On the generation of skew Hadamard matrices

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Outine:

1. skew Hadamard matrices—definition, uses, existence
2. some constructions and classification results
3. a problem—skew regular Hadamard matrices
4. two approaches: skewizing regular matrices, regularizing skew matrices
5. preliminaries—three normalization, type, skew three-normalization
6. skewizing algorithm
7. skew switching operations

Skew Hadamard matrices are (real) Hadamard matrices H satisfying $(H - I)^T = I - H$.

Equivalently, $C = H - I$ is an antisymmetric conference matrix.
A **conference matrix**

- has 0 on the diagonal, ± 1 off of the diagonal
- has orthogonal rows: $CC^T = (n-1)I$, $C^T = -C$.

Every conference matrix of size $\equiv 0 \pmod 4$ is equivalent to an antisymmetric matrix, $C^T = -C$.

Uses

- telephone conference networks (Belevitch, 1950)
- design of experiments: D-optimal weighing designs (Kounias and Chadjipantelis, 1983); edge designs (Elster and Neumaier, 1995)
- graph theory: doubly-regular tournaments (Reid and Brown, 1972)
- combinatorial designs: numerous Hadamard-matrix constructions rely on skew Hadamard matrices—see Seberry Wallis, 1972 and Seberry and Yamada, 1992

Existence

Conjecture

A skew Hadamard matrix exists in each of sizes 1, 2, $4m$, $n \in \mathbf{Z}_+$.

Until 2008, smallest open size was 188. Following construction of size 236 (Fletcher, Koukouvinos, and Seberry, 2004) and size 188 (Đoković, 2008), the smallest open size appears to be 276.

Progress continues: Đoković, Golubitsky, and Kotsiereas, 2014, find skew Hadamard matrices of sizes 852 and 2524.

Constructions: Paley

When $4m - 1$ is a prime power, Paley's construction gives a skew Hadamard matrix of size $4m$.

Equivalently, adjacency matrix of the Paley digraph is skew Hadamard.

Constructions: doubling

Theorem

(Seberry Wallis, 1971) If $C + I$ is a skew Hadamard matrix of size n, then

$$\begin{bmatrix} C+I & C+I \\ C-I & I-C \end{bmatrix}$$

is a skew Hadamard matrix of size $2n$.

Constructions: doubling

Theorem

If H is a skew Hadamard matrix of size n and A is a symmetric Hadamard matrix of size n, then

$$\begin{bmatrix} H & A \\ -A & H \end{bmatrix}$$

is a skew Hadamard matrix of size $2n$.

Constructions: Goethals–Seidel, 1970

If A, B, C, D are circulant ± 1 matrices of size m satisfying

$$AA^T + BB^T + CC^T + DD^T = 4mI$$

and R is the reversing permutation, then

$$H = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{bmatrix}$$

is a $4m \times 4m$ Hadamard matrix. If $A - I$ is skew symmetric, then H is skew Hadamard.

Constructions: Williamson array and good matrices

If A, B, C, D are ± 1 matrices of size m satisfying

$$AA^T + BB^T + CC^T + DD^T = 4mI$$

and if $A - I$ is circulant and skew symmetric while B, C, D are back circulant and symmetric, then

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

is a $4m \times 4m$ skew Hadamard matrix.

Constructions

Numerous others, including

- G matrices
- constructions of Wallis–Whiteman
- construction of Delsarte, Goethals, and Seidel

Resources

Skew Hadamard matrices can be found online at the websites of Christos Koukouvinos, Jennifer Seberry, Warren Smith (rangevoting.org), and Ted Spence.

Notions of equivalence

Two ± 1 matrices A and B are **Hadamard equivalent** if $A = PBQ$ for some signed permutation matrices P and Q.

Two ± 1 matrices A and B are **skew Hadamard equivalent** if $A = PBP$ for some signed permutation matrix P.

The partition of the set of skew Hadamard matrices into skew Hadamard equivalence classes is a refinement of the partition into Hadamard equivalence classes.

Number of Hadamard equivalence classes

n	# classes skew Hadamard	# classes Hadamard
4	1	1
8	1	1
12	1	1
16	2	5
20	2	3
24	16	60
28	54	487
32	> 2180	$13,710,027$
36	> 563	$> 18,000,000$

skew Hadamard equiv. classes larger probably around 6000–7000

Goals

- To devise tools for generating large numbers of non-equivalent skew Hadamard matrices
- To understand how the equivalence classes are related.

A motivating problem

Generate examples of skew regular Hadamard matrices of sizes 36, 100, 196, . . .

A **skew regular Hadamard matrix** is a skew Hadamard matrix whose row sums are all equal in absolute value.

Finding a skew regular Hadamard matrix of size $n = 4r^2$, r odd, allows, among other things, the construction of regular Hadamard matrices of sizes $n(n-1)^k$ for all k.

A motivating problem

Remarks

- If a row sum of a skew Hadamard matrix is s, the corresponding column sum is $2 - s$.
- The row sums of a skew regular Hadamard matrix of size $4r^2$ will equal $\pm 2r$.
- The row sums of a skew regular Hadamard matrix cannot be made all positive without sacrificing skewness.
- There will be $r(2r+1)$ positive row sums.

