A matrix approach to Yang multiplication

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Sequence and real Hadamard matrices

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A quadruple $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of complementary sequences of length $\boldsymbol{n}$ can be used to construct a Hadamard matrix of order $4n$, via the Goethals-Seidel array:

$$\boldsymbol{H} = \begin{bmatrix} A & -BR & -CR & -DR \\ BR & A & -D^\top R & C^\top R \\ CR & D^\top R & A & -B^\top R \\ DR & -C^\top R & B^\top R & A \end{bmatrix}, \quad \boldsymbol{H}\boldsymbol{H}^\top = 4n\boldsymbol{I},$$

where

$$\begin{aligned} \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} &= \text{circulant matrix with first row } \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \\ \boldsymbol{R} &= \text{back diagonal permutation matrix.} \end{aligned}$$

Quadruple of complementary sequences

Kharaghani–Koukouvinos, Part V, Chapter 8 of CRC Handbook.

1. Base seq. $BS(m, n)$: length (m, m, n, n)
2. Near normal seq. $NN(n)$: a special case of $BS(n + 1, n)$
3. Nonperiodic complementary seq. $NCS(n)$: length (n, n, n, n)
4. Golay seq. $GCP(n)$: $(n, n, 0, 0)$

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For $\{0,\pm 1\}$-sequences (ternary),

1. Normal seq. $NS(n)$: length $(n,n,n,0)$, weight $2n$,
2. T-seq. $TS(n)$: length (n,n,n,n), weight n

(with some disjointness conditions).
Work done by Craigen, Doković, Kotsireas, Seberry,. . . .

From base sequences to 4 complementary sequences

Yang (1989), Theorem 4, states

$$\begin{aligned} & BS(m+1,m) \neq \emptyset,\ BS(n+1,n) \neq \emptyset \\ & \implies NCS((2m+1)(2n+1),4) \neq \emptyset \\ & (\implies \exists H(4(2m+1)(2n+1))). \end{aligned}$$

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Is the proof difficult?
See the next page: only 9 lines.

$$[Q, R] = (\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n, \alpha_{n+1}),$$
$$[S, T] = (\gamma_1, \delta_1, \gamma_2, \delta_2, \dots, \gamma_n, \delta_n, \gamma_{n+1}),$$

where

$$\alpha_k = \begin{pmatrix} Af_k^*/Cg_k \\ Bf_k^*/Dg_k^* \end{pmatrix}, \quad \beta_k = \begin{pmatrix} -B^*e_k/Dh_k \\ Ae_k/-Ch_k^* \end{pmatrix},$$
$$\gamma_k = \begin{pmatrix} Ag_k^*/-Cf_k \\ Bg_k/-Df_k \end{pmatrix} \quad \textit{and} \quad \delta_k = \begin{pmatrix} -Bh_k/-D^*e_k \\ Ah_k^*/C^*e_k \end{pmatrix}.$$

Proof. Obviously (Q, R, S, T) are four $(1, -1)$-sequences of length $(2m+1)(2n+1)$. By letting $a = A(z^2)$, $b = B(z^2)$, $c = zC(z^2)$ and $d = zD(z^2)$, also $f = z^{-nM}F(z^{2M})$, $g = z^{-nM}G(z^{2M})$, $h = z^x H(z^{2M})$ and $e = z^y E(z^{2M})$, where $M = 2m+1$, $x = (1-n)M$ and $y = 2m + (1-n)M$, in (L), and by observing that $Q = \mathbf{q}w$, $R = \mathbf{r}w$, $S = \mathbf{s}w$ and $T = \mathbf{t}w$, where $w = z^{nM}$, we obtain

$$\begin{aligned} |Q|^2 + |R|^2 + |S|^2 + |T|^2 &= |\mathbf{s}|^2 + |\mathbf{t}|^2 + |\mathbf{q}|^2 + |\mathbf{r}|^2 \\ &= (|A|^2 + |Bq|^2 + |C|^2 + |D|^2)(|E|^2 + |F|^2 + |G|^2 + |H|^2) \\ &= 4(2m+1)(2n+1) \quad \text{for any } z \text{ on } \mathbf{K}. \end{aligned}$$

