H\overline{H}^T = vI$.
- denoted by $\mathrm{BH}(v, m)$.
- $\mathrm{BH}(v, 2)$ is so called *Hadamard matrix* of order v
- *Near Butson-Hadamard matrix* $\mathrm{BH}_\gamma(v, m)$ of type γ: $H\overline{H}^T = (v - \gamma)I + \gamma J$ for a $\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m]$.

$$H\overline{H}^T = (v-\gamma)I + \gamma J$$

$$H\overline{H}^T = \begin{bmatrix} v-\gamma & 0 & 0 & 0 & 0 \\ 0 & v-\gamma & 0 & 0 & 0 \\ 0 & 0 & v-\gamma & 0 & 0 \\ 0 & 0 & 0 & v-\gamma & 0 \\ 0 & 0 & 0 & 0 & v-\gamma \end{bmatrix} + \begin{bmatrix} \gamma & \gamma & \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma & \gamma & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{v} & \gamma & \gamma & \gamma & \gamma \\ \gamma & \boldsymbol{v} & \gamma & \gamma & \gamma \\ \gamma & \gamma & \boldsymbol{v} & \gamma & \gamma \\ \gamma & \gamma & \gamma & \boldsymbol{v} & \gamma \\ \gamma & \gamma & \gamma & \gamma & \boldsymbol{v} \end{bmatrix}$$

Main equation

Example 1

$\mathrm{BH}_\gamma(5,5)$ exists for $\gamma \in \{-\xi_5^3 - \xi_5^2 + 2, 0, 5, \xi_5^3 + \xi_5^2 + 3\}$ with $|\gamma| \in \{1.38, 0, 5, 3.61\}$, respectively.

Main equation

Example 1

$BH_\gamma(5,5)$ exists for $\gamma \in \{-\xi_5^3 - \xi_5^2 + 2, 0, 5, \xi_5^3 + \xi_5^2 + 3\}$ with $|\gamma| \in \{1.38, 0, 5, 3.61\}$, respectively. For instance, the matrix H has $\gamma = -\xi_5^3 - \xi_5^2 + 2$ with $|\gamma| = 1.38$

$$H = \begin{bmatrix} 1 & 1 & -\xi_5^2 & 1 & 1 \\ 1 & 1 & 1 & -\xi_5^2 & 1 \\ 1 & 1 & 1 & 1 & -\xi_5^2 \\ -\xi_5^2 & 1 & 1 & 1 & 1 \\ 1 & -\xi_5^2 & 1 & 1 & 1 \end{bmatrix}.$$

Main equation

Example 2

Similarly, we obtained by an exhaustive search that $BH_\gamma(8,5)$ exists for $\gamma \in \{-\xi_5^3 - \xi_5^2 + 5, -\xi_5^3 - \xi_5^2, 8, \xi_5^3 + \xi_5^2 + 1, \xi_5^3 + \xi_5^2 + 6\}$ with $|\gamma| \in \{6.61, 1.61, 8, 0.61, 4.38\}$, respectively.

Main equation

Example 2

Similarly, we obtained by an exhaustive search that $\mathrm{BH}_\gamma(8,5)$ exists for $\gamma \in \{-\xi_5^3 - \xi_5^2 + 5, -\xi_5^3 - \xi_5^2, 8, \xi_5^3 + \xi_5^2 + 1, \xi_5^3 + \xi_5^2 + 6\}$ with $|\gamma| \in \{6.61, 1.61, 8, 0.61, 4.38\}$, respectively. In particular, the matrix H has $\gamma = -\xi_5^3 - \xi_5^2 + 2$ with $|\gamma| = 0.61$

$$H = \begin{bmatrix} 1 & 1 & \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5^3 & \zeta_5 & 1 \\ 1 & 1 & 1 & \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5^3 & \zeta_5 \\ \zeta_5 & 1 & 1 & 1 & \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5^3 \\ \zeta_5^3 & \zeta_5 & 1 & 1 & 1 & \zeta_5^2 & \zeta_5^3 & 1 \\ 1 & \zeta_5^3 & \zeta_5 & 1 & 1 & 1 & \zeta_5^2 & \zeta_5^3 \\ \zeta_5^3 & 1 & \zeta_5^3 & \zeta_5 & 1 & 1 & 1 & \zeta_5^2 \\ \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5^3 & \zeta_5 & 1 & 1 & 1 \\ 1 & \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5^3 & \zeta_5 & 1 & 1 \end{bmatrix}.$$

Main equation

- $H\overline{H}^T = (v - \gamma)I + \gamma J$

Main equation

- $H\overline{H}^T = (v - \gamma)I + \gamma J$
- $\det(H) \in \mathbb{Z}[\zeta_m]$

Main equation

- $H\overline{H}^T = (v-\gamma)I + \gamma J$
- $\det(H) \in \mathbb{Z}[\zeta_m]$
- $\det(H\overline{H}^T) = \det(H)\overline{\det(H)} = ((\gamma+1)v-\gamma)(v-\gamma)^{v-1}$.

