Congruences for the Number of Transversals of Latin squares

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e.g.

$$\begin{matrix} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit & \spadesuit & \heartsuit \end{matrix}$$

is a Latin square of order 4.

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A *transversal* of a latin square is a set of entries which includes exactly one entry from each row and column and one of each symbol.

$\spadesuit$	**$\heartsuit$**	$\diamondsuit$	$\clubsuit$
$\heartsuit$	$\spadesuit$	**$\clubsuit$**	$\diamondsuit$
$\clubsuit$	$\diamondsuit$	$\heartsuit$	**$\spadesuit$**
$\diamondsuit$	$\clubsuit$	$\spadesuit$	$\heartsuit$

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Alon *et al.* [1995] lamented that

> *"There have been more conjectures than theorems on latin transversals in the literature."*

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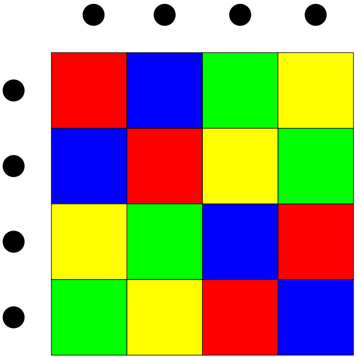

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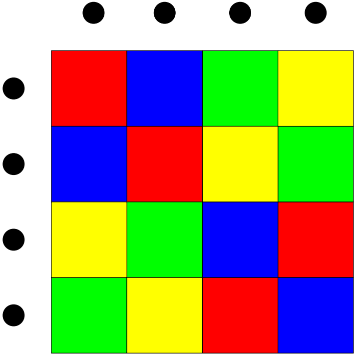

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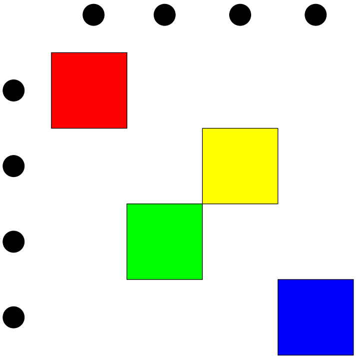

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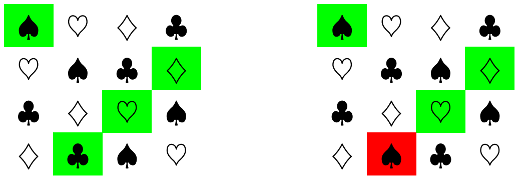

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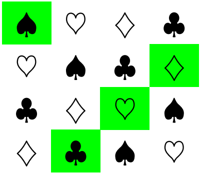

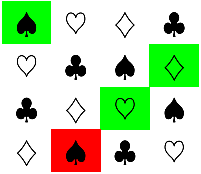

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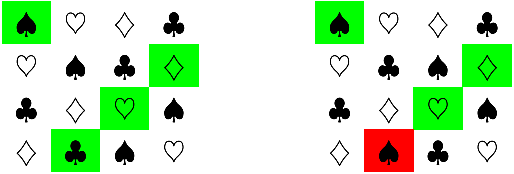

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Conj: [Brualdi] Every latin square has a near transversal.

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If true, it is barely so:

1	2	3	4	5
2	1	4	5	6
3	4	6	2	1
4	5	1	6	3
6	3	5	1	2

1	2	3	4	5	6	7
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3	4	1	2	7	5	6
4	5	6	7	1	2	3
5	3	7	6	2	1	4
6	7	2	5	3	4	1
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Permanents

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The *permanent* of A is defined by

$$\mathrm{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the sum is over all permutations σ of $\{1, 2, \ldots, n\}$.
In other words, the sum of the diagonal products.

Example:

$$\mathrm{per} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1.5.9 + 1.6.8 + 2.4.9 + 2.6.7 + 3.5.7 + 3.4.8 = 450$$

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By a result of Kräuter and Seifter we know that

$$2^{n - \lfloor \log_2(n+1) \rfloor}$$

divides $\mathrm{per}(H)$.

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Define D_n by the same formula, but using determinants.

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Conj: For $n \equiv 0 \mod 4$, we have $E_n \equiv D_n \mod 4$ and $T_{00} \equiv T_{01} \equiv T_{10} \equiv T_{11} \mod 2$.

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Thrm: Let $n \equiv 0 \mod 4$. Then $E_{2i-1} \equiv E_{2i} \mod 2$ for each i.

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Thrm: If $n \equiv 1 \mod 2$ then $t_{rc} \equiv E_n \mod 2$ for any r and c.

Is this known?

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Prf: Ryser's formula for computing the permanent of A is:

$$\text{per}(A) = \sum_{S \subseteq \{1,\dots,n\}} (-1)^{n-|S|} \prod_{i=1}^{n} X_i.$$

where

$$X_i = \sum_{j \in S} a_{ij}.$$

Consider the terms coming from S and its complement together

$$(-1)^{n-|S|} \left(\prod X_i - \prod (4k - X_i) \right) \equiv \pm 2 \prod X_i \mod 4.$$

But at least one X_i is even since $\sum_i X_i = 4k|S|$.

Conjectures

Conj: $t_{ik} + t_{jk} + t_{i\ell} + t_{j\ell} \equiv 0 \mod 4$ for all i, j, k, ℓ.

Conj: Let $n \equiv 0 \mod 2$. Then $2t_{ij} \equiv E_{n-1} \mod 4$ for all i, j.

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Conj: Let $n \equiv 1 \mod 2$. Then $E_{n-1} \equiv 0 \mod 4$ and $R_{4i+2} + 2R_{n-(4i+2)} \equiv 0 \mod 4$.

Our story ends...