Quantum Dynamical Entropy, Chaotic Unitaries & Complex Hadamards

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Overview

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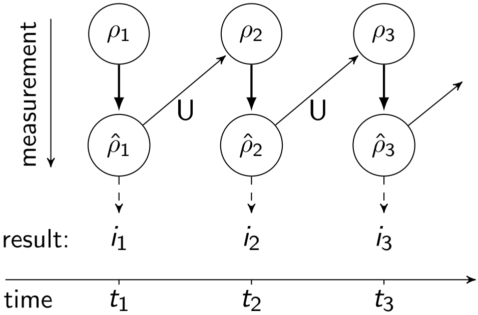
1. Successive measurements on a d-dim quantum system;
2. Measurement with k possible outcomes;
3. Unitary evolution U between two subsequent measurements;

This procedure generates strings of random measurement outcomes.



- Randomness is quantified by **quantum dynamical entropy**;
- Unitaries with maximal dynamical entropy: **chaotic**;
- They are expressed by **complex Hadamards**.

- Evolution of the system:

$$\mathcal{S}(\mathbb{C}^d) \ni \rho \mapsto U\rho\, U^* \in \mathcal{S}(\mathbb{C}^d)$$

where $\mathcal{S}(\mathbb{C}^d)$ - positive Hermitian operators on $\mathbb{C}^d$ with trace 1;
$U\colon \mathbb{C}^d \to \mathbb{C}^d$ is unitary; $\mathcal{U}(d)$ stands for the unitary group of dim d.

- Measurement:
rank-1 PVMs: collection $\{\Pi_1, \dots \Pi_d\}$ of 1-dim projections on $\mathcal{S}(\mathbb{C}^d)$ satisfying

$$\sum_{j=1}^{d} \Pi_j = \mathbb{I}.$$

Every Π_j corresponds to some normalized $|\varphi_j\rangle \in \mathbb{C}^d$,
$\{|\varphi_j\rangle\}_{j=1}^{d}$ is an **orthonormal basis** of $\mathbb{C}^d$.

Quantum entropy of U with respect to Π:

$$H(U,\Pi) := \lim_{n\to\infty}\frac{H_n}{n} = \lim_{n\to\infty}(H_{n+1}-H_n)$$

Here H_n is n-th partial entropy:

$$H_n := \sum_{i_1,\ldots,i_n=1}^{d} \eta\left(P_{i_1,\ldots,i_n}(\mathbb{I}/d)\right) \quad \text{where} \quad \eta(x) = \begin{cases} -x\ln x & x>0 \\ 0 & x=0 \end{cases}$$

and $P_{i_1,\ldots,i_n}(\mathbb{I}/d) = p_{i_1}(\mathbb{I}/d)\cdot p_{i_1,i_2}\cdot\ldots\cdot p_{i_{n-1},i_n}$ is the probability of measuring $(i_1,\ldots,i_n)$, starting from the maximally mixed state $\mathbb{I}/d$, with $p_j(\mathbb{I}/d) = 1/d$ and $p_{jl} = |\langle\varphi_j|U|\varphi_l\rangle|^2$ for $j,l = 1,\ldots,d$.

For rank-1 PVMs: $$H(U,\Pi) = \frac{1}{d}\sum_{i,j=1}^{d}\eta\left(|\langle\varphi_i|\,U\,|\varphi_j\rangle|^2\right)$$

Srinivas (1978), Pechukas (1982), Beck & Graudenz (1992) - for PVMs; Słomczyński & Życzkowski (1994) - for POVMs; Crutchfield & Wiesner (2008) - quantum entropy rate

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Quantum dynamical entropy of U (independent of measurement):

$$H_{dyn}(U) := \max_{\Pi \in PVMs} H(U,\Pi) = \max_{\substack{\{|\varphi_j\rangle\}_{j=1}^{d} \\ \text{orthonormal} \\ \text{bases of } \mathbb{C}^d}} \frac{1}{d}\sum_{i,j=1}^{d} \eta\left(|\langle\varphi_i|\, U\, |\varphi_j\rangle|^2\right)$$

- $H_{\mathrm{dyn}}(U)$ depends only on **the eigenvalues of U**;
- $0 \le H_{dyn}(U) \le \ln d$.

U is chaotic iff $H_{\mathrm{dyn}}(U) = \ln d$ $\leftrightsquigarrow$ **complex Hadamards**

Such U can produce maximally random strings of measurement results.