Main equation

- $H\overline{H}^T = (v-\gamma)I + \gamma J$
- $\det(H) \in \mathbb{Z}[\zeta_m]$
- $\det(H\overline{H}^T) = \det(H)\overline{\det(H)} = ((\gamma+1)v-\gamma)(v-\gamma)^{v-1}$.
- Therefore, we want to find criteria for the unsolvability of the equation

$$\alpha\overline{\alpha} = ((\gamma+1)v-\gamma)(v-\gamma)^{v-1} \tag{1}$$

over $\mathbb{Z}[\zeta_m]$.

Previous Work

Checking solvability of this kinds of equations was studied in:

Previous Work

Checking solvability of this kinds of equations was studied in:

- self conjugate condition [Brock,1988]
  Brock said that for a positive integer w there exists no solution α to the equation $\alpha\overline{\alpha} = w$ over $\mathbb{Q}(\zeta_m)$ if the square-free part of w is divisible by a prime which is self-conjugate modulo m in his work.

Previous Work

Checking solvability of this kinds of equations was studied in:

- self conjugate condition [Brock,1988]
  Brock said that for a positive integer w there exists no solution α to the equation $\alpha\overline{\alpha} = w$ over $\mathbb{Q}(\zeta_m)$ if the square-free part of w is divisible by a prime which is self-conjugate modulo m in his work.
- Principal ideal decomposition [(Winterhof,Yayla,Ziegler),2014]
  Existence of γ-Butson Hadamard matrices for $\gamma \in \mathbb{Z}$ condition is studied under an equation $D = \alpha\overline{\alpha}$ with parameters m, v and $\gamma \in \mathbb{Z}$.

Our Motivation

- What about the case $\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m] \backslash \mathbb{Z}$?

Our Motivation

- What about the case $\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m] \backslash \mathbb{Z}$?
- Obtain BH_γ matrices having $|\gamma|$ as small as possible

Problems

- Problem - 1 :
  For which parameters $m, v \in \mathbb{Z}^+$ does a BH_γ exists if γ is noninteger?

Problems

- Problem - 1 :
  For which parameters $m, v \in \mathbb{Z}^+$ does a BH_γ exists if γ is noninteger?
- Problem - 2 :
  What can be said about conference matrices and sequences?

Results

1. A new result stating necessary conditions for the nonexistence of a near Butson-Hadamard matrix ($\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m]$).

Results

1. A new result stating necessary conditions for the nonexistence of a near Butson-Hadamard matrix ($\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m]$). Examples of nonexistence and existence cases by computer search.

Results

1. A new result stating necessary conditions for the nonexistence of a near Butson-Hadamard matrix ($\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m]$). Examples of nonexistence and existence cases by computer search.
2. Consequences of our results applied to the concept of sequences and conference matrices. Examples of existence cases for nearly perfect sequences.

Main Theorem

A condition for the non-existence of a solution $\alpha \in \mathbb{Z}[\zeta_m]$ to this equation is presented in Theorem 3.

Theorem 3

Let $D \in \mathbb{Z}[\zeta_m] \cap \mathbb{R}$ such that $D = tq^{2e+1}$

- *where $q, t \in \mathbb{Z}[\zeta_m]$ and q is squarefree, provided that every prime ideal $\mathfrak{t} \lhd \mathbb{Z}[\zeta_m]$ with $\mathfrak{t}|(t)$ is principal,*
- *$(q) = \mathfrak{q}_1\mathfrak{q}_2$ where $\mathfrak{q}_1$, $\mathfrak{q}_2$ are non-principal prime ideals of $\mathbb{Z}[\zeta_m]$, $e > 0$ be rational integer,*
- *$gcd(2e + 1 - 2k, h_m) = 1$ for $0 \leq k \leq e - 1$ and*
- *$gcd(N(q), N(t)) = 1$.*

Then, there exists no $\alpha \in \mathbb{Z}[\zeta_m]$ satisfying $D = \alpha\bar{\alpha}$.

Proof

Proof.

