Hadamard Matrices and Clifford-Gastineau-Hills Algebras

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Dedicated with Great Respect to Anne Penfold Street

Abstract

Research into the construction of Hadamard matrices and orthogonal designs has led to deeper algebraic and combinatorial concepts. This paper surveys the place of amicability, repeat designs and the Clifford and Clifford-Gastineau-Hills algebras in laying the foundations for a *Theory of Orthogonal Designs.*

Keywords: *Orthogonal Designs; Hadamard Matrices; Clifford Algebras; Clifford-Gastineau-Hills (CGH) Algebras; 05B20.*

Research into the existence question for Hadamard matrices has been crucial in forcing the study of related theoretical results. The pioneering work by Kathy Horadam in her work on the five-fold path [10], her work with Warwick de Launey on cocyclic Hadamard matrices [4] are examples and work by Warwick de Launey and Dane Flannery [3] in their foundational work on algebraic design theory [3] yet another. Paul Leopardi has explored their relationship to amicability/anti-amicability graphs [**?**]. Other authors have concentrated further on their applications and structure in multidimensional space.

To construct Hadamard matrices Geramita and Seberry [6] used orthogonal designs. This survey discusses the path from Hadamard matrices to orthogonal designs, amicable Hadamard matrices and anti-amicable Hadamard matrices to amicable orthogonal designs and then to constructs called product designs and repeat designs. In each case the number of variables possible has been solved by converting the question into algebra. The study of the role

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of algebras in orthogonal design constructions leads us to see that product designs are subsets of repeat designs. The algebras of orthogonal designs are Clifford algebras and the algebras of repeat designs are Clifford-Gastineau-Hills algebras. The study of the algebras allows us to obtain exactly the maximum possible variables in each of the designs studied.

This leads to questions about how this knowledge when applied to Hadamard matrices of orders which are powers of two may be able to have embedded substructures to hide messages and/or improve some error correction capabilities. Conceivably such deeper knowledge may have applications in other areas such as spectrometry, sound enhancement or compression and other signal processing.

1 Introduction

Eddington in 1920, in his studies of relativity, raised the combinatorial question "What is the largest number of matrices of a given order which can anti-commute and square to $-I$, I the identity matrix" (see [5]). We will see that a set of p $n \times n$ matrices E_i which satisfy the algebraic conditions

$$\begin{array}{ll} E_i^2 = -I & (1 \le i \le p) \\ E_j E_i = -E_i E_j & (1 \le i < j \le p)\,, \end{array} \tag{1}$$

is *necessary* for the existence of an orthogonal design of order n on $p+1$ variables.

That it is *sufficient* is not immediately clear, since the E_i must satisfy other, combinatorial conditions, namely

$$\text{each } E_i \text{ is a } \{0, \pm 1\} \text{ matrix and } E_j * E_i = 0 \ (i \neq j)\,, \tag{2}$$

where "*" is the *Hadamard product* defined as

$$(a_{ij}) * (b_{ij}) = (a_{ij} b_{ij})$$

the component-wise multiplication.

An algebra, which is associative with a "1", on p generators, $\alpha_1, \ldots, \alpha_p$ say, with defining equations

$$\begin{array}{ll} \alpha_i^2 = -1 & (1 \le i \le p) \\ \alpha_j \alpha_i = -\alpha_i \alpha_j & (1 \le i < j \le p)\,. \end{array} \tag{3}$$

is an example of the well known *Clifford Algebras*.

2 Orthogonal Designs

Definition 1. An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted

$$OD(n; s_1, s_2, \ldots, s_u)$$

on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order n with real entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words it has the *additive property*,

$$AA^\top = \left(s_1 x_1^2 + \ldots + s_u x_u^2\right) I_n \tag{4}$$

where I_n is the identity matrix.

Example 1. We take the $OD(4; 1, 1, 1, 1)$, D and observe it can be written as either

$$D = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \tag{5}$$

or as the sum

$$D = aE_1 + bE_2 + cE_3 + dE_4\,,$$

where a, b, c and d are commuting variables (they do not need to be real) and

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

D is an orthogonal design if

$$DD^\top = \left(a^2 + b^2 + c^2 + d^2\right) I_4\,.$$

The algebraic conditions which make this an orthogonal design are

$$\begin{array}{ll} E_1^2 = I\,,\ E_i^2 = -I & (2 \le i \le 4) \\ E_j E_i = -E_i E_j & (1 \le i < j \le 4)\,, \end{array} \tag{6}$$

and the combinatorial conditions which make this an orthogonal design are

$$\text{each } \ E_i \text{ is a } \{0, \pm 1\} \text{matrix and } \ E_j * E_i = 0 \ (i \neq j) \,. \tag{7}$$

Thus we have linked the orthogonal design, the quadratic form and the Clifford-type algebras together. The orthogonal design has the extra properties that $E_1^2 = I$ and disjointness of matrices in the combinatorial conditions.