$BS(m+1,m) \times BS(n+1,n) \to NCS((2m+1)(2n+1),4)$

$$
\begin{aligned}
(a,b,c,d) &\in BS(m+1,m) \\
&\subset \{\pm1\}^{m+1} \times \{\pm1\}^{m+1} \times \{\pm1\}^{m} \times \{\pm1\}^{m}, \\
(f,g,h,e) &\in BS(n+1,n) \\
&\subset \{\pm1\}^{n+1} \times \{\pm1\}^{n+1} \times \{\pm1\}^{n} \times \{\pm1\}^{n},
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
(q,r,s,t) &\in NCS((2m+1)(2n+1),4) \\
&\subset (\{\pm1\}^{(2m+1)(2n+1)})^4.
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$$(a',b',c',d') \in (\{\pm1\}^{2m+1})^4,\ (f',g',h',e') \in (\{\pm1\}^{2n+1})^4.$$

Our matrix approach:

$$\begin{aligned}(Q,R,S,T) &\in (\{\pm1\}^{(2n+1)\times(2m+1)})^4,\\ (q,r,s,t) &\in NCS((2m+1)(2n+1),4),\\ Q &= f'^{*\top}a' + g'^{\top}c' - e'^{\top}b'^{*} + h'^{\top}d'.\end{aligned}$$

Lagrange identity

Let $\mathcal{R}$ be a commutative ring with involutive automorphism $*$. Let $a, b, c, d, f, g, h, e \in \mathcal{R}$. Set

$$\begin{aligned} q &= af^* + cg - b^*e + dh, \\ r &= bf^* + dg^* + a^*e - ch^*, \\ s &= ag^* - cf - bh - d^*e, \\ t &= bg - df + ah^* + c^*e. \end{aligned}$$

Then

$$\begin{aligned} &qq^* + rr^* + ss^* + tt^* \\ &= (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*). \end{aligned}$$

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We use this with

$$\mathcal{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}], \; * : x \mapsto x^{-1}, \; y \mapsto y^{-1}.$$

The Hall polynomial $f_a(x)$

Let $a = (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$.
Define the *Hall polynomial* $f_a(x) \in \mathbb{Z}[x]$ of a by

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It is more convenient to use

$$\psi_a(x) = x^{1-n} f_a(x^2).$$

Example: $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2)$

$$\begin{aligned} f_a(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3, \\ \psi_a(x) &= a_0 x^{-3} + a_1 x^{-1} + a_2 x^{1} + a_3 x^{3}, \\ f_b(x) &= b_0 + b_1 x + b_2 x^2, \\ \psi_b(x) &= b_0 x^{-2} + b_1 x^{0} + b_2 x^{2}. \end{aligned}$$

Complementary sequences

Define

$$*: \mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x^{\pm 1}], \quad x \mapsto x^{-1}.$$

A k-tuple $(a_1, \dots, a_k)$ of sequences with all entries in $\{\pm 1\}$ is said to be complementary if

$$\sum_{i=1}^{k} f_{a_i}(x) f_{a_i}^*(x) \in \mathbb{Z}.$$

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Example:

$$BS(m,n) : (a,b,c,d) \in \{\pm 1\}^m \times \{\pm 1\}^m \times \{\pm 1\}^n \times \{\pm 1\}^n,$$
$$NCS(n,4) : (q,r,s,t) \in (\{\pm 1\}^n)^4.$$

Yang Multiplication Theorem (C.H. Yang, 1989)

Let

$$(a, b, c, d) \in BS(m+1, m), \quad (f, g, h, e) \in BS(n+1, n).$$

Then $\exists (q, r, s, t) \in NCS((2m+1)(2n+1))$.

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Yang's approach:

$$\begin{aligned} f_q(x) = {} & f_a(x^2) f_{f^*}(x^{2(2m+1)}) + x f_c(x^2) f_g(x^{2(2m+1)}) \\ & - x^{2m+1} f_{b^*}(x^2) f_e(x^{2(2m+1)}) \\ & + x^{2m+2} f_d(x^2) f_h(x^{2(2m+1)}). \end{aligned}$$

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$$\begin{aligned} \psi_Q(x,y) = \psi_a(x)\psi_f^*(y) + \psi_c(x)\psi_g(y) \\ - \psi_b^*(x)\psi_e(y) + \psi_d(x)\psi_h(y). \end{aligned}$$

$\psi_Q(x, y)$ for an $n \times m$ matrix Q

Let $q_0, \dots, q_{n-1}$ denote the row vector of Q:

$$Q = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{bmatrix}.$$

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Example: Let $Q = (q_{ij})$ be a 3×4 matrix. Then $\psi_Q(x, y)$ is

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$\psi_a(x)$ and $\psi_Q(x, y)$

Lemma

For sequences $\boldsymbol{a}, \boldsymbol{b}$ regarded as row vectors,

$$\psi_{\boldsymbol{b}^\top \boldsymbol{a}}(\boldsymbol{x}, \boldsymbol{y}) = \psi_{\boldsymbol{a}}(\boldsymbol{x})\psi_{\boldsymbol{b}}(\boldsymbol{y}).$$

$\psi_{a}(x)$ and $\psi_{Q}(x, y)$

Lemma

For sequences a, b regarded as row vectors,

$$\psi_{b^\top a}(x, y) = \psi_a(x)\psi_b(y).$$

For a matrix Q, denote by $\mathrm{seq}(Q)$ the sequence obtained by concatenating the rows of Q.