We first assume that there exists $\alpha \in \mathbb{Z}[\zeta_m]$ such that $\alpha\bar{\alpha} = tq^{2e+1}$ such that

$$(\alpha) = \mathfrak{t}_1\mathfrak{q}_1^{2e+1-k}\mathfrak{q}_2^k, (\bar{\alpha}) = \mathfrak{t}_2\mathfrak{q}_1^k\mathfrak{q}_2^{2e+1-k}$$

for some $\mathfrak{t} \lhd \mathbb{Z}[\zeta_m]$. We have

$$(\alpha) = \mathfrak{t}_1\mathfrak{q}_1^{2e+1-k}\mathfrak{q}_2^k = \mathfrak{t}_1\mathfrak{q}_1^{2e+1-2k}q^k$$

We know that t_1 and q are principal ideals of $\mathbb{Z}[\zeta_m]$ but $\mathfrak{q}_1^{2e+1-2k}$ is nonprincipal since $gcd(2e+1-2k, h_m) = 1$. Hence we get a contradiction. Next, we assume that $\alpha = \mathfrak{t}_1 q^s$, $\bar{\alpha} = \mathfrak{t}_2 q^{2e+1-s}$ for some principal ideals $\mathfrak{t}_1, \mathfrak{t}_2 \lhd \mathbb{Z}[\zeta_m]$ and $s \in \mathbb{Z}^+ \cup \{0\}$, $s \leq e$. Then, $q^{2e+1-2s}|\mathfrak{t}_1$. However, this contradicts to $gcd(N(q), N(\mathfrak{t})) = 1$. □

Class Number Table

Table: The class number h_m of $\mathbb{Q}(\zeta_m)$ for $m \leq 70$ [Washington1997].

m	h_m	m	h_m	m	h_m	m	h_m	m	h_m	m	h_m	m	h_m
1	1	11	1	21	1	31	9	41	121	51	5	61	76301
2	1	12	1	22	1	32	1	42	1	52	3	62	9
3	1	13	1	23	3	33	1	43	211	53	48891	63	7
4	1	14	1	24	1	34	1	44	1	54	1	64	17
5	1	15	1	25	1	35	1	45	1	55	10	65	64
6	1	16	1	26	1	36	1	46	3	56	2	66	1
7	1	17	1	27	1	37	37	47	695	57	9	67	853513
8	1	18	1	28	1	38	1	48	1	58	8	68	8
9	1	19	1	29	8	39	2	49	43	59	41421	69	69
10	1	20	1	30	1	40	1	50	1	60	1	70	1

Example

Example 4

$$D = ((-\zeta_{23} - \zeta_{23}^{22})5 + 1 + \zeta_{23} + \zeta_{23}^{22})(6 + \zeta_{23} + \zeta_{23}^{22})^4 \in \mathbb{Z}[\zeta_{23}]$$

Example

Example 4

$D = ((-\zeta_{23} - \zeta_{23}^{22})5 + 1 + \zeta_{23} + \zeta_{23}^{22})(6 + \zeta_{23} + \zeta_{23}^{22})^4 \in \mathbb{Z}[\zeta_{23}]$
$v = 5,\ m = 23,\ \gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$

Example

Example 4

$D = ((-\zeta_{23} - \zeta_{23}^{22})5 + 1 + \zeta_{23} + \zeta_{23}^{22})(6 + \zeta_{23} + \zeta_{23}^{22})^4 \in \mathbb{Z}[\zeta_{23}]$
$v = 5,\ m = 23,\ \gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1^4 \mathfrak{p}_2 \mathfrak{p}_3^4 \mathfrak{q}_4 \mathfrak{q}_5$ where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \lhd \mathbb{Z}[\zeta_{23}]$ are principal prime ideals $\mathfrak{q}_4, \mathfrak{q}_5 \in \mathbb{Z}[\zeta_{23}]$ are the non-principal prime ideals.

Example

Example 4

$D = ((-\zeta_{23} - \zeta_{23}^{22})5 + 1 + \zeta_{23} + \zeta_{23}^{22})(6 + \zeta_{23} + \zeta_{23}^{22})^4 \in \mathbb{Z}[\zeta_{23}]$
$v = 5,\ m = 23,\ \gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1^4\mathfrak{p}_2\mathfrak{p}_3^4\mathfrak{q}_4\mathfrak{q}_5$ where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \lhd \mathbb{Z}[\zeta_{23}]$ are principal prime ideals $\mathfrak{q}_4, \mathfrak{q}_5 \in \mathbb{Z}[\zeta_{23}]$ are the non-principal prime ideals.
There is no $\alpha \in \mathbb{Z}[\zeta_m]$ satisfying $D = \alpha\bar{\alpha}$.