The fact that the structure and representation theory of the Clifford algebra (3) are known means that Eddington's problem can be solved (see Kawada and Iwahore, [11]). Moreover this representation theory is known to give a complete solution to the problem of what are the possible orders of orthogonal designs on any number of variables. As noted above, the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d, 0 \leq d < 4$, then $\rho(n) = 8c + 2^d$ [6].

We now consider Clifford algebras in a more complex context (over fields of characteristic 2: we observe that in fact characteristic $\neq 2$ is easier to deal with, and characteristic 2 is a special case). We do not treat these but refer the reader to Lam [12], O'Meara [13] and Artin [1]. The more modern view has been that Clifford algebras arise naturally from quadratic forms. In fact the class of all Clifford algebras corresponding to non-singular quadratic forms over a field F of characteristic not 2 coincides with the class of all $F-$algebras, C, on a finite number of generators $\{\alpha_i\}$ with defining equations of the form

$$\begin{array}{ll} \alpha_i^2 = k_i & (\text{some } k_i \in \mathcal{F} = F/\{0\} \\ \alpha_j \alpha_i = -\alpha_i \alpha_j & (i \neq j), \end{array} \tag{8}$$

we identify k_i in F with $k_i \in \mathcal{F}$ with $k_i 1_C$ in C.

We note the similarity of equations (1) with those of (6).

3 Amicable Orthogonal Designs

In the paper, Geramita-Geramita-Wallis [8], the following remarkable pairs of matrices are given:

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}; \qquad Y = \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix}. \tag{9}$$

and

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ x_3 & x_3 & -x_1 & -x_2 \\ x_3 & -x_3 & x_2 & -x_1 \end{bmatrix}; \qquad Y = \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ -y_3 & -y_3 & y_2 & y_1 \\ -y_3 & y_3 & y_1 & -y_2 \end{bmatrix}. \tag{10}$$

The first pair satisfy the following equations

$$\begin{aligned} XX^\top &= \left(x_1^2 + x_2^2\right) I_2 \\ YY^\top &= \left(y_1^2 + y_2^2\right) I_2 \\ XY^\top &= \begin{bmatrix} x_1y_1 + x_2y_2 & -x_1y_2 + x_2y_1 \\ x_2y_1 - x_1y_2 & -x_1y_1 - x_2y_2 \end{bmatrix} = YX^\top \end{aligned} \tag{11}$$

so the quadratic forms have unique properties since

$$\left[XY^\top\right]\left[XY^\top\right]^\top = XY^\top YX^\top = \left(x_1^2 + x_2^2\right)\left(y_1^2 + y_2^2\right) I_2\,. \tag{12}$$

The second pair satisfy the equations

$$\begin{aligned} XX^\top &= \left(x_1^2 + x_2^2 + 2x_3^2\right) I_4\,, \\ YY^\top &= \left(y_1^2 + y_2^2 + 2y_3^2\right) I_4\,, \\ XY^\top &= YX^\top\,, \end{aligned} \tag{13}$$

and

$$\left[XY^\top\right]\left[XY^\top\right]^\top = XY^\top YX^\top = \left(x_1^2 + x_2^2 + 2x_3^2\right)\left(y_1^2 + y_2^2 + 2y_3^2\right) I_4\,.$$

We then asked do any more exist? If so then how many variables can occur in each of any such pair of orthogonal designs, called *amicable orthogonal designs*, for a given order. This has been solved completely by Daniel Shapiro in his PhD thesis [18]. Orders 2, 4 and 8 are constructed in the PhD theses of Deborah J Street [19] and Ying Zhao [22]. Seberry [17, Sections 5.5, 5.9] discusses orders 2, 4, and 8 but other orders remain, as yet, un-constructed. The next problem is to determine whether orthogonal designs actually exist for these necessary conditions.

In constructing Hadamard matrices *amicability* and *anti-amicability* proved a useful tool. Its extension to orthogonal designs proved decisive in the equating and killing theorem of Geramita and Seberry [6]. Indeed it is crucial to Craigen's [2] extension to the previously known asymptotic existence results [20].

So let us be more precise and investigate further.

Definition 2. Two orthogonal designs X and Y are said to be *amicable* if $XY^\top = YX^\top$ and to be *anti-amicable* if $XY^\top = -YX^\top$. An *amicable k-set* will be used to describe a set of k matrices $X_1, \cdots, X_k$ which pairwise satisfy $X_iX_j^\top = X_jX_i^\top$ for all $1 \leq i,\ j \leq k$ and an *anti-amicable k-set* if $X_1, \cdots, X_k$ pairwise satisfy $X_iX_j^\top = -X_jX_i^\top$ for all $1 \leq i,\ j \leq k$.

