Aalto University
School of Electrical
Engineering

Classification of Butson-type Hadamard matrices using an orderly algorithm

Pekka Lampio   Ferenc Szöllősi   Patric R.J. Östergård

Aalto University

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Definition of a Butson-type Hadamard matrix

Definition (*Butson*, 1962)

A Butson-type Hadamard matrix $\mathbf{H}$ of order n is an $n \times n$ complex matrix, such that,

1. $\mathbf{H}\overline{\mathbf{H}}^T = n\mathbf{I}_n$,
2. $(\mathbf{H}_{ij})^q = 1$ for all i, j,

where q is a positive integer.

- another name: a Butson Hadamard matrix
- columns (rows) are mutually orthogonal

Example

Below is an example of a Butson-type Hadamard matrix of order 6 over complex fourth roots of unity, $\{\pm 1, \pm i\}$, denoted by BH(4,6):

1	1	1	1	1	1
1	1	i	-1	-1	-i
1	i	-i	i	-i	-1
1	-1	1	-i	-1	i
1	-1	-1	1	i	-i
1	-i	-1	-1	1	i

Log-Hadamard representation

Definition

A log-Hadamard matrix of a Butson-type Hadamard matrix H is any real matrix Φ_H satisfying:

$$H_{jk} = exp(i[\Phi]_{jk}).$$

The phases $[\Phi]_{jk}$ may be chosen to belong to $[0, 2\pi]$.

Example:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \Longrightarrow \Phi_H = \frac{2\pi}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

Equivalence of Butson-type Hadamard matrices

These operations produce a Butson-type Hadamard matrix when applied to any Butson-type Hadamard matrix:

1. Permuting the order of rows,
2. Permuting the order of columns,
3. Multiplying a row by a root of unity,
4. Multiplying a column by a root of unity.

Butson-type Hadamard matrices are equivalent if they are essentially the same in the following sense:

Definition

Butson-type Hadamard matrices A and B are equivalent, denoted by, $A \cong B$, if B can be generated from A by applying Operations 1, 2, 3, and 4.

Problems

1. Classification of Butson-type Hadamard matrices
    - Find all inequivalent matrices for the given parameters
    - Determine some basic properties of the matrices, such as, the symmetries of the matrices

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   - Isolated matrices are interesting

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   - We know that all Fourier matrices are isolated, are there any other kinds of isolated matrices?
   - Isolated matrices are interesting

3. Existence of $p \times p$ Butson-type Hadamard over qth roots of unity where p is prime and p does not divide q
   - A 7×7 matrix over 6th roots of unity exists, any other such matrices?

Rectangular BH matrices

Definition

A rectangular BH matrix $\mathbf{H}$ of order n is an $m \times n$ complex matrix, such that,

1. $1 \leq m \leq n$
2. $\mathbf{H}\overline{\mathbf{H}}^T = n\mathbf{I}_m$,
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where q is a positive integer.

- Rectangular BH matrices are a generalization of Butson-type Hadamard matrices.
- Equivalence is defined exactly the same way as for Butson-type Hadamard matrices.
- All Butson-type Hadamard matrices can be constructed by appending rows to rectangular BH matrices.

Computational methods: basic idea

Append rows to a rectangular BH matrix until it is a square matrix.

Example: BH(q=6, n=12):

0	0	0	0	0	0	0	0	0	0	0	0
.	.	.	.	.	.	.	.	.	.	.	.
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Computational methods: basic idea

Append rows to a rectangular BH matrix until it is a square matrix.

Example: BH(q=6, n=12):

0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	3	4	1	4	3	2	3	5	0
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
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0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	3	4	1	4	3	2	3	5	0
0	0	2	0	2	4	4	0	2	4	2	4
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
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.	.	.	.	.	.	.	.	.	.	.	.

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0	0	1	3	4	1	4	3	2	3	5	0
0	0	2	0	2	4	4	0	2	4	2	4
0	0	4	0	4	2	2	0	4	2	4	2
0	1	0	3	3	4	1	4	0	5	3	2
0	1	4	3	1	0	3	4	4	1	1	4
0	3	1	0	1	1	1	3	4	4	4	4
0	3	3	0	3	5	5	3	0	2	0	2
0	3	4	3	1	4	1	0	2	3	5	0
0	3	5	0	5	3	3	3	2	0	2	0
0	4	1	3	4	3	0	1	4	1	1	4
0	4	3	3	0	1	4	1	0	5	3	2

Some numbers related to BH(q=6,n=12) matrices

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- $461,683,233,537,839,796,286,862,284,208,209,920,000$ $\approx 5 \times 10^{38}$ BH(q=6,n=12) matrices in all,

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- 8703 inequivalent BH(q=6,n=12) matrices,
- $461,683,233,537,839,796,286,862,284,208,209,920,000$ $\approx 5 \times 10^{38}$ BH(q=6,n=12) matrices in all,
- $\approx 6 \times 10^{34}$ BH(q=6,n=12) matrices in an equivalence class on the average

Computational methods

Exhaustive tree search

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- Find all inequivalent $n \times n$ Butson-type Hadamard matrices over q:th roots of unity.