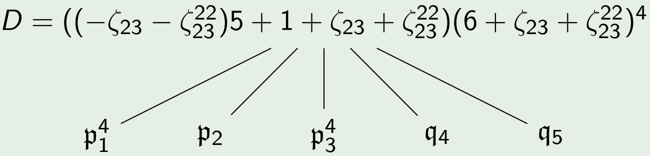

Figure: Ideal Decomposition of D for value $v = 5, \gamma = 1 - \zeta_{23} - \zeta_{23}^{22}$

Example 5

$$D = ((-\zeta_{23} - \zeta_{23}^{22})46 + 1 + \zeta_{23} + \zeta_{23}^{22})(47 + \zeta_{23} + \zeta_{23}^{22})^{45} \in \mathbb{Z}[\zeta_{23}]$$

Example 5

$D = ((-\zeta_{23} - \zeta_{23}^{22})46 + 1 + \zeta_{23} + \zeta_{23}^{22})(47 + \zeta_{23} + \zeta_{23}^{22})^{45} \in \mathbb{Z}[\zeta_{23}]$
for $v = 46,\ m = 23,\ \gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$

Example 5

$D = ((-\zeta_{23} - \zeta_{23}^{22})46 + 1 + \zeta_{23} + \zeta_{23}^{22})(47 + \zeta_{23} + \zeta_{23}^{22})^{45} \in \mathbb{Z}[\zeta_{23}]$
for $v = 46$, $m = 23$, $\gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1\mathfrak{p}_2^{45}\mathfrak{p}_3^{45}\mathfrak{q}_4\mathfrak{q}_5\mathfrak{q}_6\mathfrak{q}_7$ where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \lhd \mathbb{Z}[\zeta_{23}]$ are principal prime ideals and $\mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7 \lhd \mathbb{Z}[\zeta_{23}]$ are the non-principal ideals.

Example 5

$D = ((-\zeta_{23} - \zeta_{23}^{22})46 + 1 + \zeta_{23} + \zeta_{23}^{22})(47 + \zeta_{23} + \zeta_{23}^{22})^{45} \in \mathbb{Z}[\zeta_{23}]$
for $v = 46$, $m = 23$, $\gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1 \mathfrak{p}_2^{45} \mathfrak{p}_3^{45} \mathfrak{q}_4 \mathfrak{q}_5 \mathfrak{q}_6 \mathfrak{q}_7$ where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \lhd \mathbb{Z}[\zeta_{23}]$ are principal prime ideals and $\mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7 \lhd \mathbb{Z}[\zeta_{23}]$ are the non-principal ideals.
The methodology in Example 4 does not work for this example.

Example 5

$D = ((-\zeta_{23} - \zeta_{23}^{22})46 + 1 + \zeta_{23} + \zeta_{23}^{22})(47 + \zeta_{23} + \zeta_{23}^{22})^{45} \in \mathbb{Z}[\zeta_{23}]$
for $v = 46$, $m = 23$, $\gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1\mathfrak{p}_2^{45}\mathfrak{p}_3^{45}\mathfrak{q}_4\mathfrak{q}_5\mathfrak{q}_6\mathfrak{q}_7$ where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \lhd \mathbb{Z}[\zeta_{23}]$ are principal prime ideals and $\mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7 \lhd \mathbb{Z}[\zeta_{23}]$ are the non-principal ideals.
The methodology in Example 4 does not work for this example.
Note that $(\alpha) = \mathfrak{t}_1\mathfrak{q}_5\mathfrak{q}_7^{38}$ is a principal ideal and satisfies $D = \alpha\bar{\alpha}$ for a convenient principal ideal $\mathfrak{t}_1 \lhd \mathbb{Z}[\zeta_{23}]$ such that $t_1 \mid D$.