Remark 1. We note here that the definitions of *amicable k-set* and *anti-amicable k-set* are mentioned here for purely historical reasons. It was Wolfe's [21] inspiration in considering amicable pairs and amicable triples that led to the insight of the importance of Clifford algebras (3) in solving the question of the number of variables possible in an orthogonal design. However, as we will see, amicable k-sets or k-tuples are a special case of repeat designs.

Example 2. We give two examples of amicable triples to show their existence. Amnon Neeman found the following $(1, 7, 1)$:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ - & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & - & 0 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & - & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & - & 0 \end{array}\right], \quad \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & - & - & 1 & 1 & 1 \\ - & 0 & 1 & 1 & - & - & - & 1 \\ - & - & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & - & 0 & - & 1 & - & 1 \\ \hline 1 & 1 & - & 1 & 0 & - & 1 & 1 \\ - & 1 & - & - & 1 & 0 & - & 1 \\ - & 1 & - & 1 & - & 1 & 0 & - \\ - & - & - & - & - & - & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ \hline & & & & 0 & 1 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & 0 \end{array}\right].$$

The following three matrices give a $(2,7,1)$:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & & & & \\ - & 0 & 0 & - & & & & \\ - & 0 & 0 & 1 & & & & \\ 0 & 1 & - & 0 & & & & \\ \hline & & & & 0 & 0 & 1 & 1 \\ & & & & 0 & 0 & 1 & - \\ & & & & - & - & 0 & 0 \\ & & & & - & 1 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & - & 1 & - & - & 1 \\ - & 0 & 1 & - & - & 1 & 1 & 1 \\ - & - & 0 & 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & 0 & 1 & 1 & 1 & 1 \\ \hline - & 1 & - & - & 0 & 1 & - & - \\ 1 & - & - & - & - & 0 & - & 1 \\ 1 & - & 1 & - & 1 & 1 & 0 & - \\ - & - & - & - & 1 & - & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ \hline & & & & 0 & - & 0 & 0 \\ & & & & - & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & - \end{array}\right].$$

We note that the pair of matrices given by equation (13) may be written as

$$X = \sum_{i=0}^{2} x_i A_i\,, \qquad Y_i = \sum_{i=1}^{3} y_i B_i\,, \tag{14}$$

$$(A_i, B_j \ \{0, \pm 1, \pm 2\} \text{ matrices where } A_i * A_j = 0, B_i * B_j = 0, \text{ for } i \neq j)\,. \tag{15}$$

Substituting and comparing like terms gives:

$$\begin{cases} A_i A_i^\top = u_i I, \qquad B_j B_j^\top = v_j I, \\ A_i A_j^\top + A_j A_i^\top = 0 \ \ (i \neq j), \quad B_i B_j^\top + B_j B_i^\top = 0 \ \ (i \neq j), \\ A_i B_j^\top = B_j A_i^\top \ \text{(for all } \ i, j), \end{cases}$$

and similar equations with products reversed.

Set

$$E_i = \frac{1}{\sqrt{u_i u_0}} A_i A_0^\top\,, \qquad F_j = \frac{1}{\sqrt{v_j u_0}} B_j A_0^\top\,,$$

It is easily verified that $E_0 = I$ and E_1, E_2, F_1, F_2, F_3,

$$\begin{cases} E_i^2 = -I\,, \qquad F_j^2 = -I\,, \\ E_j E_i = -E_i E_j \ \ (i \neq j) \qquad F_i F_j = -F_j F_i \ \ (i \neq j)\,, \\ E_i F_j = F_j E_i \ \text{(for all } \ i, j)\,. \end{cases}$$

The equations can be considered as a “Clifford-like algebra" with generators $\alpha_1, \alpha_2, \beta_1, \beta_2$ and β_3,

$$\begin{array}{lll} \alpha_i^2 = -1 & \beta_i^2 = -1 & (-1 \in \mathcal{F} = F/\{0\}) \\ \alpha_j\alpha_i = -\alpha_i\alpha_j & \beta_j\beta_i = -\beta_i\beta_j & (i \neq j)\,, \\ \alpha_i\beta_j = \beta_j\alpha_i & & \end{array}$$

4 Foundational Motivating Constructions for Orthogonal Designs

Geramita and Seberry [6] gave a number of constructions, these were first named product designs and repeat designs in Robinson's PhD Thesis [14]. The next construction for orthogonal designs appears in a slightly different form in [6].