Example: discrete Fourier transform $F_d/\sqrt{d}$.

chaotic Unitaries $\leftrightsquigarrow$ complex Hadamards

The following conditions are equivalent:

1. U is chaotic;
2. there exists an orthonormal basis $\{|\varphi_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$ such that
$$\frac{1}{d}\sum_{i,j=1}^{d}\eta\left(|\langle\varphi_i|\,U\,|\varphi_j\rangle|^2\right)=\ln d;$$
3. there exists an orthonormal basis $\{|\varphi_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$ such that
$$|\langle\varphi_i|\,U\,|\varphi_j\rangle| = \frac{1}{\sqrt{d}} \quad \text{for each} \quad i,j=1,\ldots,d;$$
4. $\sqrt{d}\,U$ is represented by a **complex Hadamard matrix** (in some orthonormal basis).

Simple necessary condition:

$\sqrt{d}\,U$ is represented by a complex Hadamard $\implies$ $|\mathrm{tr}\,U|\le\sqrt{d}$.

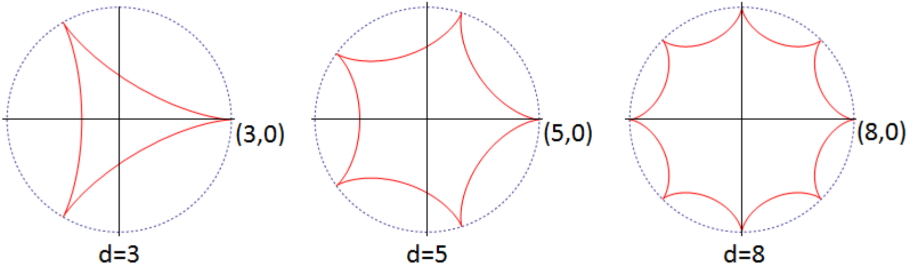
H_{dyn} is invariant under (i) *phase multiplication*: $U \to e^{i\varphi}U$ for $\varphi \in \mathbb{R}$;

(ii) *inversion*: $U \to U^{-1}$.

We focus on **special unitaries**, i.e., with determinant fixed at one.

Fact:

All possible traces of special unitaries of size d **fill in the d-hypocycloid** with one cusp at $(d, 0)$. (Charzyński et al. 2005)



Traces of chaotic unitaries = union of suitably rescaled and rotated (filled in) hypocycloids

$$\{\mathrm{tr}\, U \mid H_{dyn}(U) = \ln d,\, U \in \mathcal{U}(d)\} \;=\; \bigcup_{\substack{H\in \mathrm{Had(d)} \\ \sigma\in S_d}} \alpha_{H,\sigma} \mathrm{Hypocycloid}(d)$$

$$\text{with} \quad \alpha_{H,\sigma} := \left(\frac{\mathrm{sgn}(\sigma)\prod_{j=1}^{d} H_{j,\sigma(j)}}{\det H} \right)^{\frac{1}{d}} .$$

- rescaling factor: $|\alpha_{H,\sigma}| = 1/\sqrt{d}$;
- $\alpha_{H,\sigma}$ = any d-th root of the normalized summand corresponding to σ in the Leibnitz formula for the determinant of H:

$$\det H = \sum_{\sigma\in S_d} \mathrm{sgn}(\sigma) \prod_{j=1}^{d} H_{j,\sigma(j)};$$

- **equivalent Hadamard matrices give the same multipliers α.**

Qubits: dim $= 2$

In dim 2 all complex Hadamards are equivalent to $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Only one multiplier: $\frac{1}{\sqrt{2}}$ $\rightsquigarrow$ $U \in \mathcal{U}(2)$ is chaotic iff $|\mathrm{tr}\, U| \leq \sqrt{2}$

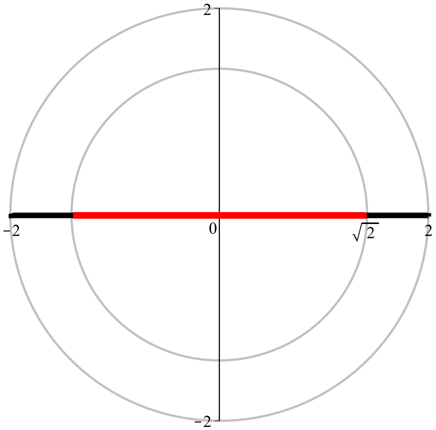
<u>Volume</u>:

$$\mathrm{Haar}(\{U \in \mathcal{U}(2) \mid H_{dyn}(U) = \ln 2\}) = \frac{2+\pi}{2\pi} \approx 0.818$$