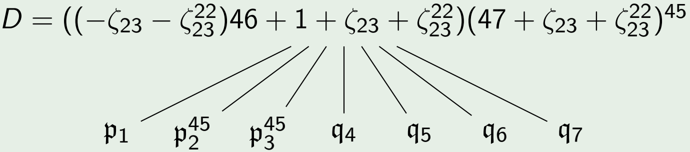

Figure: Ideal Decomposition of D for value $v = 46, \gamma = 1 - \zeta_{23} - \zeta_{23}^{22}$

Example 6

$D = ((-\zeta_{23} - \zeta_{23}^{22})39 + 1 + \zeta_{23} + \zeta_{23}^{22})(40 + \zeta_{23} + \zeta_{23}^{22})^{38} \in \mathbb{Z}[\zeta_{23}]$
for $v = 39$, $m = 23$, $\gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$

Example 6

$D = ((-\zeta_{23} - \zeta_{23}^{22})39 + 1 + \zeta_{23} + \zeta_{23}^{22})(40 + \zeta_{23} + \zeta_{23}^{22})^{38} \in \mathbb{Z}[\zeta_{23}]$
for $v = 39$, $m = 23$, $\gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4\mathfrak{p}_5^2\mathfrak{p}_6^{38}\mathfrak{p}_7^{38}\mathfrak{p}_8^{38}\mathfrak{q}_9\mathfrak{q}_{10}\mathfrak{q}_{11}^{38}\mathfrak{q}_{12}^{38}$ where
$\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5, \mathfrak{p}_6, \mathfrak{p}_7, \mathfrak{p}_8 \lhd \mathbb{Z}[\zeta_{23}]$ are principal ideals and
$\mathfrak{q}_9, \mathfrak{q}_{10}, \mathfrak{q}_{11}, \mathfrak{q}_{12} \lhd \mathbb{Z}[\zeta_{23}]$ are non-principal ideals.

Example 6

$D = ((-\zeta_{23} - \zeta_{23}^{22})39 + 1 + \zeta_{23} + \zeta_{23}^{22})(40 + \zeta_{23} + \zeta_{23}^{22})^{38} \in \mathbb{Z}[\zeta_{23}]$
for $v = 39$, $m = 23$, $\gamma = -1 - \zeta_{23} - \zeta_{23}^{22}$
$D = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4\mathfrak{p}_5^2\mathfrak{p}_6^{38}\mathfrak{p}_7^{38}\mathfrak{p}_8^{38}\mathfrak{q}_9\mathfrak{q}_{10}\mathfrak{q}_{11}^{38}\mathfrak{q}_{12}^{38}$ where
$\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5, \mathfrak{p}_6, \mathfrak{p}_7, \mathfrak{p}_8 \lhd \mathbb{Z}[\zeta_{23}]$ are principal ideals and
$\mathfrak{q}_9, \mathfrak{q}_{10}, \mathfrak{q}_{11}, \mathfrak{q}_{12} \lhd \mathbb{Z}[\zeta_{23}]$ are non-principal ideals.
Note that $(\alpha) = \mathfrak{t}_1\mathfrak{q}_{10}\mathfrak{q}_{12}^{38}$ is a principal ideal and satisfies $D = \alpha\bar{\alpha}$ for a convenient principal ideal $\mathfrak{t}_1 \lhd \mathbb{Z}[\zeta_{23}]$ such that $t_1 \mid D$.

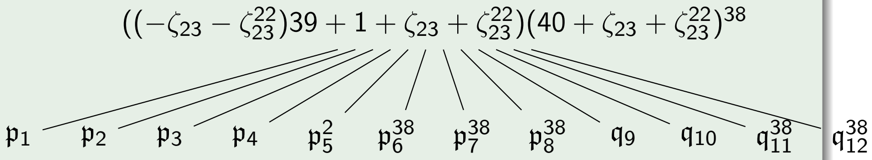

Figure: Ideal Decomposition of D for value $v = 39, \gamma = 1 - \zeta_{23} - \zeta_{23}^{22}$

Four Nonprincipal ideals

- Let $q_1, q_2, q_3, q_4 \lhd \mathbb{Z}[\zeta_m]$ be non-principal prime ideals of $\mathbb{Z}[\zeta_m]$ dividing D.
- Assume that $q_1q_2, q_3q_4, q_1q_3, q_2q_4$ are all principal in $\mathbb{Z}[\zeta_m]$.
- If $gcd(N(q_1q_2), N(q_3q_4)) = 1$, $gcd(N(q_1q_3), N(q_2q_4)) = 1$,
- Then we can conclude that there exists no solution.

Conference Matrix

Definition 7

A square matrix C of order v with 0 on the diagonal and all off-diagonal entries in $\mathcal{E}_m$ is called a *near conference matrix* $\mathrm{C}_\gamma(v, m)$ if $C\overline{C}^T = (v - 1 - \gamma)I + \gamma J$ for a $\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m]$.