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- Find all inequivalent $n \times n$ Butson-type Hadamard matrices over q:th roots of unity.
- Build matrices one row at a time starting with 2-row matrices. Prune equivalent matrices as they are encountered.

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Exhaustive tree search

- Find all inequivalent $n \times n$ Butson-type Hadamard matrices over q:th roots of unity.
- Build matrices one row at a time starting with 2-row matrices. Prune equivalent matrices as they are encountered.
- Orderly generation is used for the detection of equivalent matrices.

Ordering matrices

Definition

The binary relation $<_M$ defines a total order on the set of $m \times n$ rectangular BH matrices over qth roots of unity as follows: $A <_M B$ if $\arg(A_{ij}) \leq \arg(B_{ij})$ and $A_{rs} = B_{rs}$ for all r, s with $rn + s < in + j$.

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Example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 3 & 3 & 0 \end{bmatrix} <_M \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

Orderly generation

Definition

A rectangular BH matrix A is canonical if it is the smallest matrix in it its equivalence class, that is, if

$$A \cong B \quad \text{implies} \quad A \leq_M B.$$

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- We search for the canonical matrices and ignore all the other matrices.

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$$A \cong B \quad \text{implies} \quad A \leq_M B.$$

- There is a unique canonical matrix in every equivalence class.
- We search for the canonical matrices and ignore all the other matrices.
- If $A <_M B$, then $\begin{bmatrix} A \\ X \end{bmatrix} <_M \begin{bmatrix} B \\ Y \end{bmatrix}$.

Orderly generation (cont.)

Theorem

If a matrix A is canonical, then its rows and columns are in ascending order.

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- Unfortunately the converse of the theorem does not hold.
- We must check that none of the equivalence operations yield a smaller matrix.
- The equivalence classes are very large, but luckily there is an efficient algorithm for checking if a matrix is canonical.

Orderly generation (cont.)

Perform the following steps for each row:

1. Check that the new row is orthogonal.

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1. Check that the new row is orthogonal.
2. Check that the resulting matrix is canonical.
3. Check the second column pruning condition.

These are performed in order 3, 1, 2.

Second column pruning

Theorem

Let H be a canonical $BH(q, n)$ matrix and let S be the $n \times 2$ matrix formed by the first two columns of H. Then the transpose of S is a normalized matrix where the columns are in ascending order.

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- This theorem allows the detection of those rectangular BH matrices that can not be extended to BH matrix.

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Let H be a canonical $BH(q, n)$ matrix and let S be the $n \times 2$ matrix formed by the first two columns of H. Then the transpose of S is a normalized matrix where the columns are in ascending order.

- This theorem allows the detection of those rectangular BH matrices that can not be extended to BH matrix.
- Form the set of all normalized $2 \times n$ matrices with columns in ascending order at the beginning of the search.

Second column pruning (cont.)

Example: BH(q=4, n=10):

0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	2	2	2	3
0	0	1	2	3	0	1	2	3	2
0	1	3	3	2	0	2	0	2	1
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
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0	0
0	0
0	0
0	0
0	0
0	2
0	2
0	2
0	2
0	2

0	0
0	0
0	0
0	0
0	1
0	2
0	2
0	2
0	2
0	3

0	0
0	0
0	0
0	1
0	1
0	2
0	2
0	2
0	3
0	3

Second column pruning (cont.)

Example: BH(q=4, n=10):

0	0	0	0	0	0	0	0	0	0		0	0		0	0		0	0
0	0	0	0	1	2	2	2	2	3		0	0		0	0		0	0
0	0	1	2	3	0	1	2	3	2		0	0		0	0		0	0
0	1	3	3	2	0	2	0	2	1		0	0		0	0		0	1
0	2	0	2	2	1	0	3	1	3		0	0		0	1		0	1
.	.	.	.	.	.	.	.	.	.		0	2		0	2		0	2
.	.	.	.	.	.	.	.	.	.		0	2		0	2		0	2
.	.	.	.	.	.	.	.	.	.		0	2		0	2		0	2
.	.	.	.	.	.	.	.	.	.		0	2		0	2		0	3
.	.	.	.	.	.	.	.	.	.		0	2		0	3		0	3

Second column pruning (cont.)

Example: BH(q=4, n=10):

0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	2	2	2	3
0	0	1	2	3	0	1	2	3	2
0	1	3	3	2	0	2	0	2	1
0	2	0	2	2	1	0	3	1	3
0	2	1	3	0	3	3	2	1	1
0	2	2	0	2	3	1	1	0	3
0	2	3	1	0	2	1	3	3	1
0	3	2	2	1	1	3	1	3	0
.	.	.	.	.	.	.	.	.	.

