Informational power of the Hoggar SIC-POVM

Anna Szymusiak, Wojciech Słomczyński

Institute of Mathematics, Jagiellonian University, Kraków, Poland

5th Workshop on Real and Complex Hadamard Matrices and Applications
10-14 July, 2017, Budapest

UNIVERSITAS
JAGELLONICA
CRACOVIENSIS

Phys. Rev. A 94, 012122 (2016)

POVM,

SIC-POVM,

Hoggar's SIC-POVM

Pure quantum states:

$$\mathbb{CP}^{d-1} \text{ – rays in } \mathbb{C}^d$$

or, equivalently,

$$\mathcal{P}(\mathbb{C}^d) \text{ – rank-1 orthogonal projections.}$$

Mixed quantum states – all convex combinations of pure states

General quantum measurement – POVM (positive operator-valued measure)

Special case:

A **normalized rank-1 POVM** $\Pi = \{\Pi_j\}_{j=1}^k$ is a set of k subnormalized rank-one projections $\Pi_j = (d/k)|\psi_j\rangle\langle\psi_j|$ satisfying the identity decomposition:

$$\frac{d}{k}\sum_{j=1}^{k}|\psi_j\rangle\langle\psi_j| = \mathbb{I}.$$

$\psi_j \in \mathbb{C}^d$, $|\psi_j| = 1$ for $j = 1, \ldots, k$

One can identify such POVMs with configurations of pure quantum states.

Definition

A **symmetric informationally complete POVM (SIC-POVM)** consists of d^2 subnormalized rank-one projections $\Pi_j = |\psi_j\rangle\langle\psi_j|/d$ with equal pairwise Hilbert-Schmidt inner products:

$$\mathrm{tr}(\Pi_i^* \Pi_j) = \frac{|\langle\psi_i|\psi_j\rangle|^2}{d^2} = \frac{1}{d^2(d+1)} \quad \text{for } i \neq j.$$

A brief overwiev of SIC-POVMs

- first studied in the context of *equiangular lines* in $\mathbb{C}^d$ or *complex spherical 2-designs* (e.g. Hoggar's papers 1978-82)
- extensively examined by Zauner in his PhD Thesis (1999) under the name of *regular quantum designs with degree 1*
- independently studied by Renes et al. (2003), the notion of *SIC-POVMs* introduced
- the existence of SIC-POVMs in every dimension – still an open problem
  - analytical solutions known for $d = 2 - 24, 28, 30, 31, 35, 37, 39, 43, 48$ (results by Scott & Grassl (2010), Appleby et al. (2017) and Chien (2015))
  - numerical confirmation up to $d = 151$ plus few other up to $d = 323$ (code designed and written by Scott)
  - simple interpretation in terms of metric spaces: the equilateral dimension (i.e., the maximum number of equidistant points) of $\mathbb{CP}^{d-1}$ endowed with the Fubini-Study metric is d^2
  - difficulty: how to inscribe a regular $(d^2 - 1)$-simplex in $\mathbb{R}^{d^2-1}$ into the $(2d - 2)$-dimensional subset of the $(d^2 - 2)$-sphere?
- $d = 3$ – the only known dimension with infinite family of nonequivalent SIC-POVMs

The Hoggar SIC-POVM (the Hoggar lines)

- First construction: complexification of diameters of certain quaternionic polytope in $\mathbb{H}^4$.

S. G. Hoggar, Math. Scand. **43**, 241 (1978)

- It can be obtained by taking the orbit of fiducial vector

$$\psi = \frac{1}{\sqrt{6}}(1+i, 0, -1, 1, -i, -1, 0, 0)^T$$

under action of the three-qubit Pauli group (isomorphic to $(\mathbb{Z}_2 \otimes \mathbb{Z}_2)^3$).

G. Zauner, "Quantendesigns. Grundzüge einer nichtkommutativen Designtheorie", Ph.D. thesis, Universität Wien (1999)

- The only known SIC-POVM that is not group-covariant with respect to the finite Weyl-Heisenberg group (isomorphic to $\mathbb{Z}_8 \otimes \mathbb{Z}_8$).