Conference Matrix

Definition 7

A square matrix C of order v with 0 on the diagonal and all off-diagonal entries in $\mathcal{E}_m$ is called a *near conference matrix* $\mathrm{C}_\gamma(v, m)$ if $C\overline{C}^T = (v - 1 - \gamma)I + \gamma J$ for a $\gamma \in \mathbb{R} \cap \mathbb{Z}[\zeta_m]$.

A matrix C with entries in $\mathcal{E}_3$ and having the first row $(0, \zeta_3^2, \zeta_3^2, \zeta_3^2, 1, \zeta_3^2, \zeta_3, \zeta_3, \zeta_3^2, 1, \zeta_3^2, \zeta_3^2, \zeta_3^2)$ is an example of a circulant conference matrix.
Note that $C\overline{C}^T = 10I + 2J$.

Conference Matrix

Similar to the case near Butson-Hadamard matrices, we obtain that a near conference matrix $C = \mathrm{C}_\gamma(v, m)$ satisfies

$$\det(C)\overline{\det(C)} = (\gamma + 1)(v - 1)(v - 1 - \gamma)^{v-1}$$

and hence we have

$$\alpha\overline{\alpha} = (\gamma + 1)(v - 1)(v - 1 - \gamma)^{v-1}. \qquad (2)$$

Sequences

- $\underline{a} = (a_0, a_1, \ldots, a_{v-1}, \ldots)$ v-periodic sequence
  - an *m-ary sequence* if $a_0, a_1, \ldots, a_{v-1} \in \mathcal{E}_m = \{1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1}\}$
  - an *almost m-ary sequence* if $a_0 = 0$ and $a_1, \ldots, a_{v-1} \in \mathcal{E}_m$.

Sequences

- $\underline{a} = (a_0, a_1, \ldots, a_{v-1}, \ldots)$ v-periodic sequence
  - an *m-ary sequence* if $a_0, a_1, \ldots, a_{v-1} \in \mathcal{E}_m = \{1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1}\}$
  - an *almost m-ary sequence* if $a_0 = 0$ and $a_1, \ldots, a_{v-1} \in \mathcal{E}_m$.
- For $0 \leq t \leq v-1$, the *autocorrelation function* $C_{\underline{a}}(t)$ is defined by

$$C_{\underline{a}}(t) = \sum_{i=0}^{v-1} a_i \overline{a_{i+t}},$$

where $\overline{a}$ is the complex conjugate of a.

Sequences

- *Perfect Sequence* (PS) if $C_{\underline{a}}(t) = 0$ for all $1 \leq t \leq v - 1$.

Sequences

- *Perfect Sequence* (PS) if $C_{\underline{a}}(t) = 0$ for all $1 \leq t \leq v - 1$.
- *Nearly Perfect Sequence* (NPS) of type γ if $C_{\underline{a}}(t) = \gamma$ for all $1 \leq t \leq v - 1$.

Sequences

- *Perfect Sequence* (PS) if $C_{\underline{a}}(t) = 0$ for all $1 \leq t \leq v - 1$.
- *Nearly Perfect Sequence* (NPS) of type γ if $C_{\underline{a}}(t) = \gamma$ for all $1 \leq t \leq v - 1$.
- For instance, $(0, \zeta_3^2, \zeta_3^2, \zeta_3^2, 1, \zeta_3^2, \zeta_3, \zeta_3, \zeta_3^2, 1, \zeta_3^2, \zeta_3^2, \zeta_3^2)$ is a 3-ary NPS of period 13 and type $\gamma = 2$.

Proposition 1

If a NPS of type γ exists, then γ is a real number.

Proposition 1

If a NPS of type γ exists, then γ is a real number.

Remark 8

- *NPSs are equivalent to circulant γ-Butson-Hadamard matrices and conference matrices.*

Proposition 1

If a NPS of type γ exists, then γ is a real number.

Remark 8

- *NPSs are equivalent to circulant γ-Butson-Hadamard matrices and conference matrices.*
- *Let $\underline{a} = (a_0, a_1, \ldots, a_{v-1}, \ldots)$ be an m-ary NPS of period v.*

Proposition 1

If a NPS of type γ exists, then γ is a real number.

Remark 8

- *NPSs are equivalent to circulant γ-Butson-Hadamard matrices and conference matrices.*
- *Let $\underline{a} = (a_0, a_1, \ldots, a_{v-1}, \ldots)$ be an m-ary NPS of period v.*
- *Let $H = (h_{i,j})$ be a circulant matrix defined by $h_{0,j} = a_j$ for $j = 0, 1, \ldots, v-1$ then H is a circulant near Butson-Hadamard matrix of order v.*

Proposition 1

If a NPS of type γ exists, then γ is a real number.

