Complex Hadamard Matrices for $N = 6$

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Outline

Motivation

- Complex Hadamard orbits, N=6
  - Incomplete characterization
  - Complete set of MUBs problem
  - Matolcsi et al conjecture
- Background
  - $S_6^{(0)}$ isolated, $F_6^{(2)}$, $(F_6^{(2)})^T$ affine orbits, defect=4

    $D_6^{(1)}$ $X_6^{(2)}$
  - Special ansatz: $B_6^{(1)}$ $(X_6^{(2)})^T$ $K_6^{(3)} = \frac{1}{\sqrt{3}} \begin{bmatrix} F_2 & Z_1 & \dots \\ Z_3 & \frac{1}{2} Z_3 A Z_1 & \dots \\ \dots & \dots & \dots \end{bmatrix}$,

    $M_6^{(1)}$ $K_6^{(2)}$

    non-affine
- Extra motivation
  - Non-affine in Krakow list: $K_6^{(3)}$, $(B_9^{(0)} \to) K_9^{(2)}$, $(P_{13}^{(2)} \to)$ $P_{13}^{(4)}$, no other reported

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N=6, Brute force

- $36 \rightarrow 25$ matrix elements, $|h_{ij}| = 1/\sqrt{6}$.
- 5+4+3+2=15 non-linear complex orthogonality relations
- Best reported: Szöllősi starts with
  - $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_{22} & a_{23} & x_1 & x_2 & x_3 \\ 1 & a_{32} & a_{33} & y_1 & y_2 & y_3 \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \end{bmatrix}$$, 4 parameters, 6 unknowns

    3 (complex) orthogonality relations, non-linear in x_i, y_i
  - Clever trick (Haagerup) => one eqn, two unknowns
  - $x^3 + \alpha x^2 + \beta x + \gamma + \bar{\beta}\frac{1}{x} + \bar{\alpha}\frac{1}{x^2} + \frac{1}{x^3} = 0$
  - No general solution $\in \mathbb{T}$ known on closed form
  - Go ahead anyway $\Longrightarrow$half numeric, half analytic orbit $G_6^{(4)}$

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Definitions

- Notation

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \qquad W_A = 3A^\dagger A; \qquad U_A = 3AA^\dagger$$

- Unitarity

$$\begin{aligned} A^\dagger A + C^\dagger C &= 2E \\ AA^\dagger + BB^\dagger &= 2E \\ A^\dagger B + C^\dagger D &= 0 \end{aligned}$$

- Equivalence

$$\tilde{H} \sim P_1 U_1 H U_2 P_2; \qquad \text{Dephased: } F_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

- Defect

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Rephasing invariants

- Notation: $(G = A,\ B,\ C \text{ or } D)$

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$$W_G = 3G^\dagger G = \begin{bmatrix} 3 & w_3 & \bar{w}_2 \\ \bar{w}_3 & 3 & w_1 \\ w_2 & \bar{w}_1 & 3 \end{bmatrix} \qquad U_G = 3GG^\dagger = \begin{bmatrix} 3 & u_3 & \bar{u}_2 \\ \bar{u}_3 & 3 & u_1 \\ u_2 & \bar{u}_1 & 3 \end{bmatrix}$$

- Rephasing example:

$$\begin{bmatrix} g_{21} & g_{22} & g_{23} \end{bmatrix} \to \exp(i\phi) \begin{bmatrix} g_{21} & g_{22} & g_{23} \end{bmatrix}$$

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$$u_1 \bar{u}_1,\ u_2 \bar{u}_2,\ u_3 \bar{u}_3,\ u_1 u_2 u_3,\ w_1 \bar{w}_1,\ w_2 \bar{w}_2,\ w_3 \bar{w}_3,\ w_1 w_2 w_3$$

- Intrinsic properties of A, B, C and D for phase-equivalent Hadamard matrices

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- Intrinsic properties of $A,\ B,\ C$ and D for phase-equivalent Hadamard matrices

Rephasing invariants (cont)

Definition:

$$
\begin{aligned}
p_1 &= w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3 \\
p_2 &= w_1\bar{w}_1 w_2\bar{w}_2 + w_2\bar{w}_2 w_3\bar{w}_3 + w_3\bar{w}_3 w_1\bar{w}_1 \\
p_3 &= w_1\bar{w}_1 w_2\bar{w}_2 w_3\bar{w}_3 \\
p_4 &= w_1 w_2 w_3 + \bar{w}_1\bar{w}_2\bar{w}_3
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
p_{1w} &= p_{1u} \\
p_{2w} - p_{3w} &= p_{2u} - p_{3u} \\
p_{4w} &= p_{4u}
\end{aligned}
}
$$