Construction 1. *Let x_1, x_2 commuting variables and W, Y_1 and Y_2 be the matrices of order n described by*

1. *$W * Y_i = 0$, for $i = 1, 2$, (* the Hadamard product);*
2. *$Y_1Y_2^\top = Y_2Y_1^\top$ more precisely $AOD(n : (u_1, u_2, \ldots ; v_1, v_2; \ldots ; w))$;*
3. *W is an $OD(n : w)$; and*
4. *$Y_iW^\top = -WY_i^\top$ for $i = 1, 2$.*

Then the following matrix is an $OD(2n; (w, w, u_1, u_2, \ldots, v_1, v_2, \ldots))$

$$\begin{bmatrix} Y_1 + x_1W & Y_2 + x_2W \\ Y_2 - x_2W & -Y_1 + x_1W \end{bmatrix}.$$

Construction 2 (Geramita-(Seberry)Wallis [9])**.** *Let Y_1, Y_2, Y_3 be skew-symmetric orthogonal designs of types $(p_{i1}, p_{i2}, ..)$, $i = 1, 2, 3$ in order n, and Z a symmetric $OD(n\colon h_1, h_2, ..)$. Further, suppose $Y_iY_j^\top = Y_jY_i^\top$ and $Y_kZ^\top = ZY_k^\top$. Then*

$$\begin{bmatrix} x_1I_n + Y_1 & x_2I_n + Y_2 & x_3I_n + Y_3 & Z \\ -x_2I_n + Y_2 & x_1I_n - Y_1 & Z & -x_3I_n - Y_3 \\ -x_3I_n + Y_3 & -Z & x_1I_n - Y_1 & x_2I_n + Y_2 \\ -Z & x_3I_n - Y_3 & -x_2I_n + Y_2 & x_1I_n + Y_1 \end{bmatrix}$$

is an $OD(4n; (1, p_{11}, p_{12}, \ldots, 1,\ p_{21}, p_{22}, \ldots, 1,\ p_{31}, p_{32}, \ldots, h_1, h_2, \ldots))$.

Proof. By straightforward verification. □

Closer study of these two constructions shows that if we replace W by the identity matrix and Z by the zero matrix O the matrices satisfy the same equations. The first was previously used as an illustration of a product design and the second given as an illustration of a repeat design. We now proceed to study the more general concept of repeat designs.

5 Repeat Orthogonal Designs

Robinson and Seberry [16] defined a *repeat design*, but we prefer to give the formal definition in an alternative form:

Definition 3. Suppose $X, Y_1, \ldots, Y_k, Z$ are orthogonal designs of order n, types $(u_1, \ldots, u_p)$, $(v_{11}, \ldots, v_{1q_1}), \ldots, (v_{k1}, \ldots, v_{kq_k})$, and $(w_1, \ldots, w_r)$ on the variables $(x_1, \ldots, x_p)$, $(y_{11}, \ldots, y_{1q_1}), \ldots, (y_{k1}, \ldots, v_{kp_k})$, and $(z_1, \ldots, z_r)$ respectively, and that

(i) $X * Y_i = 0$ (for all i)
(ii) $Y_i X^\top = -X Y_i^\top$,
(iii) $Y_j Y_i^\top = Y_i Y_j^\top$, $\quad ZX^\top = XZ^\top$, $\quad ZY_i^\top = Y_i Z^\top$ (all i, j)

Then we call the $(k+2)$-set $(X, Y_1, \ldots, Y_k, Z)$ a *repeat design* of order n, type $(u_1, \ldots, u_p; v_{11}, \ldots, v_{1q_1}; \ldots; v_{k1}, \ldots, v_{kq_k}; w_1, \ldots, w_r)$ on the variables $(x_1, \ldots, x_p;\ y_{11}, \ldots, y_{1q_1}; \ldots; y_{k1}, \ldots, v_{kp_k}; z_1, \ldots, z_r)$.

Of course $X, Y_1, \ldots, Y_k, Z$ in Definition 3 corresponds to $R, P_1, \ldots, H$ respectively in [6]. Otherwise, apart from the fact that we have allowed X in Definition 3 to be on more than one variable, the conditions are equivalent.

Product designs [7] may be regarded as particular cases of repeat designs, given by $k = 2$, $r = 0$ and $Z = 0$ (zero matrix, which may be regarded as an orthogonal design on no variables).

Similarly a theory of repeat designs should yield a theory of amicable k-sets, if we can allow $X = Z = 0$. In the immediate following we assume that X has at least one variable (while allowing $Y_1, \ldots, Y_k, Z$ to have as few as no variables each), but it will be found that this restriction may be removed painlessly.

Remark 2. We note that this indicates that the existence problem for triples (R, S, H) which are repeat designs $(I; (R; S); H)$ is very difficult and far from resolved.

When

$$
\begin{array}{lll}
XX^\top = \left(\sum_0^p u_j x_j^2\right) I\,, & Y_iY_i^\top = \left(\sum_1^{q_i} v_{ij} y_{ij}^2\right) I\,, & ZZ^\top = \left(\sum_1^r w_j z_j^2\right) I \\
Y_iX^\top = -XY_i^\top\,, & & \\
Y_jY_i^\top = Y_iY_j^\top \ \ (i \neq j) & Y_iZ^\top = ZY_i^\top & XZ^\top = ZX^\top\,,
\end{array}
\tag{16}
$$

and similar equations with $X^\top X$, etc., in place of $XX^\top$, etc.

