Open problems on Butson matrices

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Butson-type Hadamard matrices

This talk is based on the preprint

Orderly generation of Butson-type Hadamard matrices.

`https://arxiv.org/pdf/1707.02287.pdf`

Joint work with P. Lampio and P. Östergård.

The team is hiring an exceptional doctoral student in the field of Information and Communications Technology (the actual work will be "computational design theory").

See `http://www.hict.fi/`. DL: July 30.

Summary of results

Classification of $\mathrm{BH}(n, q)$ matrices for $n \leq 11$ and $q \leq 17$, and several further cases.

Online database of Butson matrices complementing the [BTŻ] catalogue:

`https://wiki.aalto.fi/display/Butson`

(It is under construction...)

The database was compiled in order to aid us in attacking various problems, however this remains a work without much progress...

In this talk I review some open problems we encountered.

Extend the database further

Problem 1

Classify $\mathrm{BH}(n, q)$ matrices beyond $n \leq 11$ and $q \leq 17$.

[Problem A]

Give an independent verification of the $\mathrm{BH}(32, 2)$ classification.

[Problem B]

Extend Orrick's work on the $\mathrm{BH}(36, 2)$ classification.

[Orrick's observation]

Every *known* $\mathrm{BH}(36, 2)$ has an equivalent form with constant row and column sums.

[Problem C]

Classify the *regular* $\mathrm{BH}(36, 2)$ matrices (i.e., the $(36, 15, 6)$-designs).

Classify all Butson matrices with a fixed n

Uniqueness of Hadamard matrices for $n = 1, 2, 3, 5$.

Note the Haagerup invariant set:

$$\Lambda(H) = \{h_{ij}h_{k\ell}\overline{h_{i\ell}h_{kj}} \colon i, j, k, \ell \in \{1, \ldots, n\}\}$$

Consequently $F_4 \not\sim F_2 \otimes F_2$, $S_6 \not\sim F_6$, $S_6 \not\sim D_6$, $D_6 \not\sim F_6$.
Moreover $F_4(a) \not\sim F_4(b)$ for "generic" choice of a and b, etc.

$$F_4(a) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & a & -a \\ 1 & -1 & -a & a \end{bmatrix}$$

$$\Lambda(F_4(a)) = \{\pm 1, \pm a, \pm \overline{a}\}.$$

Problem 2[Grassl and Wanless, during the conference]

What are the $\mathrm{BH}(6, q)$ matrices?

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Theorem[Banica–Bischon–Schlenker, 2009]

The $\mathrm{BH}(6, q)$ matrices are S_6, $D_6(c)$, $F_6(a, b)$, $F_6(a, b)^T$, or... a hypothetical Butson matrix whose first two rows in log-form read

$$\frac{2\pi\mathbf{i}}{30}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 7 & 13 & 19 & 20 \end{bmatrix}.$$

Let $\zeta := exp(2\pi\mathbf{i}/q) \in \mathbb{C}$. Banica et al. assumed that the inner product between any two rows, which is a 6-term vanishing sum of qth roots of unity, $w(g) := g^{i_1} + g^{i_2} + g^{i_3} + g^{i_4} + g^{i_5} + g^{i_6}$, $w(\zeta) = 0$ decomposes as "sum of cycles", that is $w(g) = w_1(g) + w_2(g)$ with $w_1(\zeta) = w_2(\zeta) = 0$, or $w(g) = w_3(g) + w_4(g) + w_5(g)$, with $w_3(\zeta) = w_4(\zeta) = w_5(\zeta) = 0$.

Conjecture[B-B-S]

There are no Butson matrices where the inner products do not decompose as sum of cycles.

Spectral set conjecture in $\mathbb{R}^2$ holds?

Let $H \in \mathrm{BH}(n, q)$, and assume that $L(H) = AB \pmod{q}$, where $L(H)$ is the log-form of H, and A is $n \times r$ and B is $r \times n$. The smallest possible r is the “mod-q rank” of $L(H)$.