Brute force search strategies

1. Generate a large number of regular Hadamard matrices. For each such matrix, determine whether the matrix is equivalent to a skew Hadamard matrix. (If so, row negations produce a skew regular matrix.)
2. Generate a large number of skew Hadamard matrices. For each such matrix, attempt to find symmetric line negations that produce row sums of constant absolute value.

Search results

Note: although sizes $4r^2$ with r even are not relevant to the application, we include them as a test of the search strategies.

- $n = 4$ ($r = 1$): $H = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$ is an example
- $n = 16$ ($r = 2$): no example exists
- $n = 36$ ($n = 3$): every skew Hadamard matrix tested is equivalent to a skew regular Hadamard matrix.
- $n \geq 64$ ($r \geq 4$): no example yet known

Preliminary notions: 3-normalization, type

3-normalization: normalize columns so that $3 \times n$ submatrix consisting of rows 1, 2, 3 has an **odd** number of elements -1 in each column.

Consequences:

- column set partitioned into four classes of size $n/4$ according to 3-element prefixes.
- in each of rows 4–n, each column class contains the same number of elements -1.
- define the **type** of row quadruple $\{1, 2, 3, j\}$ to be $\min(r, n/4 - r)$, where r is the number of elements -1 in each column class in row j. Type ranges from 0 to $n/8$.
- If $n \equiv 4 \pmod 8$, $n > 4$, then type 0 cannot occur.

Preliminary notions: skew 3-normalization

If H is skew Hadamard and 3-normalized on rows 1, 2, 3, then columns 1, 2, 3 have a sort of anti-3-normalization: except for row 1, 2, 3, each row prefix contains and **even** number of elements -1.

Top left 3×3 submatrix takes one of eight forms; reduced to two under compatible normalizations of lines 1, 2, 3; reduced to one under permutations of lines 1, 2, 3.

Relation of row and column types

Observation: an a skew Hadamard matrix, the types of rows $1, 2, 3, j$ and columns $1, 2, 3, j$ are equal or differ by ± 1, depending on their pattern of overlap.

Skewizing algorithm

Idea: fix the row permutation and both row and column normalizations (using skew 3-normalization). Search reduced to finding suitable column permutation.

A bit more detail:

- there are $8 \cdot (n/4)^3$ choices for columns 1, 2, 3.
- many of these are eliminated due to incompatibility in the number of row quadruples of a given type compared with the number of column quadruples of a given type
- among the compatible choices, we recursively search for the desired column permutation
- as more columns are fixed, the search space typically reduces quite rapidly.

Application of skewizing algorithm: $n = 32, 36$

We have not yet attempted an exhaustive search through all $13,710,027$ equivalence classes in size 32, but it appears that approximately one in every 2000 classes contains a skew Hadamard matrix.

Equivalence classes in size 36 contain a skew Hadamard matrix extremely rarely. Improvements in implementation may yield results, but no example found yet.

Switching operations

If H if size n, W is $r \times s$, and $rs = n$ then

$$H = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \longrightarrow \begin{bmatrix} -W & X \\ Y & Z \end{bmatrix}$$

is a switching operation. (W is a maximal rank-1 block.)

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Columns are still orthogonal.

Special switching operation: Hall sets

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Hall-set switching requires negation of *two* rank-1 blocks.

Hall-set switching

$$\begin{bmatrix} 1 & - & - & - & j^T & j^T & j^T & j^T \\ - & 1 & - & - & j^T & j^T & -j^T & -j^T \\ - & - & 1 & - & j^T & -j^T & j^T & -j^T \\ - & - & - & 1 & j^T & -j^T & -j^T & j^T \\ j & j & j & j & A_1 & B_{12} & B_{13} & B_{14} \\ -j & -j & j & j & B_{21} & A_2 & B_{23} & B_{24} \\ -j & j & -j & j & B_{31} & B_{32} & A_3 & B_{24} \\ -j & j & j & -j & B_{41} & B_{42} & B_{43} & A_4 \end{bmatrix}$$

j is a column of $\frac{n}{4} - 1$ ones.

A_i have row/column sums 2.

B_{ij} have row/column sums 0.

Hall-set switching

$$\begin{bmatrix} 1 & - & - & - & \color{red}{-j^T} & j^T & j^T & j^T \\ - & 1 & - & - & \color{red}{-j^T} & j^T & -j^T & -j^T \\ - & - & 1 & - & \color{red}{-j^T} & -j^T & j^T & -j^T \\ - & - & - & 1 & \color{red}{-j^T} & -j^T & -j^T & j^T \\ \color{red}{-j} & \color{red}{-j} & \color{red}{-j} & \color{red}{-j} & A_1 & B_{12} & B_{13} & B_{14} \\ -j & -j & j & j & B_{21} & A_2 & B_{23} & B_{24} \\ -j & j & -j & j & B_{31} & B_{32} & A_3 & B_{34} \\ -j & j & j & -j & B_{41} & B_{42} & B_{43} & A_4 \end{bmatrix}$$

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Application of switching to skew Hadamard matrices

Need to preserve skewness.

In $n \equiv 0 \pmod 8$, focus on Hall sets. Hall columns of a Hall set are type-0, and rank-1 switching applies. Under suitable conditions, types of row quad and column quad are interchanged by switching, and the corresponding column switching operation may then be applied, preserving skewness.

A similar procedure applies when $n \equiv 4 \pmod 8$: need two Hall sets in which Hall column indices are disoint form Hall row indices. Two Hall set switching operations are then applied successively.