Lemma

If Q has m columns, then

$$\psi_{\mathrm{seq}(Q)}(x) = \psi_Q(x, x^m).$$

Our approach

Recall that our matrix approach was:

$$\psi_Q(x, y) = \psi_a(x)\psi_f^*(y) + \psi_c(x)\psi_g(y) - \psi_b^*(x)\psi_e(y) + \psi_d(x)\psi_h(y).$$

This is achieved by defining

$$Q = f^{*\top}a + g^\top c - e^\top b^* + h^\top d,$$

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This is achieved by defining

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where f^* denotes the reverse of f. Note $\psi_f^*(y) = \psi_{f^*}(y)$.

$$\psi_{\mathrm{seq}(Q)}(x) = \psi_a(x)\psi_f^*(x^m) + \psi_c(x)\psi_g(x^m) - \psi_b^*(x)\psi_e(x^m) + \psi_d(x)\psi_h(x^m).$$

Complementary sequences

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$$f_a(x^2)f_a^*(x^2) = \psi_a(x)\psi_a^*(x).$$

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$$\begin{aligned} &a_1, \dots, a_k\text{: complementary} \\ \iff &\sum_{i=1}^{k} f_{a_i}(x)f_{a_i}^*(x) \in \mathbb{Z} \\ \iff &\sum_{i=1}^{k} \psi_{a_i}(x)\psi_{a_i}^*(x) \in \mathbb{Z}. \end{aligned}$$

Recall the Lagrange identity

Let $a, b, c, d, f, g, h, e \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. Set

$$\begin{aligned} q &= af^* + cg - b^*e + dh, \\ r &= bf^* + dg^* + a^*e - ch^*, \\ s &= ag^* - cf - bh - d^*e, \\ t &= bg - df + ah^* + c^*e. \end{aligned}$$

Then

$$\begin{aligned} &qq^* + rr^* + ss^* + tt^* \\ &= (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*). \end{aligned}$$

The Lagrange identity (consequence)

Let $a, b, c, d \in \mathbb{Z}^m$, $f, g, h, e \in \mathbb{Z}^n$,

$$\begin{aligned} Q &= f^{*t}a + g^t c - e^t b^* + h^t d, \\ R &= f^{*t}b + g^{*t}d - e^t a^* - h^{*t}c, \\ S &= g^{*t}a - f^t c - h^t b + e^t d^*, \\ T &= g^t b - f^t d - h^{*t}a + e^t c^*. \end{aligned}$$

Then $Q, R, S, T \in \mathbb{Z}^{n\times m}$.

$$\begin{aligned} &(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, y) \\ &= (\psi_a\psi_a^* + \psi_b\psi_b^* + \psi_c\psi_c^* + \psi_d\psi_d^*)(x) \\ &\quad \times (\psi_e\psi_e^* + \psi_f\psi_f^* + \psi_g\psi_g^* + \psi_h\psi_h^*)(y). \end{aligned}$$

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Then

$$\begin{aligned} &(\psi_{\mathrm{seq}(Q)}\psi^*_{\mathrm{seq}(Q)} + \psi_{\mathrm{seq}(R)}\psi^*_{\mathrm{seq}(R)} \\ &+ \psi_{\mathrm{seq}(S)}\psi^*_{\mathrm{seq}(S)} + \psi_{\mathrm{seq}(T)}\psi^*_{\mathrm{seq}(T)})(x, x^m) \\ &= (\psi_a\psi_a^* + \psi_b\psi_b^* + \psi_c\psi_c^* + \psi_d\psi_d^*)(x) \\ &\quad \times (\psi_e\psi_e^* + \psi_f\psi_f^* + \psi_g\psi_g^* + \psi_h\psi_h^*)(x^m). \end{aligned}$$

Interleaving

For $a = (a_0, \ldots, a_{m-1})$, define

$$a/0 = (a_0, 0, a_1, 0, \ldots, 0, a_{m-1}) \quad \text{(length } 2m-1\text{)},$$
$$0/a = (0, a_0, 0, \ldots, 0, a_{m-1}, 0) \quad \text{(length } 2m+1\text{)}.$$

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Lemma

$$\psi_{a/0}(x) = \psi_{0/a}(x) = \psi_a(x^2).$$

Yang's Theorem

Theorem

Let $(a, b, c, d) \in BS(m+1, m)$, $(f, g, h, e) \in BS(n+1, n)$. *Then there exists* $(q, r, s, t) \in NCS((2n+1)(2m+1))$.