H. Zhu, "Some decision theoretic generalizations of information measures", Ph.D. thesis, National University of Singapore (2012)

- There exist exactly 3 *supersymmetric* SIC-POVMs (any two elements can be transformed into any two elelments via symmetry group action): $d = 2$, $d = 3$ (the Hesse configuration), $d = 8$ (the Hoggar lines).

H. Zhu, Ann. Phys. (N.Y.) **362**, 311 (2015)

- New simple construction using Hadamard matrices.

J. Jedwab, A. Wiebe, in *Algebraic Design Theory and Hadamard Matrices*, ed. by C. Colbourn (Springer Verlag, 2015), pp. 159-169

Construction by Jedwab and Wiebe

- Take an Hadamard matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

- Pick a row

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Construction by Jedwab and Wiebe

- Multiply one of the entries by some $v \in \mathbb{C}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -v & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Construction by Jedwab and Wiebe

- Take an Hadamard matrix
- Pick a row
- Multiply one of the entries by some $v \in \mathbb{C}$
- Do the same with all rows and all entries

Denote by $H(v) := \{H_{jk}(v)\}_{j,k=1}^{d}$ the obtained set of d^2 vectors such that $H_{jk}(v)$ is the j-th row of complex Hadamard matrix H with the k-th coordinate multiplied by $v \in \mathbb{C}$.

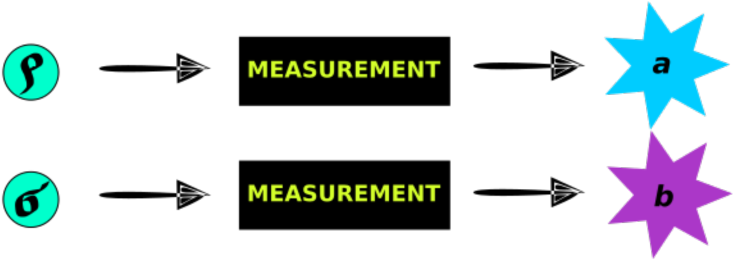
Theorem (Jedwab,Wiebe)

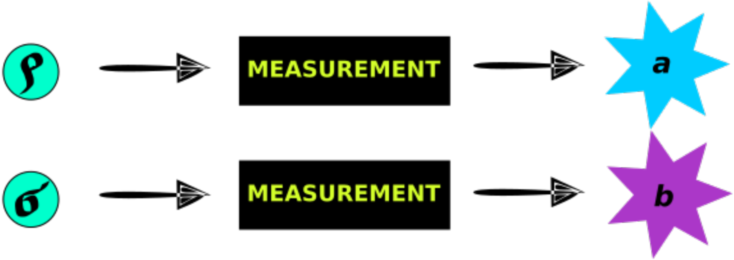
$H(v)$ is a set of d^2 equiangular lines (i.e. $H(v)$ defines a SIC-POVM) if and only if

- $d = 2$ *and* $v \in \{(1/2)(1 \pm \sqrt{3})(1+i), (-1/2)(1 \pm \sqrt{3})(1+i), (1/2)(1 \pm \sqrt{3})(1-i), (-1/2)(1 \pm \sqrt{3})(1-i)\}$, *or*
- $d = 3$ *and* $v \in \{0, -2, 1 \pm \sqrt{3}i\}$, *or*
- $d = 8$, *H is equivalent to a real Hadamard matrix and* $v \in \{-1 \pm 2i\}$.

In particular, the obtained set of equiangular lines for $d = 8$ is unitarily equivalent to the set of Hoggar lines.

Informational power of POVM


ρ
MEASUREMENT
a
σ
MEASUREMENT
b



How to quantify the **indeterminacy** of the measurement outcomes?

MEASURE OF RANDOMNESS – THE SHANNON ENTROPY

The state before measurement: ρ.
The probability of obtaining j-th outcome is given by $\mathrm{tr}(\rho\Pi_j) = (d/k)\langle\psi_j|\rho|\psi_j\rangle$.