Write

$$
X = \sum_0^p x_j A_j\,, \qquad Y_i = \sum_1^{q_i} y_{ij} B_{ij}\,, \qquad Z = \sum_1^r z_j C_j \tag{17}
$$

$$
(A_j, B_{ij}, C_j \ \ \{0 \pm 1\} \text{ matrices}) \tag{18}
$$

Substituting into (16) and comparing like terms gives:

$$
\begin{cases}
A_jA_j^\top = u_j I, \qquad B_{ij}B_{ij}^\top = v_{ij} I, \qquad C_jC_j^\top = w_j I, \\
A_iA_j^\top + A_jA_i^\top = 0 \ \ (i \neq j), \qquad B_{ij}B_{ik}^\top + B_{ik}B_{ij}^\top = 0 \ \ (j \neq k), \\
C_iC_j^\top + C_jC_i^\top = 0 \ \ (i \neq j), \\
B_{jk}A_i^\top = -A_iB_{jk}^\top\,, \\
B_{k\ell}B_{ij}^\top = B_{ij}B_{k\ell}^\top \ \ (i \neq k)\,, \quad C_kB_{ij}^\top = B_{ij}C_k^\top\,, \quad C_jA_i^\top = A_iC_j^\top\,,
\end{cases}
$$

and similar equations with products reversed.

Set

$$
E_i = \frac{1}{\sqrt{u_i u_0}} A_i A_0^\top\,, \quad F_{ij} = \frac{1}{\sqrt{v_{ij} u_0}} B_{ij} A_0^\top\,, \quad G_i = \frac{1}{\sqrt{w_i u_0}} C_i A_0^\top\,.
$$

It is easy to verify $E_0 = I$ and $E_1, \ldots, E_p$, $F_{11}, \ldots, F_{1p_1}$, $F_{k1}, \ldots, F_{kp_k}$, $G_1, \ldots, G_r$ satisfy

$$
\begin{cases}
E_i^2 = -I\,, \qquad F_{ij}^2 = -I\,, \qquad G_i^2 = I \\
E_jE_i = -E_iE_j \ \ (i \neq j) \qquad F_{ik}F_{ij} = -F_{ik}F_{ij} \ \ (j \neq k)\,, \\
G_jG_i = -G_jG_i \ \ (i \neq j) \\
F_{jk}E_i = -E_iF_{jk}\,, \qquad G_jE_i = -E_iG_j\,, \qquad G_kF_{ij} = -F_{ij}G_k \\
F_{k\ell}F_{ij} = F_{ij}F_{k\ell} \ \ (i \neq k)\,,
\end{cases}
$$

Thus we have arrived at an order n representation of a real algebra which is “Clifford-like”, with the one “non-Clifford” property that some pairs of distinct generators commute.

This algebra will be called a *Clifford-Gastineau-Hills algebra (CGH-algebra).*

Definition 4. A *Clifford-Gastineau-Hills algebra* is the real algebra on $p+q_1+\cdots q_k+r$ generators $\alpha_1,\ldots,\alpha_p$, $\beta_{11},\ldots,\beta_{1q_1},\ldots,\beta_{k1},\ldots,\beta_{kq_k},\gamma_1,\ldots,\gamma_r$, with defining equations:

$$\begin{cases} \alpha_i^2=-1\,, \quad \beta_{ij}^2=-1\,, \quad \gamma_i^2=1\,, \\ \alpha_j\alpha_i=-\alpha_i\alpha_j \;\; (i\neq j)\,, \quad \beta_{ik}\beta_{ij}=-\beta_{ij}\beta_{ik} \;\; (j\neq k)\,, \\ \gamma_j\gamma_i=-\gamma_i\gamma_j \;\; (i\neq j)\,, \\ \beta_{jk}\alpha_i=-\alpha_i\beta_{jk}\,, \quad \gamma_j\alpha_i=-\alpha_i\gamma_j\,, \quad \gamma_k\beta_{ij}=-\beta_{ij}\gamma_k\,, \\ \beta_{k\ell}\beta_{ij}=\beta_{ij}\beta_{k\ell} \;\; (i\neq k)\,. \end{cases} \tag{19}$$

For a repeat design of order n on $p+1,q_1,\ldots,q_k,r$ variables to exist it is *necessary* for a real order n representation of this algebra to exist.

Gastineau-Hills [5] answers completely the questions of just what are the possible orders of representations of (19), and whether the existence of an order n representation of (19) is sufficient for the existence of repeat design (16).

Observe that the case of product designs is included in what we have just done — we simply take $k=2$ and $r=0$.