Construction of the matrices Q, R, S, T

Let $(a, b, c, d) \in BS(m+1, m)$, $(f, g, h, e) \in BS(n+1, n)$. Then

$$a, b \in \{\pm 1\}^{m+1}, c, d \in \{\pm 1\}^{m}, f, g \in \{\pm 1\}^{n+1}, h, e \in \{\pm 1\}^{n}.$$

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Set

$$a' = a/0,\ b' = b/0,\ c' = 0/c,\ d' = 0/d \in \{0, \pm 1\}^{2m+1},$$
$$f' = f/0,\ g' = g/0,\ h' = 0/h,\ e' = 0/e \in \{0, \pm 1\}^{2n+1}.$$

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Define $(2n+1) \times (2m+1)$ matrices with entries in $\{\pm 1\}$:

$$Q = f'^{*t}a' + g'^{t}c' - e'^{t}b'^{*} + h'^{t}d',$$
$$R = f'^{*t}b' + g'^{*t}d' - e'^{t}a'^{*} - h'^{*t}c',$$
$$S = g'^{*t}a' - f'^{t}c' - h'^{t}b' + e'^{t}d'^{*},$$
$$T = g'^{t}b' - f'^{t}d' - h'^{*t}a' + e'^{t}c'^{*}.$$

$(a, b, c, d), (f, g, h, e) \to (Q, R, S, T) \to$

Set $q = \mathrm{seq}(Q),\ r = \mathrm{seq}(R),\ s = \mathrm{seq}(S),\ t = \mathrm{seq}(T)$. Then

$$(\psi_q \psi_q^* + \psi_r \psi_r^* + \psi_s \psi_s^* + \psi_t \psi_t^*)(x)$$

$(a,b,c,d),(f,g,h,e) \to (Q,R,S,T) \to$

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$$(\psi_q\psi_q^* + \psi_r\psi_r^* + \psi_s\psi_s^* + \psi_t\psi_t^*)(x)$$
$$= (\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, x^{2m+1})$$

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Set $q = \mathrm{seq}(Q)$, $r = \mathrm{seq}(R)$, $s = \mathrm{seq}(S)$, $t = \mathrm{seq}(T)$. Then

$$\begin{aligned}
&(\psi_q\psi_q^* + \psi_r\psi_r^* + \psi_s\psi_s^* + \psi_t\psi_t^*)(x) \\
&= (\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, x^{2m+1}) \\
&= (\psi_{a'}\psi_{a'}^* + \psi_{b'}\psi_{b'}^* + \psi_{c'}\psi_{c'}^* + \psi_{d'}\psi_{d'}^*)(x) \\
&\quad \times (\psi_{e'}\psi_{e'}^* + \psi_{f'}\psi_{f'}^* + \psi_{g'}\psi_{g'}^* + \psi_{h'}\psi_{h'}^*)(x^{2m+1})
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&\in \mathbb{Z}.
\end{aligned}$$

Thus $(q, r, s, t) \in NCS((2m + 1)(2n + 1))$. This proves Yang's theorem (see arXiv:1705.05062 for details).

Another result of Yang (1983)

Theorem

Let $(a, b, c, d) \in BS(m, n)$. *Suppose* $f, g \in \{0, \pm 1\}^k$ *and* $e \in \{0, \pm 1\}^{k-1}$ *satisfy*

1. (e, f, g) *is complementary with weight* $2k + 1$,
2. $(0|f), (e|00) \in \{0, \pm 1\}^{k+1}$ *are disjoint,*
3. g *and* g^* *have the same support.*

Then $\exists (q, r, s, t) \in TS((2k + 1)(m + n))$.

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Remarks:

1. Yang (1983) shows this only for $k = 6$ with e, f, g given.
2. This is different from better known Yang multiplication (1989): $NS(k) \neq \emptyset,\ BS(m,n) \neq \emptyset \implies TS((2k+1)(m+n))$.
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