Note:

p_1, p_2 and p_3 even in w_i

p_4 odd in w_i

Outline

Unitarity + Rephasing Invariants

Recall unitarity:

$$A^\dagger A + C^\dagger C = 2E$$
$$W_A + W_C = 6E$$
$$w_i|_A + w_i|_C = 0$$

For a complex Hadamard matrix $H = \frac{1}{\sqrt{2}} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$,

p_1 and p_{2-3} are the same for A, B, C and D

$$p_4|_A = -p_4|_B = -p_4|_C = p_4|_D$$

Intrinsic properties for rephasing-equivalent Hadamard matrices.

Construction outline

$$
\left.\begin{array}{ccccc}
U_A & \overset{(B^\dagger)^\dagger(B^\dagger)=2E-\frac{1}{3}U_A}{\longrightarrow} & B & \overset{D^\dagger D=2E-\frac{1}{3}W_B}{\longrightarrow} & D \\
\uparrow & & & & \\
A & & & & \\
\downarrow & & & & \\
W_A & \overset{C^\dagger C=2E-\frac{1}{3}W_A}{\longrightarrow} & & C &
\end{array}\right\} H = \tfrac{1}{\sqrt{2}} \left[\begin{array}{cc} A & B \\ C & D \end{array} \right]
$$

- Needed: solutions to

$$X^\dagger X = \frac{1}{3} W$$

for $X = C$ and $W = 6E - W_A$, etc

Construction overview (cont)

- $$X^\dagger X = \frac{1}{3} W \qquad |X_{ij}| = \frac{1}{\sqrt{3}}$$
- General solution PUX with

$$X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \qquad x_i, y_i \in \mathbb{T}$$

P permutation, U rephasing matrix

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Basic equation

Basic system:

$$\begin{aligned} \frac{y_1}{x_1} + \frac{y_2}{x_2} + \frac{y_3}{x_3} &= w_1 \\ \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} &= w_2 \\ x_1 + x_2 + x_3 &= w_3 \end{aligned}$$

$x_i, y_i \in \mathbb{T}, \qquad w_i \in \mathbb{C}$

- Define:

$$\xi^3 = x_1 x_2 x_3 = x_i \frac{x_j + x_k}{\bar{x}_j + \bar{x}_k} = x_i \frac{x_i - w_3}{\bar{x}_i - \bar{w}_3} \in \mathbb{T}$$

- Rephasing invariants:

$$\hat{x}_i = x_i/\xi \qquad \sigma_3 = w_3/\xi$$

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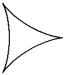
$$\hat{x}_i = x_i / \xi \qquad \sigma_3 = w_3 / \xi$$

Basic equation (cont)

$$\begin{aligned} \hat{x}_1 + \hat{x}_2 + \hat{x}_3 &= \sigma_3 \\ \hat{x}_1 \hat{x}_2 \hat{x}_3 &= 1 \\ x^3 - \sigma_3 x^2 + \bar{\sigma}_3 x - 1 &= 0 \end{aligned}$$



Roots $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ in $\mathbb{T}$ iff σ_3 in deltoid

Remains to find ξ and y_i.

Identity in $\mathbb{T}$:

$$(x_i + x_j)\left(\frac{1}{y_i} + \frac{1}{y_j}\right)\left(\frac{y_i}{x_i} + \frac{y_j}{x_j}\right) = |x_i + x_j|^2 + \left|\frac{1}{y_i} + \frac{1}{y_j}\right|^2 + \left|\frac{y_i}{x_i} + \frac{y_j}{x_j}\right|^2 - 4$$

$$\left(w_1 - \frac{y_i}{x_i}\right)\left(w_2 - \frac{1}{y_i}\right)(w_3 - x_i) = \left|w_1 - \frac{y_i}{x_i}\right|^2 + \left|w_2 - \frac{1}{y_i}\right|^2 + |w_3 - x_i|^2 - 4$$

Basic equation (cont)

Eliminate $1/y_i$, and then y_i using $y_i\bar{y}_i = 1$. B_3, A_i and R combinations of w_i

.