If we also rewrite $q_1,q_2,\beta_{1j},\beta_{2j}$ as q,r,β_j,γ_j respectively we find that the existence of an order n product design on $(p+1,q,r)$ variables implies the existence of an order n representation of the real algebra on $p+q+r$ generators $\alpha_1,\ldots,\alpha_p$, $\beta_1,\ldots,\beta_q,\gamma_1,\ldots,\gamma_r$ with defining equations.

$$\begin{cases} \alpha_i^2=\beta_j^2=\gamma_k^2=-1 \\ \alpha_j\alpha_i=-\alpha_i\alpha_j\,, \quad \beta_j\beta_i=-\beta_i\beta_j\,, \quad \gamma_j\gamma_i=-\gamma_i\gamma_j \;\; (i\neq j) \\ \beta_j\alpha_i=-\alpha_i\beta_j\,, \quad \gamma_j\alpha_i=-\alpha_i\gamma_j\,, \quad \gamma_j\beta_i=\beta_i\gamma_j\,, \end{cases} \tag{20}$$

again a "not-quite-Clifford" algebra.

Note that (20) is not quite the same as equation (3.10) in [5, p.20], so that a theory of amicable triples need not necessarily by itself yield a theory of product designs.

In fact not even equation (3.8) in [5, p.18] (the algebra corresponding to more general amicable k-sets), seems to contain (20) as a particular case.

Then we have

Theorem 1. *Let* $(L;M_1+M_2+\cdots+M_s;N)$ *be product designs* $POD(n:a_1,\ldots,a_p;\, b_{11},\ldots,b_{1_{q_1}},\, b_{21},\ldots,\, b_{2_{q_2}},\ldots,\, b_{s1},\ldots,\, b_{sq_s};\, c_1,\ldots,c_t)$, *where* M_i *is of type* $(b_{i1},\ldots,b_{i_{q_i}})$.

Further, let $(X; (Y_1; Y_2; \ldots; Y_u); Z)$ *be repeat orthogonal designs,*

$$ROD(m : (r_1, \ldots, r_w); (p_{11}, \ldots, p_{1_{v_1}}; p_{21}, \ldots, p_{2_{v_2}}; p_{u1}, \ldots, p_{uv_u}); \\ h_1, \ldots, h_x)\,.$$

Then

$$L \times X + M_1 \times Y_{j1} + \cdots + M_k \times P_{jk} + N \times Z$$

is an orthogonal design of order mn *and type* 1 *of*

(i) $(a_1 r, \ldots, a_p r,\ b_1 p_{11}, \ldots,\ b_1 p_{1_{v_1}}, \ldots,\ b_s p_{s1}, \ldots,\ b_s p_{sq_s},\ ch_1, \ldots, ch_x)$,

(ii) $(a_1 r, \ldots, a_p r,\ b_1 p_{11}, \ldots,\ b_1 p_{1_{v_1}}, \ldots,\ b_s p_{s1}, \ldots,\ b_s p_{sq_s},\ c_1 h, \ldots, c_t h)$,

(iii) $(ar_1, \ldots, ar_w,\ b_1 p_{11}, \ldots,\ b_1 p_{1_{v_1}}, \ldots,\ b_s p_{s1}, \ldots,\ b_s p_{sq_s},\ ch_1, \ldots, ch_x)$,

(iv) $(ar_1, \ldots, ar_w,\ b_1 p_{11}, \ldots,\ b_1 p_{1_{v_1}}, \ldots,\ b_s p_{s1}, \ldots,\ b_s p_{sq_s},\ c_1 h, \ldots, c_t h)$.

where a*,* c*,* r*,* h *are the sum of some or all of the* a_i*,* c_i*,* r_i*,* h_i*, respectively, and* $b_i = b_{i1} + \cdots + b_{iq_i}$.

This construction is at first sight quite formidable, but as we shall see, it does lead to new orthogonal designs.

Geramita and Seberry [6] using many results by Peter J Robinson [14] give many results on the productivity of the previously mentioned product designs. However we need to give some repeat designs as our argument is that product designs are a subset of repeat designs. First we see that repeat designs do lead to new designs:

Example 3. These repeat designs are an example of creating new designs

ROD	Design
$ROD(4 : (1; (1; 3); 1, 3))$	$(I; (T_1; T_4); T_0)$
$ROD(4 : (1; (2; 3); 1, 3))$	$(I; (T_3; T_4); T_0)$
$ROD(4 : (1; (1; 2); 1, 1, 2))$	$(I; (T_1; T_3); T_3)$
$ROD(4 : (1; (2; 1, 2); 1, 2))$	$(I; (T_2; T_6); T_7)$

where

$$T_0 = \begin{bmatrix} x & y & y & y \\ y & -x & y & y \\ y & -y & y & -x \\ y & y & -x & -y \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & - \\ - & - & 0 & 0 \\ - & + & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & - & + \\ - & + & 0 & 0 \\ - & - & 0 & 0 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix}, \quad T_5 = \begin{bmatrix} u & v & w & w \\ v & -u & -w & w \\ w & -w & v & -u \\ w & w & -u & -v \end{bmatrix},$$

$$T_6 = \begin{bmatrix} 0 & a & b & b \\ -a & 0 & -b & b \\ -b & b & 0 & -a \\ -b & -b & a & 0 \end{bmatrix}, \quad T_7 = \begin{bmatrix} u & 0 & w & w \\ 0 & -u & -w & w \\ w & -w & 0 & -u \\ w & w & -u & 0 \end{bmatrix}.$$

These repeat designs can be constructed using Theorem 2.