For $U_\theta \sim \begin{bmatrix} \exp\left(\frac{\theta}{2}\mathrm{i}\right) & 0 \\ 0 & \exp\left(-\frac{\theta}{2}\mathrm{i}\right) \end{bmatrix}$

we have $\mathrm{tr}\, U_\theta = 2\cos\frac{\theta}{2}$.

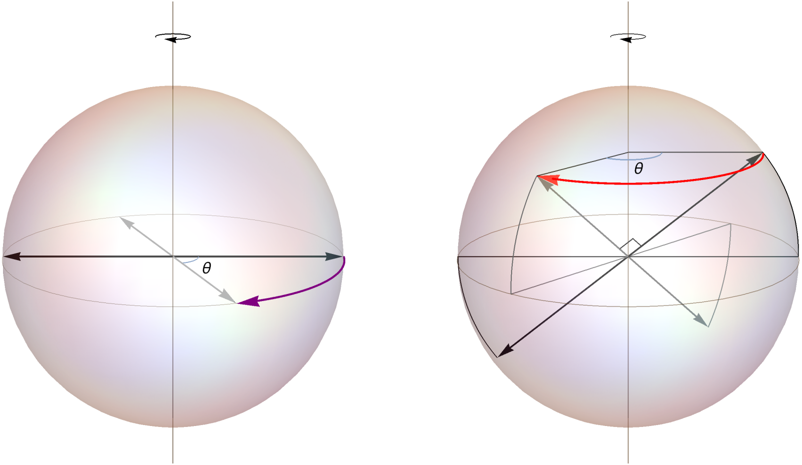
$U_\theta \in \mathcal{U}(2)$ is chaotic iff $\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$



Geometric interpretation

$\mathcal{S}(\mathbb{C}^2) \rightsquigarrow S^2 \subset \mathbb{R}^3, \quad SU(2) \rightsquigarrow SO(3).$

Chaotic unitary/rotation – the image of <u>some</u> axis under the rotation is perpendicular to the original axis.



Qutrits: dim = 3

All Hadamards of order 3 are equivalent to F_3 (Craigen 1991).

Two multipliers:

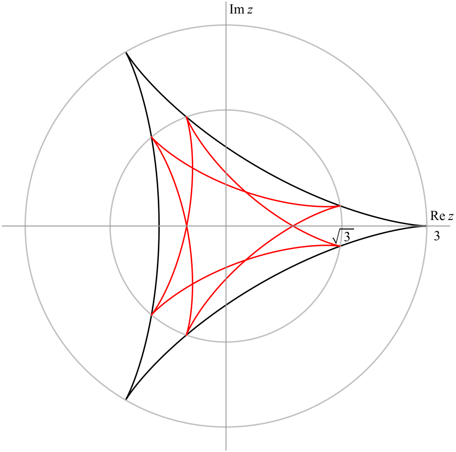
$$\frac{\exp(\mathrm{i}\pi/18)}{\sqrt{3}} \quad \text{and} \quad \frac{\exp(-\mathrm{i}\pi/18)}{\sqrt{3}}$$

This fully characterizes chaotic unitaries in dim 3, since tr U fully characterizes the eigenvalues of U.

Volume ≈ 0.592

Regions filled in by all possible traces of

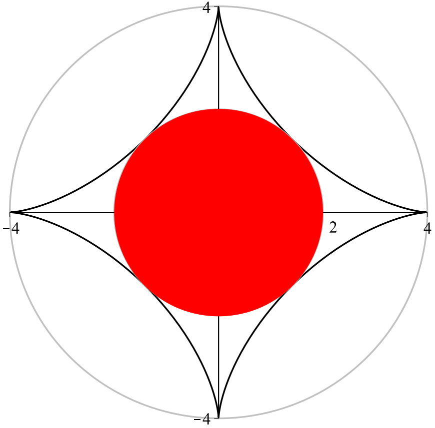
- special unitaries,
- **special complex Hadamards:**