Remark 8

- *NPSs are equivalent to circulant γ-Butson-Hadamard matrices and conference matrices.*
- *Let $\underline{a} = (a_0, a_1, \ldots, a_{v-1}, \ldots)$ be an m-ary NPS of period v.*
- *Let $H = (h_{i,j})$ be a circulant matrix defined by $h_{0,j} = a_j$ for $j = 0, 1, \ldots, v-1$ then H is a circulant near Butson-Hadamard matrix of order v.*
- *Similarly, an almost m-ary NPS is equivalent to a circulant near conference matrix.*

Corollary 9

Let $v, m \in \mathbb{Z}^+$ and $\gamma \in \mathbb{Z}[\zeta_m] \cap \mathbb{R}$ such that $D = ((\gamma+1)v - \gamma)(v-\gamma)^{v-1}$ and $D = tq^{2e+1}$ where $q, t \in \mathbb{Z}[\zeta_m]$ and q is squarefree. Suppose that (i) to (iv) are satisfied.

(i) *Every prime ideal $\mathfrak{t} \lhd \mathbb{Z}[\zeta_m]$ with $\mathfrak{t}|(t)$ is principal.*

(ii) *$(q) = \mathfrak{q}_1\mathfrak{q}_2$ where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are non-principal prime ideals of $\mathbb{Z}[\zeta_m]$.*

(iii) *$e > 0$ be rational integer, $gcd(2e+1-2k, h_m) = 1$ for $0 \le k \le e-1$.*

(iv) *$gcd(N(q), N(t)) = 1$.*

Corollary 9

Let $v, m \in \mathbb{Z}^+$ and $\gamma \in \mathbb{Z}[\zeta_m] \cap \mathbb{R}$ such that $D = ((\gamma+1)v - \gamma)(v-\gamma)^{v-1}$ and $D = tq^{2e+1}$ where $q, t \in \mathbb{Z}[\zeta_m]$ and q is squarefree. Suppose that (i) to (iv) are satisfied.

- (i) *Every prime ideal $\mathfrak{t} \lhd \mathbb{Z}[\zeta_m]$ with $\mathfrak{t}|(t)$ is principal.*
- (ii) *$(q) = \mathfrak{q}_1\mathfrak{q}_2$ where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are non-principal prime ideals of $\mathbb{Z}[\zeta_m]$.*
- (iii) *$e > 0$ be rational integer, $gcd(2e+1-2k, h_m) = 1$ for $0 \leq k \leq e-1$.*
- (iv) *$gcd(N(q), N(t)) = 1$.*

Then the following hold:

- (i) *there exists no $BH_\gamma(v, m)$.*
- (ii) *there exists no v-periodic m-ary NPS of type γ.*

Corollary 10

Let $v, m \in \mathbb{Z}^+$, $\gamma \in \mathbb{Z}[\zeta_m] \cap \mathbb{R}$ *such that* $D = (\gamma + 1)(v - 1)(v - 1 - \gamma)^{v-1}$ *and* $D = tq^{2e+1}$ *where* $q, t \in \mathbb{Z}[\zeta_m]$ *and* q *is squarefree. If the conditions (i) - (iv) given in Corollary 9 are satisfied, then*

Corollary 10

Let $v, m \in \mathbb{Z}^+$, $\gamma \in \mathbb{Z}[\zeta_m] \cap \mathbb{R}$ such that $D = (\gamma + 1)(v - 1)(v - 1 - \gamma)^{v-1}$ and $D = tq^{2e+1}$ where $q, t \in \mathbb{Z}[\zeta_m]$ and q is squarefree. If the conditions (i) - (iv) given in Corollary 9 are satisfied, then

- *There exists no $\mathrm{C}_\gamma(v, m)$,*
- *There exists no v-periodic an almost m-ary NPS of type γ.*

Example 11

Consider $\mathrm{BH}_\gamma(67, 23)$, $\gamma = -1 - \zeta_{23}$, $v = 67$ and $m = 23$.

$$\alpha\overline{\alpha} = (1 - 66\zeta_{23})(68 + \zeta_{23})^{66}$$

Every prime ideal dividing $(68 + \zeta_{23})^{66}$ is principal. $(1 - 66\zeta_{23})$ has the non-principal ideal decomposition over $\mathbb{Z}[\zeta_{23}]$. Hence, $\mathrm{BH}_\gamma(67, 23)$ does not exist by Corollary 9.