$ROD(4:(1;(1,1;1,1);1))$ $\quad ROD(4:(1;(1,1;1,2);2))$
$ROD(4:(1;(1,1;2);1,2))$ $\quad ROD(4:(1;(1;1,2);2,2))$
$ROD(4:(1;(1,2;1,2);4))$

Example 4. There are product designs $POD(8:1,1,2,3;1,3,3;1)$, $POD(8:2,2;1,1,1,1;4)$ and $POD(8:1,1,1;1,1,1;5)$. Then using the repeat design $ROD(4:1;(2;3);1,3)$ with the matrix of weight 2 used once only, we have $OD(32;(1,1,2,3,2,9,9,1,3))$, $OD(32;(2,2,2,3,3,3,4,12))$ and $OD(32;(1,1,1,2,3,3,5,15))$.

Since all of these have weight 31, we use the Geramita-Verner theorem to obtain the following orthogonal designs: $OD(32;1,1,1,1,2,2,3,3,9,9)$, $OD(32;1,2,2,2,3,3,3,4,12)$ and $OD(32;1,1,1,1,2,3,3,5,15)$. These last two designs are exciting.

The product designs $POD(4\colon 1,1,1;1,1,1;1)$ can be used with the repeat designs of types $(1;(p;3);1,3)$, $p=1,2$, to obtain $OD(16;1,1,1,1,p,p,3,3)$, $p=1,2$. These were first given in Geramita and Seberry [6].

Remark 3. In the preceding example we have concentrated on constructing orthogonal designs with no zero. There is considerable scope to exploit these constructions to look, for other orthogonal designs in order 32 and higher powers of 2.

We can collect the results from Example 3 in the following statement:

Proposition 1. *In order* 4 *there exist repeat designs of types* $(1; (r; s); h)$ *for* $0 \leq r,\ s \leq 3,\ 0 \leq h \leq 4$.

Noting that the repeat designs $(R; (P); H)$ are just amicable orthogonal designs $R + P$ and H, we see that:

Corollary 1. *There exist* $AOD(4; (1, r), (h))$ *for* $0 \leq r \leq 3,\ 0 \leq h \leq 4$.

Remark 4. The non-existence of $AOD(8; (1, 7), (5))$ and $AOD(16; (1, 15), (1))$ means there are no repeat designs of types $(1; (r; 7); 5)$ in order 8 and $(1; (r; 15); 1)$ in order 16 (see Robinson [15]).

The construction and replication lemmas given later allow us to say:

Comment 1. In order 8 there, in fact, exist repeat designs $(1; (r); h)$ for all $0 \leq r \leq 7$ and $0 \leq h \leq 8$, except $r = 7$, $h = 5$ (which cannot exist).

In order 16 there exist repeat designs $(1; (r); h)$ for all $r = 1, 2, 3, \ldots, 15$, $h = 1, 2, \ldots, 16$, except possibly the following pairs (r, h): $(13, 1)$, $(13, 5)$, $(13, 9)$, $(15, 7)$, $(15, 9)$, $(15, 15)$ which are undecided and $(15, 1)$ which does not exist.

5.1 Construction and Replication of Repeat Designs

We now show that many repeat designs can be constructed.

Lemma 1. *Suppose* $AOD(n_1 : (a); (b_1, b_2)$ *and* $AOD(n_2 : (c); (d_1, d_2)$ *are amicable orthogonal designs. Then there is a repeat design in order* $n_1 n_2$ *of type* $ROD(n_1 n_2 : (b_1 d_1;\ (ad_2, b_2 d_1; b_2 c, b_1 d_2); ac)$.

Proof. Let $A, x_1 B_1 + x_2 B_2$ and $C, y_1 D_1 + y_2 D_2$ be the amicable orthogonal designs. Then $(B_1 \times D_1;(xA \times D_2 + yB_2 \times D_1;uB_2 \times C + wB_1 \times D_2);A \times C)$ are the required repeat designs. □

Example 5. Let $A = C = \left[\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right]$, $B_1 = D_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $B_2 = D_2 = \left[\begin{smallmatrix} 0 & 1 \\ - & 0 \end{smallmatrix}\right]$. Then the repeat design in order 4 and type $(1; (1, 2; 1, 2); 4)$ is

$$\left(I_4\,;\ \left(\left[\begin{array}{cc|cc} 0 & y & x & x \\ \bar{y} & 0 & x & \bar{x} \\ \hline \bar{x} & \bar{x} & 0 & y \\ \bar{x} & x & \bar{y} & 0 \end{array}\right] ; \left[\begin{array}{cc|cc} 0 & u & w & u \\ \bar{u} & 0 & \bar{u} & w \\ \hline \bar{w} & u & 0 & \bar{u} \\ \bar{u} & \bar{w} & u & 0 \end{array}\right] \right) ; \quad z \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ \hline 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right] \right).$$

Before we proceed to our uses of repeat designs, we first note some replication results.