$$B_3(-2x_i\bar{w}_3 + 3 + w_3\bar{w}_3 - 2w_3\frac{1}{x_i})y_i$$
$$+B_3(\bar{w}_2\bar{w}_3 - w_1)x_i + B_3(w_3w_1 - \bar{w}_2)$$
$$+A_2x_i^2 - w_3A_2x_i + \bar{w}_3A_1 - A_1\frac{1}{x_i} = 0$$

$$\begin{aligned}
(A_2x_i\frac{(x_i - w_3)}{(\frac{1}{x_i} - \bar{w}_3)} - A_1)(\bar{A}_2\frac{1}{x_i}\frac{\frac{1}{x_i} - \bar{w}_3}{(x_i - w_3)} - \bar{A}_1) - B_3^2R &= 0 \\
\xi^3 &= x_i\frac{x_i - w_3}{\bar{x}_i - \bar{w}_3} \\
(A_2\xi^3 - A_1)(\bar{A}_2\frac{1}{\xi^3} - \bar{A}_1) - B_3^2R &= 0 \\
\sigma_3 &= \frac{w_3}{\xi}
\end{aligned}$$

Basic equation (cont)

- Second order in ξ^3 (or σ_3^3) , two solutions for $\xi^3 \in \mathbb{T}$ iff discriminant $(w_1\bar{w}_1 - w_2\bar{w}_2)^2\Delta_2 \leq 0$
- $\Delta_2(p_1, p_2 - p_3, p_4)$ rephasing invariant

$$\begin{aligned}\Delta_2 &= (p_2-p_3)^2 - 2(p_2-p_3)(p_1^2 - p_1p_4 - 27p_1 + 15p_4 + 162)\\ &\quad +(p_1-p_4)(p_1^3 - p_1^2p_4 + 18p_1^2 - 6p_1p_4 - 4p_4^2 - 81p_1 - 27p_4),\end{aligned}$$

- $\xi \to \sigma_3 = w_3/\xi \to \hat{x}_i \to x_i = \xi\hat{x}_i \to y_i \to X = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$
- W gives two closed form solutions $P_1U_1X_1$ and $P_2U_2X_2$ if $\sigma_3 \in$deltoid and $\Delta_2 \leq 0$.
- Go ahead, construct H !

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Construction of H

- Apply to H : $A \to C_1, C_2$, $A \to B_1, B_2$ etc: same Δ_2 condition at every step.
  Recall: p_1, p_2, p_3 even, p_4 odd in w_i:

$$\Delta_2(p_1, p_2 - p_3, p_4)|_C = \Delta_2(p_1, p_2 - p_3, -p_4)|_A$$
$$\Delta_2(p_1, p_2 - p_3, \pm p_4)| \leq 0$$

Construction of H, cont 1

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} A & B \\ C & D \end{bmatrix};$$

- 8 candidate H before last step
  - Final unitarity condition selects 2 two complex Hadamards for each A that satisfies $\Delta_2 \leq 0$.

$$U_A \overset{\Delta_2 \leq 0}{\longrightarrow} \left\{ \begin{array}{l} B_1 \overset{\Delta_2 \leq 0}{\longrightarrow} \left\{ \begin{array}{l} D_{11} \\ D_{12} \end{array} \right. \\ B_2 \overset{\Delta_2 \leq 0}{\longrightarrow} \left\{ \begin{array}{l} D_{21} \\ D_{22} \end{array} \right. \end{array} \right.$$

$$\uparrow$$

$$A$$

$$\downarrow$$

$$W_A \overset{\Delta_2 \leq 0}{\longrightarrow} \left\{ \begin{array}{l} C_1 \\ C_2 \end{array} \right.$$

$$\left. \right\} \overset{C^\dagger D + A^\dagger B = 0}{\longrightarrow} \left\{ \begin{array}{l} H_1 \\ H_2 \end{array} \right.$$

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Construction of H, Δ_2 cond

- $\Delta_2 \leq 0$ implicit condition on the parameters
- For 10^6 random A, 320526 passed this condition.
- Work in progress, room for improvements.
  - reconsider choice of parameters =>rephasing invariants?
  - detail: $\Delta_2 \leq 0$ implies σ_3 in deltoid.
  - reformulate $\Delta_2 \leq 0$
- Matolcsi et al conjecture - confirmed numerically
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Summary

- Main points:
  - rephasing invariants p_1, $p_2 - p_3$, p_4
  - Overcome the sextic polynomial roots obstacle
  - Closed form expressions in terms of square and cubic roots for any $N = 6$ complex Hadamard
  - Parameter condition: $\Delta_2(p_1, p_2 - p_3, \pm p_4) \leq 0$
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Outline