Note: $A := L(H)$ and $B := I$ always work, since $L(H) = L(H)I$.

Example[Fourier matrices]

Let $A := [0, 1, \ldots, n-1]$, $B := A^T$. Then $L(F_n) = AB$. The q-rank of the Fourier matrices is 1.

Example[Kronecker products]

Let $A := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, and $B := A^T$. Then $L(F_2 \otimes F_2) = AB$. The q-rank of Kronecker products is the number of terms in the product.

Spectral set conjecture in $\mathbb{R}^2$ holds? (ctd.)

Example[Kolountzakis–Matolcsi, 2004]

Let $A := \begin{bmatrix} 0 & 4 & 2 & 6 & 6 & 2 \\ 0 & 2 & 4 & 1 & 5 & 6 \\ 0 & 6 & 3 & 4 & 2 & 7 \end{bmatrix}$, $B^T := \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 7 & 0 \end{bmatrix}$. Then $L(H) := AB$ is the log form of a $\mathrm{BH}(6, 8)$, and $6 \nmid 8^3$.

Theorem[Sz., 2012]

If $H \in \mathrm{BH}(n, q)$ with q-rank r, then for every m there is a Dita-type $\mathrm{BH}(mn, mq)$ with q-rank at most r.

Theorem[Sz., 2012]

Let q be even, and let $H \in \mathrm{BH}(n, q)$ with q-rank r. Assume that $L(H) = AB$ and A has a row with even coordinates. Then there is a $\mathrm{BH}(2n, q)$ with q-rank exactly r.

Problem 3

Is there a $\mathrm{BH}(n, q)$ with $n \nmid q^2$ such that $L(H) = AB$ with q-rank 2?

Analogue for Petrescu's block construction?

Petrescu in his PhD thesis came up with an array

$$P(t) := \begin{bmatrix} X(t) & Y(t) & T \\ Y(t) & X(t) & T \\ T^* & T^* & D \end{bmatrix}$$

of order $n = s + s + (s+1) = 3s+1$. Analytic examples for $n = 7, 13, 19, 31, 79$, and numerical examples of higher orders.

Here the ansatz is that T forms s rows of a $\mathrm{BH}(s+1, q)$, and D is normal, regular, and $DD^* = (s-1)I + 2J$.

Problem 4

Construct $\mathrm{BH}(n, q)$ matrices for $n \equiv 5 \pmod 6$.

Theorem[Haagerup]

For $n = 5$ F_5 is unique.

Theorem[Lampio–Östergård–Sz.]

There are no $\mathrm{BH}(11, q)$ for $q <= 17$ and $q \neq 11$.

Uniqueness of the Fourier matrices?

Problem 5

Is it true that $F_p \in \mathrm{BH}(p,p)$ is unique up to monomial equivalence?

Positive results, based on computation:

Theorem[Hirasaka–Kim–Mizoguchi]

The Fourier matrix F_p is the unique $\mathrm{BH}(p,p)$ for primes $p \leq 17$.

Negative results based on the McNulty–Weigert construction:

Lemma

The $\mathrm{BH}(2p,p)$ matrices are not unique for $p = 7$.

Theorem[See H–K–M]

If F_p is not unique up to equivalence, then there is a non-Desarguesian projective plane of order p.

Classification of Butson-type MUBs?

Lemma

There are exactly 24 $\mathrm{BH}(6,30)$ up to equivalence.

Lemma

There are exactly 41 $\mathrm{BH}(6,36)$ up to equivalence.

Lemma

There are exactly 40 $\mathrm{BH}(6,42)$ up to equivalence.

For $n=6$ I expect that $q\sim 256$ can be done with reasonable efforts.

Problem

Classify pairs of Butson matrices $H,K\in\mathrm{BH}(n,q)$ such that $HK^*/\sqrt{n}\in\mathrm{BH}(n,q)$.

Idea: in a hypothetical complete set of MUBs in $\mathbb{C}^6$ approximate the first basis with a Butson-matrix...

More problems? More questions?

Thank you for your kind attention!