Theorem 2. *Suppose there are repeat designs $ROD(n : (r; (p_1, \dots, p_i; q_1, \dots, q_j); h_1, \dots, h_k))$ called X, Y, Z where $h_1 + h_2 + \cdots + h_k = h$ and $p_1 + \cdots + p_i = p$. Further suppose $A+B$ and $C+D$ are $AOD(m; (a, b), (c, d))$. Then there are repeat designs of order mn and types*

1. $(ar; (cp_1, cp_2, \dots, b_r; aq_1, aq_2, \dots, bh); ch)$,
2. $(ar; (ap_1, ap_2, \dots; cq_1, cq_2, \dots); ah_1, ah_2, \dots, ch_i, \dots)$,
3. $(ar; (ap_1, ap_2, \dots, bh_1; cq_1, cq_2, \dots); ch_1, ah_2, ah_3, \dots)$,
4. $(ar; (bh_1, bh_2, \dots; rb + pd, cq_1, cq_2, \dots); rd + bp)$, *where* $d = b$,
5. $(ar; (cq_1, cq_2, \dots; cp); ah_1, ah_2, \dots, bp)$,
6. $(ar; (br, dp_1, dp_2, \dots; aq_1, aq_2, \dots, bh); dh)$,
7. $(ar; (cp_1, \dots; cq_1, \dots); ch_1, ch_2, \dots, dr)$,
8. $(ar; (cp_1, \dots, dq_1, \dots); ah_1, ah_2, \dots, bp_1, bp_2, \dots)$,
9. $(ar; (ap_1, \dots; aq_1, \dots); ch, dh)$,
10. $(cr; (br; bh_1, bh_2, \dots); ar)$,
11. $(cr; (br; bh); ar, abrh)$.

Proof. Use the following constructions:

1. $(A \times X; (C \times Y + xB \times X; yA \times Q + zB \times Z); C \times Z)$,
2. $(A \times X; (A \times Y; C \times Q); xA \times V + C \times W)$,
3. $(A \times X; (A \times Y + xB \times V; C \times Q); C \times V + yA \times W)$,
4. $(A \times X; (B \times Z; xB \times X + yC \times Q - xD \times Y); D \times Z + B \times Y)$,
5. $(A \times X; (C \times Q; C \times Y); xA \times Z + yB \times Y)$,
6. $(A \times X; (B \times X + wD \times Y; xA \times Q + yB \times Z); D \times Z)$,
7. $(A \times X; (C \times Y; C \times Q); C \times Z + yD \times X)$,
8. $(A \times X; (C \times Y + xD \times Q); A \times Z + yB \times Y)$,

9. $(A \times X; (A \times Y; A \times Q); C \times Z + yD \times Z)$,

10. $(C \times X; (B \times X; B \times Z); A \times X)$,

11. use Lemma on the result (x). □

Corollary 2. *There are repeat designs of type* $ROD(2^t : 1; (1, 2, \ldots, 2^{t-1}; 1, 2, \ldots, 2^{t-1}); 2^t)$.

Proof. Use part (i) of Lemma 2 repeatedly with repeat designs $ROD(4 : 1; (1, 2; 1, 2); 4)$ and type $AOD(2; (1, 1), (2))$. □

5.2 Construction of Orthogonal Designs

The use of repeat designs is so powerful a source of orthogonal designs that for us, it is quite impossible to indicate all the designs constructed here. We use Robinson's Ph.D. thesis [14] and Seberry [17] as a source for product designs.

The constructions using these methods [6] allow us to say

Theorem 3. *All orthogonal designs of type* $(2^t; a, b, c, 2^t - a - b - c)$ *and of type* (a, b, c), $0 \leq a + b + c \leq 2^t$, *exist for* $t = 2, 3, 4, 5, 6, 7, 8, 9$.

Remark 5. We believe these results do, in fact, allow the construction of all full orthogonal designs (that is, with no zero) with four variables in every power of 2, but we have not been able to prove this result.

Example 6. There is a product design of type $(1, 1, 1, 1, 2, 4, \ldots, 2^{t-4}; 2, 2^{t-3}; 2, 4, \ldots, 2^{t-4}, 2^{t-3}, 2^{t-3})$ in order 2^t. So using an amicable pair of weights (a, b) in order n gives an $OD(2^t n\colon (1, 1, 1, 1, 2, 4, \ldots, 2^{t-4},\ 2a, 2^{t-3}a,\ 2b, 4b, \ldots, 2^{t-4}b,\ 2^{t-3}b, 2^{t-3}b))$.

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