N	isolated	affine (def)	non-affine	composite
2	$F_2^{(0)}$	-	-	-
3	$F_3^{(0)}$	-	-	-
4	-	$F_4^{(1)}(1)$	-	$\begin{pmatrix} F_2 & \Delta F_2 \\ F_2 & -\Delta F_2 \end{pmatrix}$
5	$F_5^{(0)}$	-	-	-
6	$S_6^{(0)}$	$F_6^{(2)}(4)$	$K_6^{(3)}(4)$ $G_6^{(4)}(4)$	$K_6^{(3)} = \begin{pmatrix} F_2 & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$ $G_6^{(4)}$ no
7	$F_7^{(0)}, Q_7^{(0)}$ $C_{7A...D}^{(0)}$	$P_7^{(1)}$ 2(!)		
8	$A_8^{(0)}$ $V_{8A-D}^{(0)}$	$F_8^{(5)}$ (5) $S_8^{(4)}$ (5) $D_{8A...B}^{(6)}$ (5)	?	
9	$S_9^{(0)}$ $N_9^{(0)}$ $(B_9^{(0)})$	$F_9^{(4)}(4)$	$K_9^{(2)}(2)$	$F_9^{(4)}$ yes $K_9^{(2)}$ no

Hadamard orbits, Krakow listings

- Isolated matrices:

$$F_2,\ F_3,\ F_5,\ S_6^{(0)},\ F_7^{(0)},\ Q_7^{(0)},\ C_{7A-D}^{(0)},\ A_8^{(0)},\ V_{8A-D}^{(0)},\ S_9^{(0)},\ N_9^{(0)},\ \ldots$$

(many more $N \geq 10$)

- Affine orbits:

$$H = H_0 \circ \exp(i\phi_1 R_1 + i\phi_2 R_2 + \ldots)/\sqrt{N} \qquad R_i \text{ const}$$

$$F_4^{(1)},\ F_6^{(2)},\ (F_6^{(2)})^T,\ P_7^{(1)},\ S_8^{(4)},\ F_8^{(5)},\ D_8^{(6)},\ BC_{9A}^{(1)},\ F_9^{(4)},\ G_{10}^{(3)},\ D_{10}^{(3)},$$
$$N_{10B}^{(3)},\ F_{10}^{(4)},\ (F_{10}^{(4)})^T,\ D_{10A}^{(7)},\ D_{10B}^{(7)},\ S_{12}^{(5)},\ ..\ (\text{many more } N \geq 12)$$

Composite construct (example):

$$F_6^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} F_3 & UF_3 \\ F_3 & -UF_3 \end{bmatrix}; \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{bmatrix}$$

Hadamard orbits, Krakow listings

- Isolated matrices:

$$F_2,\quad F_3,\quad F_5,\quad S_6^{(0)},\quad F_7^{(0)},\quad Q_7^{(0)},\quad C_{7A-D}^{(0)},\ A_8^{(0)},\quad V_{8A-D}^{(0)},\quad S_9^{(0)},\quad N_9^{(0)},\ \ldots$$

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Hadamard orbits, Krakow listing (cont)

- Non-affine orbits

$$K_6^{(3)} \text{ and suborbits },\ G_6^{(4)},\ K_9^{(2)},\ P_{13}^{(4)} \text{ (no other known)}$$

Composite constructs: example, N=12, 11 parameters:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} K_{6A}^{(3)} & UK_{6B}^{(3)} \\ K_{6A}^{(3)} & -UK_{6B}^{(3)} \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_5 \end{bmatrix}$$

Hadamard orbits, Krakow listing (cont)

- $K_6^{(3)} = \begin{bmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & \frac{1}{2} Z_3 A Z_1 & \frac{1}{2} Z_3 B Z_2 \\ Z_4 & \frac{1}{2} Z_4 B Z_1 & \frac{1}{2} Z_4 A Z_2 \end{bmatrix}; \quad A = \begin{bmatrix} A_{11} & A_{12} \\ \bar{A}_{12} & -\bar{A}_{11} \end{bmatrix}$

$$\begin{aligned} A_{11} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}(\cos\theta + \exp(-i\phi)\sin\theta) \\ A_{12} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}(-\cos\theta + \exp(i\phi)\sin\theta) \\ B &= -F_2 - A \\ Z &= \ldots \end{aligned}$$