dim $= 4$

One-parameter family of unequivalent Hadamards: (Craigen 1991)

$$F_4^{(1)}(\varphi) := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & ie^{i\varphi} & -1 & -ie^{i\varphi} \\ 1 & -1 & 1 & -1 \\ 1 & -ie^{i\varphi} & -1 & ie^{i\varphi} \end{bmatrix}, \quad \varphi \in [0, 2\pi)$$



$$\Downarrow$$

They generate all possible multipliers:

$$\frac{1}{2}e^{i\psi}, \ \psi \in [0, 2\pi).$$

dim = 5

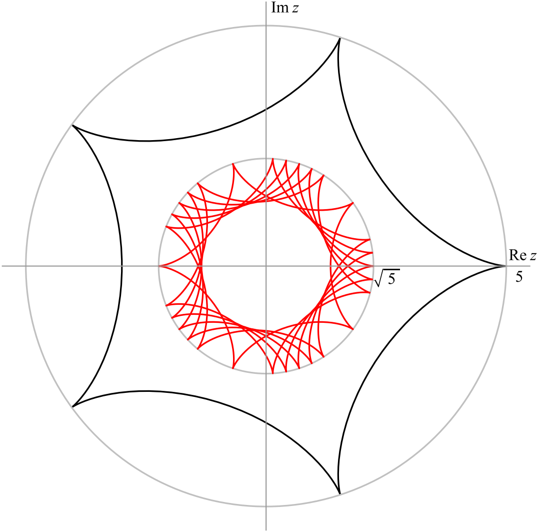
All Hadamards equivalent to F_5.

(Haagerup 1996)

F_5 generates six multipliers:

$$\frac{1}{\sqrt{5}}, \quad \frac{e^{\pi i/25}}{\sqrt{5}}, \quad \frac{e^{2\pi i/25}}{\sqrt{5}},$$

$$-\frac{1}{\sqrt{5}}, \quad \frac{e^{-\pi i/25}}{\sqrt{5}}, \frac{e^{-2\pi i/25}}{\sqrt{5}},$$



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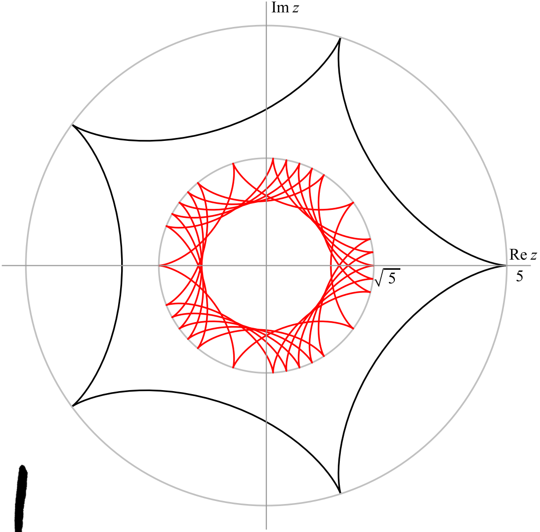

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→ POVMs

Measurement: **rank-1 POVMs:**

set $\{\Pi_1, \ldots \Pi_k\}$ of positive non-zero Hermitian operators such that $\sum_{j=1}^{k} \Pi_j = \mathbb{I}$ and $\Pi_j = \frac{d}{k} |\varphi_j\rangle \langle\varphi_j|$ (i.e. suitably rescaled 1-dim projection).

Such measurement generates additional randomness!

Quantum entropy of U with respect to Π for rank-1 POVMs:

$$H(U, \Pi) = \ln \frac{k}{d} + \frac{d}{k^2} \sum_{i,\, j=1}^{k} \eta \left(| \langle\varphi_i| U |\varphi_j\rangle |^2 \right)$$

Quantum dynamical entropy of U with respect to Π for rank-1 POVMs:

$$\underbrace{H_{dyn}(U, \Pi)}_{\text{evolution entropy}} := \underbrace{H(U, \Pi)}_{\text{total entropy}} - \underbrace{H(\mathbb{I}, \Pi)}_{\text{measurement entropy}}$$

$$H_{\text{dyn}}(U, \Pi) = \ln d \quad \Longrightarrow \quad \Pi \text{ is a PVM.}$$

U is chaotic iff U is chaotic w.r.t. a PVM iff $\sqrt{d}U \sim$ complex Hadamard