The **Shannon entropy of measurement** Π is defined as the Shannon entropy of the probability distribution of the measurement outcomes:

$$H(\rho, \Pi) := \sum_{j=1}^{k} \eta(\mathrm{tr}(\rho\Pi_j)),$$

for an initial state ρ, where $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

- $H(\cdot, \Pi)$ attains its minima in pure states.
- $H(\rho, \Pi)$ is maximal for maximally mixed state $\rho_* := \frac{1}{d}\mathbb{I}$.
- Our aim: to find the minimizers of the entropy among pure states.

What if we consider an *ensemble* of possible initial states?

$V := \{\pi_i, \rho_i\}_{i=1}^{l}$ – an ensemble of initial states, $\Pi = \{\Pi_j\}_{j=1}^{k}$ – a POVM

How much information can be extracted from V by measurement Π?

Definition

The **mutual information between V and Π** is given by

$$I(V,\Pi) := \sum_{i=1}^{l} \eta\left(\sum_{j=1}^{k} P_{ij}\right) + \sum_{j=1}^{k} \eta\left(\sum_{i=1}^{l} P_{ij}\right) - \sum_{i=1}^{l}\sum_{j=1}^{k} \eta(P_{ij}),$$

where $P_{ij} = \pi_i \mathrm{tr}(\rho_i \Pi_j)$ for $i = 1, \ldots, l$ and $j = 1, \ldots, k$, and $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$.

What is the capability of extracting information by given measurement?

Definition

The **informational power** of Π is denoted by $W(\Pi)$ and given by

$$W(\Pi) := \max_{V-\text{ensemble}} I(V,\Pi).$$

M. Dall'Arno, G.M. D'Ariano, and M.F. Sacchi, Phys. Rev. A **83**, 062304 (2011)
O. Oreshkov, J. Calsamiglia, R. Muñoz Tapia, and E. Bagan, New J. Phys. **13**, 073032 (2011)

- There exists a maximally informative ensemble consisting of pure states only
- The informational power and the Shannon entropy of measurement Π are *in many cases* related by

$$W(\Pi) = \ln k - \min_{\rho} H(\rho, \Pi)$$

- The cases in which the informational power has been computed analytically so far:
    - all highly symmetric POVMs in dimension 2: seven sporadic measurements[3], including the 'tetrahedral' SIC-POVM[1,2], and one infinite series[2,3],
    - all SIC-POVMs in dimension three[4],
    - the POVM consisting of four MUBs in dimension three[5],
    - the Hoggar SIC-POVM[6]

[1] M. Dall'Arno, G.M. D'Ariano, and M.F. Sacchi, Phys. Rev. A **83**, 062304 (2011)

[2] O. Oreshkov, J. Calsamiglia, R. Muñoz Tapia, and E. Bagan, New J. Phys. **13**, 073032 (2011)

[3] W. Słomczyński and AS, Quantum Inf. Process. **15**, 565-606 (2016)

[4] AS, J. Phys. A **47**, 445301 (2014)

[5] M. Dall'Arno, Phys. Rev. A **90**, 052311 (2014)

[6] AS and W. Słomczyński, Phys. Rev. A **94**, 012122 (2016)

- Upper bound for 2-designs: $\ln \frac{2d}{d+1}$.

M. Dall'Arno, Phys. Rev. A **92**, 012328 (2015)

- The optimal probability distribution of the measurement outcomes for SIC-POVM, if achievable, should be of the form:

$$\left(\frac{2}{d(d+1)}, \dots, \frac{2}{d(d+1)}, 0, \dots, 0\right)$$

with $\frac{d(d-1)}{2}$ zeros.

P. Harremoës and F. Topsøe, IEEE Trans. Inform. Theory **47**, 2944 (2001)

- The upper bound is satisfied for $d = 2, 3$.
- Numerical calculations in low dimensions higher than 3 indicate that it is not always the case.
- Our result: the upper bound is achieved again in dimension 8 for the Hoggar SIC-POVM.

Theorem

Let H' be a complex Hadamard matrix in dimension $d \in \{2, 8\}$, equivalent to a real Hadamard matrix H, and v such that $H'(v) = H'_{jk}(v)$ is a set of equiangular vectors. Then the entropy of SIC-POVM generated by these vectors is minimized by states defined by $H'(\bar{v})$.