Example 11

Consider $\mathrm{BH}_\gamma(67,23)$, $\gamma = -1-\zeta_{23}$, $v = 67$ and $m = 23$.

$$\alpha\overline{\alpha} = (1-66\zeta_{23})(68+\zeta_{23})^{66}$$

Every prime ideal dividing $(68+\zeta_{23})^{66}$ is principal. $(1-66\zeta_{23})$ has the non-principal ideal decomposition over $\mathbb{Z}[\zeta_{23}]$. Hence, $\mathrm{BH}_\gamma(67,23)$ does not exist by Corollary 9. Furthermore, we conclude that a 23-ary NPS of period 67 and $\gamma = -1-\zeta_{23}$ does not exist.

Table: Samples of perfect sequences with non-integer correlations

v	m	γ	$\lvert\gamma\rvert$	a
3	5	$\zeta_5^3+\zeta_5^2+1$	0.61	$1, 1, \zeta_5^2$
3	7	$\zeta_7^5+\zeta_7^2+1$	0.55	$\zeta_7^2, \zeta_7^2, 1$
4	5	$\zeta_5^3+\zeta_5^2+2$	0, 38	$1, 1, 1, \zeta_5^2$
4	7	$\zeta_7^4+\zeta_7^3+2$	0, 19	$\zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^5$
5	5	$\zeta_5^3+\zeta_5^2+3$	1, 38	$1, 1, 1, 1, \zeta_5^2$
5	7	$-\zeta_7^5-\zeta_7^2$	0, 44	$\zeta_7^2, \zeta_7^2, \zeta_7^3, \zeta_7^6, \zeta_7^3$
25	5	$\zeta_5^3+\zeta_5^2+23$	21, 38	$1, \ldots, 1, \zeta_5^2$
125	5	$\zeta_5^3+\zeta_5^2+123$	121, 38	$1, \ldots, 1, \zeta_5^2$
6	5	$\zeta_5^3+\zeta_5^2+4$	2, 38	$1, 1, 1, 1, 1, \zeta_5^2$
6	6	-1	1	$\zeta_6^4, 1, \zeta_6^4, \zeta_6^2, \zeta_6, \zeta_6^2$
6	7	$\zeta_7^4+\zeta_7^3+4$	2, 19	$\zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^5$
7	5	$2\zeta_5^3+2\zeta_5^2+3$	0, 23	$1, 1, 1, \zeta_5^2, 1, \zeta_5^2, \zeta_5^2$
7	7	$2\zeta_7^4+2\zeta_7^3+3$	0, 60	$\zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^3, \zeta_7^2, \zeta_7^3, \zeta_7^3$
8	5	$\zeta_5^3+\zeta_5^2+1$	0, 61	$1, 1, 1, \zeta_5^2, \zeta_5^3, 1, \zeta_5^3, \zeta_5$
8	7	$\zeta_7^4+\zeta_7^3+6$	4, 19	$\zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^5$
8	8	0	0	$\zeta_8^5, \zeta_8^5, 1, \zeta_8^5, \zeta_8^7, \zeta_8^7, 1, \zeta_8^7$
9	7	$\zeta_7^4+\zeta_7^3+7$	5, 19	$\zeta_7^6, \zeta_7^5 \ldots$
9	9	$\zeta_9^5+\zeta_9^4+7$	5, 12	$\zeta_9^6, \zeta_9^6, \zeta_9^6, \zeta_9^6, \zeta_9^6, \zeta_9^6, \zeta_9^6, \zeta_9^6, \zeta_9^2$
10	5	$\zeta_5^3+\zeta_5^2+8$	6, 38	$\zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, \zeta_5^2, 1$
10	7	$\zeta_7^4+\zeta_7^3+8$	6, 19	$\zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^2, \zeta_7^5$
10	10	$\zeta_{10}^3-\zeta_{10}^2+7$	6, 38	$\zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^2$
11	11	$3\zeta_{11}^6+3\zeta_{11}^5+5$	0.75	$1, 1, 1, \zeta_{11}^6, 1, 1, \zeta_{11}^6, 1, \zeta_{11}^6, \zeta_{11}^6, \zeta_{11}^6$
11	11	0	0	$1, 1, \zeta_{11}^6, \zeta_{11}^7, \zeta_{11}^3, \zeta_{11}^5, \zeta_{11}^2, \zeta_{11}^5, \zeta_{11}^3, \zeta_{11}^7, \zeta_{11}^6$
11	11	$\zeta_{11}^6+\zeta_{11}^5+9$	7, 08	$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \zeta_{11}^6$

Thanks for your attention.