0	0
0	0
0	0
0	0
0	0
0	2
0	2
0	2
0	2
0	2

0	0
0	0
0	0
0	0
0	1
0	2
0	2
0	2
0	2
0	3

0	0
0	0
0	0
0	1
0	1
0	2
0	2
0	2
0	3
0	3

Second column pruning (cont.)

Example: BH(q=4, n=10):

0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	2	2	2	3
0	0	1	2	3	0	1	2	3	2
0	1	3	3	2	0	2	0	2	1
0	2	0	2	2	1	0	3	1	3
0	2	1	3	0	3	3	2	1	1
0	2	2	0	2	3	1	1	0	3
0	2	3	1	0	2	1	3	3	1
0	3	2	2	1	1	3	1	3	0
.	.	.	.	.	.	.	.	.	.

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	1
0	0	0	1	0	1
0	2	0	2	0	2
0	2	0	2	0	2
0	2	0	2	0	2
0	2	0	2	0	3
0	2	0	3	0	3

Second column pruning (cont.)

Example: BH(q=4, n=10):

0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	2	2	2	3
0	0	1	2	3	0	1	2	3	2
0	1	3	3	2	0	2	0	2	1
0	2	.	.	.	.	.	.	.	.
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0	0		0	0		0	0
0	0		0	0		0	0
0	0		0	0		0	0
0	0		0	0		0	1
0	0		0	1		0	1
0	2		0	2		0	2
0	2		0	2		0	2
0	2		0	2		0	2
0	2		0	2		0	3
0	2		0	3		0	3

Size of the search tree for BH(q=4, n=14) matrices

r	**Total**	**With pruning**
1	1	1
2	4	4
3	42	42
4	10,141	9,142
5	1,601,560	637,669
6	21,311,746	2,118,948
7	17,175,324	189,721
8	4,234,669	155,777
9	1,675,882	108,598
10	716,604	56,103
11	249,716	17,992
12	62,739	5,558
13	9,776	3,039
14	752	752

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14	752	752

Some preliminary results (part 1)

Classification of $n \times n$ Butson-type Hadamard matrices over qth roots of unity. ("-" means that no 2-row matrices exist)

$n \setminus q$	2	3	4	5	6	7	8	9
2	1	-	1	-	1	-	1	-
3	-	1	-	-	1	-	-	1
4	1	-	2	-	2	-	3	-
5	-	-	-	1	0	-	-	-
6	0	1	1	-	4	-	3	1
7	-	-	-	-	2	1	-	-
8	1	-	15	-	36	-	143	-
9	-	3	-	-	17	-	-	23
10	0	-	10	1	34	-	60	-

Some preliminary results (part 2)

Classification of $n \times n$ Butson-type Hadamard matrices over qth roots of unity. (“-” means that no 2-row matrices exist)

$n \setminus q$	10	11	12	13	14	15	16	17
2	1	-	1	-	1	-	1	-
3	-	-	1	-	-	1	-	-
4	3	-	4	-	4	-	5	-
5	1	-	0	-	-	1	-	-
6	0	-	11	-	0	1	5	-
7	0	-	4	-	1	-	-	-
8	299	-	756	-	1412	0	2807	-
9	1	-	65	-	0	93	-	-
10	51	-	577	-	0	1	310	-

Some preliminary results (part 3)

Classification of $n \times n$ Butson-type Hadamard matrices over qth roots of unity. (“-” means that no 2-row matrices exist)

$n \setminus q$	2	3	4	5	6	7	8	9
11	-	-	-	-	-	-	-	-
12	1	2	319	-	8703	-	53024	8
13	-	-	-	-	436	-	-	-
14	0	-	752	-	167776	3	E	-
15	-	0	-	0	0	-	-	0
16	5	-	1786763	-	E	-	E	-
17	-	-	-	-	0	-	-	-
18	0	85	E	-	E	-	E	E
19	-	-	-	-	E	-	-	-
20	3	-	E	E	E	-	E	-
21	-	72	-	-	E	0	-	E

Some preliminary results (part 4)

Classification of $n \times n$ Butson-type Hadamard matrices over qth roots of unity. (“-” means that no 2-row matrices exist)

$n \setminus q$	10	11	12	13	14	15	16	17
11	0	1	0	-	0	0	-	-
12	293123	-	E	-	E	E	E	-
13	0	-	E	1	?	?	-	-
14	E	-	E	-	E	?	E	-
15	?	-	?	-	?	E	-	-
16	E	-	E	-	E	?	E	-
17	?	-	?	-	?	?	-	1
18	?	-	E	-	?	?	E	-
19	?	-	E	-	?	?	-	-
20	E	-	E	-	E	E	E	-
21	?	-	E	-	0	E	-	-

Status of the work

- A preprint available at arXiv:1707.02287
- The work continues.
- Further performance improvements are possible.

Thank you!