Sketch of the proof.

- The general case is easily reduced to the one concerning a real Hadamard matrix.
- For real Hadamard matrix H we write

$$H_{jk}(v) = \sum_{l=1}^{d} h_{jl} e_l + (v - 1) h_{jk} e_k,$$

where the canonical basis in $\mathbb{C}^d$ is denoted by $(e_l)_{l=1}^{d}$.
- We show that for every $m, n = 1, \ldots, d$ the sequence $T_{mn} := (|H'_{jk}(v) \cdot H'_{mn}(\bar{v})|^2)_{j,k=1}^{d}$ consists of two elements, one of which is 0, appearing with the desired multiplicity $(d-1)d/2$. □

Theorem

Under the assumptions of previous theorem,

the informational power of the SIC-POVM corresponding to $H'(v)$ is equal to $\ln(2d/(d+1))$, and the elements of $H'(\bar{v})$ constitute an equiprobable maximally informative ensemble.

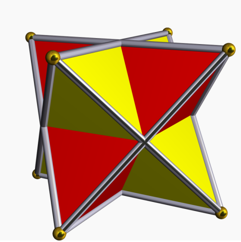
Particularly, the informational power of Hoggar lines is $2\ln(4/3)$.

How does the set $H(\bar{v})$ looks like and what is its relation with $H(v)$?

- $H(\bar{v})$ is also a SIC-POVM ('tetrahedral' for $d = 2$ and Hoggar's for $d = 8$)
- Let $C : \mathbb{C}^d \to \mathbb{C}^d$ be a complex conjugation with respect to the basis $(e'_l)_{l=1}^d$, i.e., an antiunitary involutive map keeping the basis invariant, given by $C(\sum_{l=1}^d x_l e'_l) = \sum_{l=1}^d \bar{x}_l e'_l$. Then

$$H_{jk}(\bar{v}) = C(H_{jk}(v)) \quad \text{for } j, k = 1, \ldots, d$$

- In the Bloch representation:
  - For $d = 2$ we get two dual regular tetrahedra, that together form *stella octangula*.



  - For $d = 8$ we get two regular 63-dimensional simplices inscribed in the unit sphere in a 63-dimensional real vector space, where one is the image of the other under a reflection through a 35-dimensional linear subspace.

A closer look at the localization of 28 zeros for 64 minimizers:

- Label the elements of $H(v)$ by the elements of $\Sigma := \mathbb{Z}_2^3 \otimes \mathbb{Z}_2^3$ (binary notation).
- Assume that H is the (real) Sylvester-Hadamard matrix H_3; in this case we have
$$h_{\iota\kappa} = (-1)^{\iota_1\kappa_1+\iota_2\kappa_2+\iota_3\kappa_3} \quad \text{for } \iota,\kappa \in \mathbb{Z}_2^3.$$
- Consider the blocks of zeros of $T_{\mu\nu}$:
$$B_{\mu\nu} := \{(\iota,\kappa) : H_{\iota\kappa}(v) \cdot H_{\mu\nu}(\bar{v}) = 0,\ \iota,\kappa \in \mathbb{Z}_2^3\}.$$
- Then $(\iota,\kappa) \in B_{\mu\nu}$ iff $\iota \neq \mu, \kappa \neq \nu$ and $h_{\mu+\iota,\nu+\kappa} = -1$ for $\iota,\kappa \in \mathbb{Z}_2^3$.
- The subset $\{B_{\mu\nu}\}_{\mu,\nu\in\mathbb{Z}_2^3} \subset \Sigma$ constitutes a symmetric (Menon) $(64, 28, 12)$-design.

Conclusions and problems

- The upper bound for informational power of SIC-POVMs is satisfied in dimensions 2,3 and 8 (for the Hoggar lines).
- The minimizers for $d = 2, 8$ are related to a SIC-POVM by complex conjugation in some basis.
- Is upper bound satisfied for any other SIC-POVM?
- Is there a relation between satisfying the upper bound and the "supersymmetry" of a SIC-POVM?
- Can $d = 8$ be an example of dimension in which we get different values of informational power for different (nonequivalent) SIC-POVMs?